AN INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

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Introduction

1.1 Preliminaries

A partial differential equation (PDE) describes a relation between an unknown function and its partial derivatives. PDEs appear frequently in all areas of physics and engineering. Moreover, in recent years we have seen a dramatic increase in the use of PDEs in areas such as biology, chemistry, computer sciences (particularly in relation to image processing and graphics) and in economics (finance). In fact, in each area where there is an interaction between a number of independent variables, we attempt to define functions in these variables and to model a variety of processes by constructing equations for these functions. When the value of the unknown function(s) at a certain point depends only on what happens in the vicinity of this point, we shall, in general, obtain a PDE. The general form of a PDE for a function $u(x_1, x_2, \ldots, x_n)$ is

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_{11}}, \ldots) = 0, \tag{1.1}$$

where $x_1, x_2, \ldots, x_n$ are the independent variables, u is the unknown function, and u_{x_i} denotes the partial derivative $\partial u/\partial x_i$. The equation is, in general, supplemented by additional conditions such as initial conditions (as we have often seen in the theory of ordinary differential equations (ODEs)) or boundary conditions.

The analysis of PDEs has many facets. The classical approach that dominated the nineteenth century was to develop methods for finding explicit solutions. Because of the immense importance of PDEs in the different branches of physics, every mathematical development that enabled a solution of a new class of PDEs was accompanied by significant progress in physics. Thus, the method of characteristics invented by Hamilton led to major advances in optics and in analytical mechanics. The Fourier method enabled the solution of heat transfer and wave

propagation, and Green's method was instrumental in the development of the theory of electromagnetism. The most dramatic progress in PDEs has been achieved in the last 50 years with the introduction of numerical methods that allow the use of computers to solve PDEs of virtually every kind, in general geometries and under arbitrary external conditions (at least in theory; in practice there are still a large number of hurdles to be overcome).

The technical advances were followed by theoretical progress aimed at understanding the solution's structure. The goal is to discover some of the solution's properties before actually computing it, and sometimes even without a complete solution. The theoretical analysis of PDEs is not merely of academic interest, but rather has many applications. It should be stressed that there exist very complex equations that cannot be solved even with the aid of supercomputers. All we can do in these cases is to attempt to obtain qualitative information on the solution. In addition, a deep important question relates to the formulation of the equation and its associated side conditions. In general, the equation originates from a model of a physical or engineering problem. It is not automatically obvious that the model is indeed consistent in the sense that it leads to a solvable PDE. Furthermore, it is desired in most cases that the solution will be unique, and that it will be stable under small perturbations of the data. A theoretical understanding of the equation enables us to check whether these conditions are satisfied. As we shall see in what follows, there are many ways to solve PDEs, each way applicable to a certain class of equations. Therefore it is important to have a thorough analysis of the equation before (or during) solving it.

The fundamental theoretical question is whether the problem consisting of the equation and its associated side conditions is well posed. The French mathematician Jacques Hadamard (1865–1963) coined the notion of *well-posedness*. According to his definition, a problem is called well-posed if it satisfies all of the following criteria

1. **Existence** The problem has a solution.
2. **Uniqueness** There is no more than one solution.
3. **Stability** A small change in the equation or in the side conditions gives rise to a small change in the solution.

If one or more of the conditions above does not hold, we say that the problem is *ill-posed*. One can fairly say that the fundamental problems of mathematical physics are all well-posed. However, in certain engineering applications we might tackle problems that are ill-posed. In practice, such problems are unsolvable. Therefore, when we face an ill-posed problem, the first step should be to modify it appropriately in order to render it well-posed.

1.2 Classification

We pointed out in the previous section that PDEs are often classified into different types. In fact, there exist several such classifications. Some of them will be described here. Other important classifications will be described in Chapter 3 and in Chapter 9.

- **The order of an equation**
  The first classification is according to the *order* of the equation. The order is defined to be the order of the highest derivative in the equation. If the highest derivative is of order k, then the equation is said to be of order k. Thus, for example, the equation $u_{tt} - u_{xx} = f(x, t)$ is called a second-order equation, while $u_t + u_{xxxx} = 0$ is called a fourth-order equation.
- **Linear equations**
  Another classification is into two groups: linear versus nonlinear equations. An equation is called *linear* if in (1.1), F is a linear function of the unknown function u and its derivatives. Thus, for example, the equation $x^7 u_x + e^{xy} u_y + \sin(x^2 + y^2)u = x^3$ is a linear equation, while $u_x^2 + u_y^2 = 1$ is a nonlinear equation. The nonlinear equations are often further classified into subclasses according to the type of the nonlinearity. Generally speaking, the nonlinearity is more pronounced when it appears in a higher derivative. For example, the following two equations are both nonlinear:

$$u_{xx} + u_{yy} = u^3, \tag{1.2}$$

$$u_{xx} + u_{yy} = |\nabla u|^2 u. \tag{1.3}$$

  Here $|\nabla u|$ denotes the norm of the gradient of u. While (1.3) is nonlinear, it is still linear as a function of the highest-order derivative. Such a nonlinearity is called *quasilinear*. On the other hand in (1.2) the nonlinearity is only in the unknown function. Such equations are often called *semilinear*.
- **Scalar equations versus systems of equations**
  A single PDE with just one unknown function is called a *scalar equation*. In contrast, a set of m equations with l unknown functions is called a *system* of m equations.

1.3 Differential operators and the superposition principle

A function has to be k times differentiable in order to be a solution of an equation of order k. For this purpose we define the set $C^k(D)$ to be the set of all functions that are k times continuously differentiable in D. In particular, we denote the set of continuous functions in D by $C^0(D)$, or $C(D)$. A function in the set C^k that satisfies a PDE of order k, will be called a *classical* (or *strong*) solution of the PDE. It should be stressed that we sometimes also have to deal with solutions that are not classical. Such solutions are called *weak* solutions. The possibility of weak solutions and their physical meaning will be discussed on several occasions later,

see for example Sections 2.7 and 10.2. Note also that, in general, we are required to solve a problem that consists of a PDE and associated conditions. In order for a strong solution of the PDE to also be a strong solution of the full problem, it is required to satisfy the additional conditions in a smooth way.

Mappings between different function sets are called *operators*. The operation of an operator L on a function u will be denoted by $L[u]$. In particular, we shall deal in this book with operators defined by partial derivatives of functions. Such operators, which are in fact mappings between different C^k classes, are called *differential operators*.

An operator that satisfies a relation of the form

$$L[a_1u_1 + a_2u_2] = a_1L[u_1] + a_2L[u_2],$$

where a_1 and a_2 are arbitrary constants, and u_1 and u_2 are arbitrary functions is called a *linear operator*. A linear differential equation naturally defines a linear operator: the equation can be expressed as $L[u] = f$, where L is a linear operator and f is a given function.

A linear differential equation of the form $L[u] = 0$, where L is a linear operator, is called a *homogeneous equation*. For example, define the operator $L = \partial^2/\partial x^2 - \partial^2/\partial y^2$. The equation

$$L[u] = u_{xx} - u_{yy} = 0$$

is a homogeneous equation, while the equation

$$L[u] = u_{xx} - u_{yy} = x^2$$

is an example of a *nonhomogeneous equation*.

Linear operators play a central role in mathematics in general, and in PDE theory in particular. This results from the important property (which follows at once from the definition) that if for $1 \leq i \leq n$, the function u_i satisfies the linear differential equation $L[u_i] = f_i$, then the linear combination $v := \sum_{i=1}^{n} \alpha_i u_i$ satisfies the equation $L[v] = \sum_{i=1}^{n} \alpha_i f_i$. In particular, if each of the functions $u_1, u_2, \ldots, u_n$ satisfies the homogeneous equation $L[u] = 0$, then every linear combination of them satisfies that equation too. This property is called the *superposition principle*. It allows the construction of complex solutions through combinations of simple solutions. In addition, we shall use the superposition principle to prove uniqueness of solutions to linear PDEs.

1.4 Differential equations as mathematical models

PDEs are woven throughout science and technology. We shall briefly review a number of canonical equations in different areas of application. The fundamental

laws of physics provide a mathematical description of nature's phenomena on a variety of scales of time and space. Thus, for example, very large scale phenomena (astronomical scales) are controlled by the laws of gravity. The theory of electromagnetism controls the scales involved in many daily activities, while quantum mechanics is used to describe phenomena on the atomic scale. It turns out, however, that many important problems involve interaction between a large number of objects, and thus it is difficult to use the basic laws of physics to describe them. For example, we do not fall to the floor when we sit on a chair. Why? The fundamental reason lies in the electric forces between the atoms constituting the chair. These forces endow the chair with high rigidity. It is clear, though, that it is not feasible to solve the equations of electromagnetism (Maxwell's equations) to describe the interaction between such a vast number of objects. As another example, consider the flow of a gas. Each molecule obeys Newton's laws, but we cannot in practice solve for the evolution of an Avogadro number of individual molecules. Therefore, it is necessary in many applications to develop simpler models.

The basic approach towards the derivation of these models is to define new quantities (temperature, pressure, tension,. . .) that describe average macroscopic values of the fundamental microscopic quantities, to assume several fundamental principles, such as conservation of mass, conservation of momentum, conservation of energy, etc., and to apply the new principles to the macroscopic quantities. We shall often need some additional ad-hoc assumptions to connect different macroscopic entities. In the optimal case we would like to start from the fundamental laws and then average them to achieve simpler models. However, it is often very hard to do so, and, instead, we shall sometimes use experimental observations to supplement the basic principles. We shall use x, y, z to denote spatial variables, and t to denote the time variable.

1.4.1 The heat equation

A common way to encourage scientific progress is to confer prizes and awards. Thus, the French Academy used to set up competitions for its prestigious prizes by presenting specific problems in mathematics and physics. In 1811 the Academy chose the problem of heat transfer for its annual prize. The prize was awarded to the French mathematician Jean Baptiste Joseph Fourier (1768–1830) for two important contributions. (It is interesting to mention that he was not an active scientist at that time, but rather the governor of a region in the French Alps – actually a politician!). He developed, as we shall soon see, an appropriate differential equation, and, in addition developed, as we shall see in Chapter 5, a novel method for solving this equation.

The basic idea that guided Fourier was conservation of energy. For simplicity we assume that the material density and the heat capacity are constant in space and time, and we scale them to be 1. We can therefore identify heat energy with temperature. Let D be a fixed spatial domain, and denote its boundary by ∂D. Under these conditions we shall write down the change in the energy stored in D between time t and time $t + \Delta t$:

$$\int_D [u(x, y, z, t + \Delta t) - u(x, y, z, t)] \, \mathrm{d}V = \int_t^{t+\Delta t} \int_D q(x, y, z, t, u) \mathrm{d}V \, \mathrm{d}t - \int_t^{t+\Delta t} \int_{\partial D} \vec{B}(x, y, z, t) \cdot \hat{n} \mathrm{d}S \mathrm{d}t, \quad (1.4)$$

where u is the temperature, q is the rate of heat production in D, $\vec{B}$ is the heat flux through the boundary, $\mathrm{d}V$ and $\mathrm{d}S$ are space and surface integration elements, respectively, and $\hat{n}$ is a unit vector pointing in the direction of the outward normal to ∂D. Notice that the heat production can be negative (a refrigerator, an air conditioner), as can the heat flux.

In general the heat production is determined by external sources that are independent of the temperature. In some cases (such as an air conditioner controlled by a thermostat) it depends on the temperature itself but not on its derivatives. Hence we assume $q = q(x, y, z, t, u)$. To determine the functional form of the heat flux, Fourier used the experimental observation that 'heat flows from hotter places to colder places'. Recall from calculus that the direction of maximal growth of a function is given by its gradient. Therefore, Fourier postulated

$$\vec{B} = -k(x, y, z)\vec{\nabla}u. \quad (1.5)$$

The formula (1.5) is called *Fourier's law of heat conduction*. The (positive!) function k is called the *heat conduction (or Fourier) coefficient*. The value(s) of k depend on the medium in which the heat diffuses. In a homogeneous domain k is expected to be constant. The assumptions on the functional dependence of q and $\vec{B}$ on u are called *constitutive laws*.

We substitute our formula for q and $\vec{B}$ into (1.4), approximate the t integrals using the mean value theorem, divide both sides of the equation by Δt, and take the limit $\Delta t \to 0$. We obtain

$$\int_D u_t \mathrm{d}V = \int_D q(x, y, z, t, u) \mathrm{d}V + \int_{\partial D} k(x, y, z)\vec{\nabla}u \cdot \hat{n} \mathrm{d}S. \quad (1.6)$$

Observe that the integration in the second term on the right hand side is over a different set than in the other terms. Thus we shall use Gauss' theorem to convert

the surface integral into a volume integral:

$$\int_D [u_t - q - \vec{\nabla} \cdot (k\vec{\nabla}u)]\mathrm{d}V = 0, \tag{1.7}$$

where $\vec{\nabla}\cdot$ denotes the divergence operator. The following simple result will be used several times in the book.

Lemma 1.1 *Let $h(x, y, z)$ be a continuous function satisfying $\int_\Omega h(x, y, z)\mathrm{d}V = 0$ for every domain Ω. Then $h \equiv 0$.*

Proof Let us assume to the contrary that there exists a point $P = (x_0, y_0, z_0)$ where $h(P) \neq 0$. Assume without loss of generality that $h(P) > 0$. Since h is continuous, there exists a domain (maybe very small) D_0, containing P and $\epsilon > 0$, such that $h > \epsilon > 0$ at each point in D_0. Therefore $\int_{D_0} h\mathrm{d}V > \epsilon \mathrm{Vol}(D_0) > 0$ which contradicts the lemma's assumption. □

Returning to the energy integral balance (1.7), we notice that it holds for any domain D. Assuming further that all the functions in the integrand are continuous, we obtain the PDE

$$u_t = q + \vec{\nabla} \cdot (k\vec{\nabla}u). \tag{1.8}$$

In the special (but common) case where the diffusion coefficient is constant, and there are no heat sources in D itself, we obtain the classical heat equation

$$u_t = k\Delta u, \tag{1.9}$$

where we use Δu to denote the important operator $u_{xx} + u_{yy} + u_{zz}$. Observe that we have assumed that the solution of the heat equation, and even some of its derivatives are continuous functions, although we have not solved the equation yet. Therefore, in principle we have to reexamine our assumptions a posteriori. We shall see examples later in the book in which solutions of a PDE (or their derivatives) are *not* continuous. We shall then consider ways to provide a meaning for the seemingly absurd process of substituting a discontinuous function into a differential equation. One of the fundamental ways of doing so is to observe that the integral balance equation (1.6) provides a more fundamental model than the PDE (1.8).

1.4.2 Hydrodynamics and acoustics

Hydrodynamics is the physical theory of fluid motion. Since almost any conceivable volume of fluid (whether it is a cup of coffee or the Pacific Ocean) contains a huge number of molecules, it is not feasible to describe the fluid using the law of electromagnetism or quantum mechanics. Hence, since the eighteenth century

scientists have developed models and equations that are appropriate to macroscopic entities such as temperature, pressure, effective velocity, etc. As explained above, these equations are based on conservation laws.

The simplest description of a fluid consists of three functions describing its state at any point in space-time:

- the density (mass per unit of volume) $\rho(x, y, z, t)$;
- the velocity $\vec{u}(x, y, z, t)$;
- the pressure $p(x, y, z, t)$.

To be precise, we must also include the temperature field in the fluid. But to simplify matters, it will be assumed here that the temperature is a known constant. We start with conservation of mass. Consider a fluid element occupying an arbitrary spatial domain D. We assume that matter neither is created nor disappears in D. Thus the total mass in D does not change:

$$\frac{\partial}{\partial t}\int_D \rho \mathrm{dV} = 0. \tag{1.10}$$

The motion of the fluid boundary is given by the component of the velocity $\vec{u}$ in the direction orthogonal to the boundary ∂D. Thus we can write

$$\int_D \frac{\partial}{\partial t}\rho \mathrm{d}V + \int_{\partial D} \rho\vec{u}\cdot\hat{n}\mathrm{d}S = 0, \tag{1.11}$$

where we denoted the unit external normal to ∂D by $\hat{n}$. Using Gauss' theorem we obtain

$$\int_D [\rho_t + \vec{\nabla}\cdot(\rho\vec{u})]\mathrm{d}V = 0. \tag{1.12}$$

Since D is an arbitrary domain we can use again Lemma 1.1 to obtain the mass *transport equation*

$$\rho_t + \vec{\nabla}\cdot(\rho\vec{u}) = 0. \tag{1.13}$$

Next we require the fluid to satisfy the momentum conservation law. The forces acting on the fluid in D are gravity, acting on each point in the fluid, and the pressure applied at the boundary of D by the rest of the fluid outside D. We denote the density per unit mass of the gravitational force by $\vec{g}$. For simplicity we neglect the friction forces between adjacent fluid molecules. Newton's law of motion implies an equality between the change in the fluid momentum and the total forces acting on the fluid. Thus

$$\frac{\partial}{\partial t}\int_D \rho\vec{u}\mathrm{d}V = -\int_{\partial D} p\hat{n}\mathrm{d}s + \int_D \rho\vec{g}\mathrm{d}V. \tag{1.14}$$

Let us interchange again the t differentiation with the spatial integration, and use (1.13) to obtain the integral balance

$$\int_D [\rho \vec{u}_t + \rho(\vec{u} \cdot \vec{\nabla})\vec{u}]\mathrm{d}V = \int_D (-\vec{\nabla} p + \rho \vec{g})\mathrm{d}V. \tag{1.15}$$

From this balance we deduce the PDE

$$\vec{u}_t + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla} p + \vec{g}. \tag{1.16}$$

So far we have developed two PDEs for three unknown functions $(\rho, \vec{u}, p)$. We therefore need a third equation to complete the system. Notice that conservation of energy has already been accounted for by assuming that the temperature is fixed. In fact, the additional equation does not follow from a conservation law, rather one imposes a constitutive relation (like Fourier's law from the previous subsection). Specifically, we postulate a relation of the form

$$p = f(\rho), \tag{1.17}$$

where the function f is determined by the specific fluid (or gas). The full system comprising (1.13), (1.16) and (1.17) is called the *Euler fluid flow equations*. These equations were derived in 1755 by the Swiss mathematician Leonhard Euler (1707–1783).

If one takes into account the friction between the fluid molecules, the equations acquire an additional term. This friction is called *viscosity*. The special case of viscous fluids where the density is essentially constant is of particular importance. It characterizes, for example, most phenomena involving the flow of water. This case was analyzed first in 1822 by the French engineer Claude Navier (1785–1836), and then studied further by the British mathematician George Gabriel Stokes (1819–1903). They derived the following set of equations:

$$\rho(\vec{u}_t + (\vec{u} \cdot \vec{\nabla})\vec{u}) = \mu \Delta \vec{u} - \vec{\nabla} p, \tag{1.18}$$

$$\vec{\nabla} \cdot \vec{u} = 0. \tag{1.19}$$

The parameter μ is called the fluid's viscosity. Notice that (1.18)–(1.19) form a quasilinear system of equations. The Navier–Stokes system lies at the foundation of hydrodynamics. Enormous computational efforts are invested in solving them under a variety of conditions and in a plurality of applications, including, for example, the design of airplanes and ships, the design of vehicles, the flow of blood in arteries, the flow of ink in a printer, the locomotion of birds and fish, and so forth. Therefore it is astonishing that the well-posedness of the Navier–Stokes equations has not yet been established. Proving or disproving their well-posedness is one of the most

important open problems in mathematics. A prize of one million dollars awaits the person who solves it.

An important phenomenon described by the Euler equations is the propagation of sound waves. In order to construct a simple model for sound waves, let us look at the Euler equations for a gas at rest. For simplicity we neglect gravity. It is easy to check that the equations have a solution of the form

$$\begin{aligned} \vec{u} &= 0, \\ \rho &= \rho_0, \\ p &= p_0 = f(\rho_0), \end{aligned} \tag{1.20}$$

where ρ_0 and p_0 are constants describing uniform pressure and density. Let us perturb the gas by creating a localized pressure (for example by producing a sound out of our throats, or by playing a musical instrument). Assume that the perturbation is small compared with the original pressure p_0. One can therefore write

$$\begin{aligned} \vec{u} &= \epsilon \vec{u}^1, \\ \rho &= \rho^0 + \epsilon \rho^1, \\ p &= p_0 + \epsilon p^1 = f(\rho^0) + \epsilon f'(\rho^0)\rho^1, \end{aligned} \tag{1.21}$$

where we denoted the perturbation to the density, velocity and pressure by $\vec{u}^1$, ρ^1, and p^1, respectively, ϵ denotes a small positive parameter, and we used (1.17). Substituting the expansion (1.21) into the Euler equations, and retaining only the terms that are linear in ϵ, we find

$$\begin{aligned} \rho^1_t + \rho_o \vec{\nabla} \cdot \vec{u}^1 &= 0, \\ \vec{u}^1_t + \frac{1}{\rho^0} \vec{\nabla} p^1 &= 0. \end{aligned} \tag{1.22}$$

Applying the operator $\vec{\nabla}\cdot$ to the second equation in (1.22), and substituting the result into the time derivative of the first equation leads to

$$\rho^1_{tt} - f'(\rho^0)\Delta\rho^1 = 0. \tag{1.23}$$

Alternatively we can use the linear relation between p^1 and ρ^1 to write a similar equation for the pressure

$$p^1_{tt} - f'(\rho^0)\Delta p^1 = 0. \tag{1.24}$$

The equation we have obtained is called a *wave equation*. We shall see later that this equation indeed describes waves propagating with speed $c = \sqrt{f'(\rho^0)}$. In particular, in the case of waves in a long narrow tube, or in a long and narrow tunnel, the pressure

only depends on time and on a single spatial coordinate x along the tube. We then obtain the one-dimensional wave equation

$$p^1_{tt} - c^2 p^1_{xx} = 0. \tag{1.25}$$

Remark 1.2 Many problems in chemistry, biology and ecology involve the spread of some substrate being convected by a given velocity field. Denoting the concentration of the substrate by $C(x, y, z, t)$, and assuming that the fluid's velocity does not depend on the concentration itself, we find that (1.13) in the formulation

$$C_t + \vec{\nabla} \cdot (C\vec{u}) = 0 \tag{1.26}$$

describes the spread of the substrate. This equation is naturally called the *convection equation*. In Chapter 2 we shall develop solution methods for it.

1.4.3 Vibrations of a string

Many different phenomena are associated with the vibrations of elastic bodies. For example, recall the wave equation derived in the previous subsection for the propagation of sound waves. The *generation* of sound waves also involves a wave equation – for example the vibration of the sound chords, or the vibration of a string or a membrane in a musical instrument.

Consider a uniform string undergoing transversal motion whose amplitude is denoted by $u(x, t)$, where x is the spatial coordinate, and t denotes time. We also use ρ to denote the mass density per unit length of the string. We shall assume that ρ is constant. Consider further a small interval $(-\delta, \delta)$. Just as in the previous subsection, we shall consider two forces acting on the string: an external given force (e.g. gravity) acting only in the transversal (y) direction, whose density is denoted by $f(x, t)$, and an internal force acting between adjacent string elements. This internal force is called *tension*. It will be denoted by $\vec{T}$. The tension acts on the string element under consideration at its two ends. A tension $\vec{T}_+$ acts at the right hand end, and a tension $\vec{T}_-$ acts at the left hand end. We assume that the tension is in the direction tangent to the string, and that it is proportional to the string's elongation. Namely, we assume the constitutive law

$$\vec{T} = d\sqrt{1 + u_x^2}\hat{e}_\tau, \tag{1.27}$$

where d is a constant depending on the material of which the string is made, and $\hat{e}_\tau$ is a unit vector in the direction of the string's tangent. It is an empirical law, i.e. it stems from experimental observations. Projecting the momentum conservation

equation (Newton's second law) along the y direction we find:

$$\int_{-\delta}^{\delta} \rho u_{tt} \mathrm{d}l = \int_{-\delta}^{\delta} f(x,t)\mathrm{d}l + \hat{e}_2 \cdot (\vec{T}_+ - \vec{T}_-) = \int_{-\delta}^{\delta} f(x,t)\mathrm{d}l + \int_{-\delta}^{\delta} (\hat{e}_2 \cdot \vec{T})_x \mathrm{d}x,$$

where $\mathrm{d}l$ denotes a length element, and $\hat{e}_2 = (0, 1)$. Using the constitutive law for the tension and the following formula for the tangent vector $\hat{e}_\tau = (1, u_x)/\sqrt{1+u_x^2}$, we can write

$$\hat{e}_2 \cdot \vec{T} = d\sqrt{1+u_x^2}\hat{e}_2 \cdot \hat{e}_\tau = du_x.$$

Substituting this equation into the momentum equation we obtain the integral balance

$$\int_{-\delta}^{\delta} \rho u_{tt}\sqrt{1+u_x^2}\mathrm{d}x = \int_{-\delta}^{\delta} \left[f\sqrt{1+u_x^2} + du_{xx} \right] \mathrm{d}x.$$

Since this equation holds for arbitrary intervals, we can use Lemma 1.1 once again to obtain

$$u_{tt} - \frac{c^2}{\sqrt{1+u_x^2}} u_{xx} = \frac{f(x,t)}{\rho}, \tag{1.28}$$

where the wave speed is given by $c = \sqrt{d/\rho}$. A different string model will be derived in Chapter 10. The two models are compared in Remark 10.5.

In the case of weak vibrations the slopes of the amplitude are small, and we can make the simplifying assumption $|u_x| \ll 1$. We can then write an approximate equation:

$$u_{tt} - c^2 u_{xx} = \frac{1}{\rho} f(x,t). \tag{1.29}$$

Thus, the wave equation developed earlier for sound waves is also applicable to describe certain elastic waves. Equation (1.29) was proposed as early as 1752 by the French mathematician Jean d'Alembert (1717–1783). We shall see in Chapter 4 how d'Alembert solved it.

Remark 1.3 We have derived an equation for the transversal vibrations of a string. What about its longitudinal vibrations? To answer this question, project the momentum equation along the tangential direction, and again use the constitutive law. We find that the density of the tension force in the longitudinal direction is given by

$$\frac{\partial}{\partial x}\left(d\frac{\sqrt{1+u_x^2}}{\sqrt{1+u_x^2}} \right) = 0.$$

This implies that the constitutive law we used is equivalent to assuming the string does not undergo longitudinal vibrations!

1.4.4 Random motion

Random motion of minute particles was first described in 1827 by the British biologist Robert Brown (1773–1858). Hence this motion is called *Brownian motion*. The first mathematical model to describe this motion was developed by Einstein in 1905. He proposed a model in which a particle at a point (x, y) in the plane jumps during a small time interval δt to a nearby point from the set $(x \pm \delta x, y \pm \delta x)$. Einstein showed that under a suitable assumption on δx and δt, the probability that the particle will be found at a point (x, y) at time t satisfies the heat equation. His model has found many applications in physics, biology, chemistry, economics etc. We shall demonstrate now how to obtain a PDE from a typical problem in the theory of Brownian motion.

Consider a particle in a two-dimensional domain D. For simplicity we shall limit ourselves to the case where D is the unit square. Divide the square into N^2 identical little squares, and denote their vertices by $\{(x_i, y_j)\}$. The size of each edge of a small square will be denoted by δx. A particle located at an internal vertex (x_i, y_j) jumps during a time interval δt to one of its nearest neighbors with equal probability. When the particle reaches a boundary point it dies.

Question What is the life expectancy $u(x, y)$ of a particle that starts its life at a point (x, y) in the limit

$$\delta x \to 0, \quad \delta t \to 0, \quad \frac{(\delta x)^2}{2\delta t} = k? \tag{1.30}$$

We shall answer the question using an intuitive notion of the expectancy. Obviously a particle starting its life at a boundary point dies at once. Thus

$$u(x, y) = 0, \quad (x, y) \in \partial D. \tag{1.31}$$

Consider now an internal point (x, y). A particle must have reached this point from one of its four nearest neighbors with equal probability for each neighbor. In addition, the trip from the neighboring point lasted a time interval δt. Therefore u satisfies the difference equation

$$u(x, y) = \delta t + \frac{1}{4}[u(x - \delta x, y) + u(x + \delta x, y) + u(x, y - \delta x) + u(x, y + \delta x)]. \tag{1.32}$$

We expand all functions on the right hand side into a Taylor series, assuming $u \in C^4$. Dividing by δt and taking the limit (1.30) we obtain (see also Chapter 11)

$$\Delta u = -\frac{1}{k}, \quad (x, y) \in D. \tag{1.33}$$

An equation of the type (1.33) is called a *Poisson equation*. We shall elaborate on such equations in Chapter 7.

The model we just investigated has many applications. One of them relates to the analysis of variations in stock prices. Many models in the stock market are based on assuming that stocks prices vary randomly. Assume for example that a broker buys a stock at a certain price m. She decides in advance to sell it if its price reaches an upper bound m_2 (in order to cash in her profit) or a lower bound m_1 (to minimize losses in case the stock dives). How much time on average will the broker hold the stock, assuming that the stock price performs a Brownian motion? This is a one-dimensional version of the model we derived. The equation and the associated boundary conditions are

$$ku''(m) = -1, \quad u(m_1) = u(m_2) = 0. \tag{1.34}$$

The reader will be asked to solve the equation in Exercise 1.6.

1.4.5 Geometrical optics

We have seen two derivations of the wave equation – one for sound waves, and another one for elastic waves. Yet there are many other physical phenomena controlled by wave propagation. Two notable examples are electromagnetic waves and water waves. Although there exist many analytic methods for solving wave equations (we shall learn some of them later), it is not easy to apply them in complex geometries. One might be tempted to proceed in such cases to numerical methods (see Chapter 11). The problem is that in many applications the waves are of very high frequency (or, equivalently, of very small wavelength). To describe such waves we need a resolution that is considerably smaller than a single wavelength. Consider for example optical phenomena. They are described by a wave equation; a typical wavelength for the visible light part of the spectrum is about half a micron. Assuming that we use five points per wavelength to describe the wave, and that we deal with a three-dimensional domain with linear dimension of 10^{-1} meters, we conclude that we need altogether about 10^{17} points! Even storing the data is a difficult task, not to mention the formidable complexity of solving equations with so many unknowns (Chapter 11).

Fortunately it is possible to turn the problem around and actually *use* the short wavelength to derive approximate equations that are much simpler to solve, and, yet, provide a fair description of optics. Consider for this purpose the wave equation in $\mathbb{R}^3$:

$$v_{tt} - c^2(\vec{x})\Delta v = 0. \tag{1.35}$$

Notice that the wave's speed need not be constant. We expect solutions that are oscillatory in time (see Chapter 5). Therefore we seek solutions of the form

$$v(x, y, z, t) = \mathrm{e}^{\mathrm{i}\omega t}\psi(x, y, z).$$

It is convenient to introduce at this stage the notation $k = \omega/c_0$ and $n = c_0/c(x)$, where c_0 is an average wave velocity in the medium. Substituting v into (1.35) yields

$$\Delta\psi + k^2 n^2(\vec{x})\psi = 0. \tag{1.36}$$

The function $n(x)$ is called the *refraction index*. The parameter k is called the *wave number*. It is easy to see that k^{-1} has the dimension of length. In fact, the wavelength is given by $2\pi k^{-1}$. As was explained above, the wavelength is often much smaller than any other length scale in the problem. For example, spectacle lenses involve scales such as 5 mm (thickness), 60 mm (radius of curvature) or 40 mm (frame size), all of them far greater than half a micron which is a typical wavelength. We therefore assume that the problem is scaled with respect to one of the large scales, and hence k is a very large number. To use this fact we seek a solution to (1.36) of the form:

$$\psi(x, y, z) = A(x, y, z; k)\mathrm{e}^{\mathrm{i}kS(x,y,z)}. \tag{1.37}$$

Substituting (1.37) into (1.36), and assuming that A is bounded with respect to k, we get

$$A[|\vec{\nabla}S|^2 - n^2(\vec{x})] = O\left(\frac{1}{k}\right).$$

Thus the function S satisfies the *eikonal equation*

$$|\vec{\nabla}S| = n(\vec{x}). \tag{1.38}$$

This equation, postulated in 1827 by the Irish mathematician William Rowan Hamilton (1805–1865), provides the foundation for *geometrical optics*. It is extremely useful in many applications in optics, such as radar, contact lenses, projectors, mirrors, etc. In Chapter 2 we shall develop a method for solving eikonal equations. Later, in Chapter 9, we shall encounter the eikonal equation from a different perspective.

1.4.6 Further real world equations

- **The Laplace equation**
  Many of the models we have examined so far have something in common – they involve the operator

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

This operator is called the *Laplacian*. Probably the 'most important' PDE is the *Laplace equation*

$$\Delta u = 0. \tag{1.39}$$

The equation, which is a special case of the Poisson equation we introduced earlier, was proposed in 1780 by the French mathematician Pierre-Simon Laplace (1749–1827) in his work on gravity. Solutions of the Laplace equation are called *harmonic functions*. Laplace's equation can be found everywhere. For example, in the heat conduction problems that were introduced earlier, the temperature field is harmonic when temporal equilibrium is achieved. The equation is also fundamental in mechanics, electromagnetism, probability, quantum mechanics, gravity, biology, etc.

- **The minimal surface equation**
  When we dip a narrow wire in a soap bath, and then lift the wire gently out of the bath, we can observe a thin membrane spanning the wire. The French mathematician Joseph-Louis Lagrange (1736–1813) showed in 1760 that the surface area of the membrane is smaller than the surface area of any other surface that is a small perturbation of it. Such special surfaces are called *minimal surfaces*. Lagrange further demonstrated that the graph of a minimal surface satisfies the following second-order nonlinear PDE:

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0. \tag{1.40}$$

  When the slopes of the minimal surface are small, i.e. $u_x, u_y \ll 1$, we see at once that (1.40) can be approximated by the Laplace equation. We shall return to the minimal surface equation in Chapter 10.
- **The biharmonic equation**
  The equilibrium state of a thin elastic plate is provided by its amplitude function $u(x, y)$, which describes the deviation of the plate from its horizontal position. It can be shown that the unknown function u satisfies the equation

$$\Delta^2 u = \Delta(\Delta u) = u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0. \tag{1.41}$$

  For an obvious reason this equation is called the *biharmonic equation*. Notice that in contrast to all the examples we have seen so far, it is a fourth-order equation. We further point out that almost all the equations we have seen here, and also other important equations such as Maxwell's equations, the Schrödinger equation and Newton's equation for the gravitational field are of second order. We shall return to the plate equation in Chapter 10.
- **The Schrödinger equation**
  One of the fundamental equations of quantum mechanics, derived in 1926 by the Austrian physicist Erwin Schrödinger (1887–1961), governs the evolution of the wave function u of a particle in a potential field V:

$$\mathrm{i}\hbar\frac{\partial u}{\partial t} = -\frac{\hbar}{2m}\Delta u + Vu. \tag{1.42}$$

Here V is a known function (potential), m is the particle's mass, and $\hbar$ is Planck's constant divided by 2π. We shall consider the Schrödinger equation for the special case of an electron in the hydrogen atom in Chapter 9.

- **Other equations**
  There are many other PDEs that are central to the study of different problems in science and technology. For example we mention: the Maxwell equations of electromagnetism; reaction–diffusion equations that model chemical reactions; the equations of elasticity; the Korteweg–de Vries equation for solitary waves; the nonlinear Schrödinger equation in nonlinear optics and in superfluids; the Ginzburg–Landau equations of superconductivity; Einstein's equations of general relativity, and many more.

1.5 Associated conditions

PDEs have in general infinitely many solutions. In order to obtain a unique solution one must supplement the equation with additional conditions. What kind of conditions should be supplied? It turns out that the answer depends on the type of PDE under consideration. In this section we briefly review the common conditions, and explain through examples their physical significance.

1.5.1 Initial conditions

Let us consider the transport equation (1.26) in one spatial dimension as a prototype for equations of first order. The unknown function $C(x, t)$ is a surface defined over the (x, t) plane. It is natural to formulate a problem in which one supplies the concentration at a given time t_0, and then to deduce from the equation the concentration at later times. Namely, we solve the problem consisting of the convection equation

$$C_t + \vec{\nabla} \cdot (C\vec{u}) = 0,$$

and the condition

$$C(x, t_0) = C_0(x). \tag{1.43}$$

This problem is called an *initial value problem*. Geometrically speaking, condition (1.43) determines a curve through which the solution surface must pass. We can generalize (1.43) by imposing a curve Γ that must lie on the solution surface, so that the projection of Γ on the (x, t) plane is not necessarily the x axis. In Chapter 2 we shall show that under suitable assumptions on the equation and Γ, there indeed exists a unique solution.

Another case where it is natural to impose initial conditions is the heat equation (1.9). Here we provide the temperature distribution at some initial time (say $t = 0$),

and solve for its distribution at later times, namely, the initial condition for (1.9) is of the form $u(x, y, z, 0) = u_0(x, y, z)$.

The last two examples involve PDEs with just a first derivative with respect to t. In analogy with the theory of initial value problems for ODEs, we expect that equations that involve second derivatives with respect to t will require two initial conditions. Indeed, let us look at the wave equation (1.29). As explained in the previous section, this equation is nothing but Newton's second law, equating the mass times the acceleration and the forces acting on the string. Therefore it is natural to supply two initial conditions, one for the initial location of the string, and one for its initial velocity:

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x). \tag{1.44}$$

We shall indeed prove in Chapter 4 that these conditions, together with the wave equation lead to a well-posed problem.

1.5.2 Boundary conditions

Another type of constraint for PDEs that appears in many applications is called *boundary conditions*. As the name indicates, these are conditions on the behavior of the solution (or its derivative) at the boundary of the domain under consideration. As a first example, consider again the heat equation; this time, however, we limit ourselves to a given spatial domain Ω:

$$u_t = k\Delta u \quad (x, y, z) \in \Omega, \quad t > 0. \tag{1.45}$$

We shall assume in general that Ω is *bounded*. It turns out that in order to obtain a unique solution, one should provide (in addition to initial conditions) information on the behavior of u on the boundary $\partial\Omega$. Excluding rare exceptions, we encounter in applications three kinds of boundary conditions. The first kind, where the values of the temperature on the boundary are supplied, i.e.

$$u(x, y, z, t) = f(x, y, z, t) \quad (x, y, z) \in \partial\Omega, \quad t > 0, \tag{1.46}$$

is called a *Dirichlet condition* in honor of the German mathematician Johann Lejeune Dirichlet (1805–1859). For example, this condition is used when the boundary temperature is given through measurements, or when the temperature distribution is examined under a variety of external heat conditions.

Alternatively one can supply the normal derivative of the temperature on the boundary; namely, we impose (as usual we use here the notation ∂_n to denote the outward normal derivative at $\partial\Omega$)

$$\partial_n u(x, y, z, t) = f(x, y, z, t) \quad (x, y, z) \in \partial\Omega, \quad t > 0. \tag{1.47}$$

This condition is called a *Neumann condition* after the German mathematician Carl Neumann (1832–1925). We have seen that the normal derivative $\partial_n u$ describes the flux through the boundary. For example, an insulating boundary is modeled by condition (1.47) with $f = 0$.

A third kind of boundary condition involves a relation between the boundary values of u and its normal derivative:

$$\alpha(x, y, z)\partial_n u(x, y, z, t) + u(x, y, z, t) = f(x, y, z, t) \quad (x, y, z) \in \partial D, \quad t > 0. \tag{1.48}$$

Such a condition is called *a condition of the third kind*. Sometimes it is also called the Robin condition.

Although the three types of boundary conditions defined above are by far the most common conditions seen in applications, there are exceptions. For example, we can supply the values of u at some parts of the boundary, and the values of its normal derivative at the rest of the boundary. This is called a *mixed boundary condition*. Another possibility is to generalize the condition of the third kind and replace the normal derivative by a (smoothly dependent) directional derivative of u in any direction that is not tangent to the boundary. This is called an *oblique boundary condition*. Also, one can provide a *nonlocal boundary condition*. For example, one can provide a boundary condition relating the heat flux at each point on the boundary to the integral of the temperature over the whole boundary.

To illustrate further the physical meaning of boundary conditions, let us consider again the wave equation for a string:

$$u_{tt} - c^2 u_{xx} = f(x, t) \quad a < x < b, \ t > 0. \tag{1.49}$$

When the locations of the end points of the string are known, we supply Dirichlet boundary conditions (Figure 1.1(a)):

$$u(a, t) = \beta_1(t), \quad u(b, t) = \beta_2(t), \quad t > 0. \tag{1.50}$$

Another possibility is that the tension at the end points is given. From our derivation of the string equation in Subsection 1.4.3 it follows that this case involves a

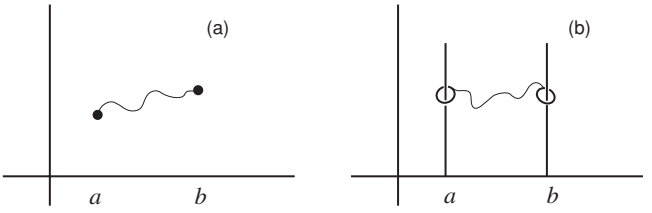

Figure 1.1 Illustrating boundary conditions for a string.

Neumann condition:

$$u_x(a,t) = \beta_1(t), \quad u_x(b,t) = \beta_2(t), \quad t > 0. \tag{1.51}$$

Thus, for example, when the end points are free to move in the transversal direction (Figure 1.1(b)), we shall use a homogeneous Neumann condition, i.e. $\beta_1 = \beta_2 = 0$.

1.6 Simple examples

Before proceeding to develop general solution methods, let us warm up with a few very simple examples.

Example 1.4 Solve the equation $u_{xx} = 0$ for an unknown function $u(x, y)$. We can consider the equation as an ODE in x, with y being a parameter. Thus the general solution is $u(x, y) = A(y)x + B(y)$. Notice that the solution space is huge, since $A(y)$ and $B(y)$ are arbitrary functions.

Example 1.5 Solve the equation $u_{xy} + u_x = 0$. We can transform the problem into an ODE by setting $v = u_x$. The new function $v(x, y)$ satisfies the equation $v_y + v = 0$. Treating x as a parameter, we obtain $v(x, y) = C(x)\mathrm{e}^{-y}$. Integrating v we construct the solution to the original problem: $u(x, y) = D(x)\mathrm{e}^{-y} + E(y)$.

Example 1.6 Find a solution of the wave equation $u_{tt} - 4u_{xx} = \sin t + x^{2000}$. Notice that we are asked to find *a* solution, and not the most general solution. We shall exploit the linearity of the wave equation. According to the superposition principle, we can split $u = v + w$, such that v and w are solutions of

$$v_{tt} - 4v_{xx} = \sin t, \tag{1.52}$$

$$w_{tt} - 4w_{xx} = x^{2000}. \tag{1.53}$$

The advantage gained by this step is that solutions for each of these equations can be easily obtained:

$$v(x,t) = -\sin t, \qquad w(x,t) = -\frac{1}{4 \times 2001 \times 2002} x^{2002}.$$

Thus

$$u(x,t) = -\sin t - \frac{1}{4 \times 2001 \times 2002} x^{2002}.$$

There are many other solutions. For example, it is easy to check that if we add to the solution above a function of the form $f(x - 2t)$, where $f(s)$ is an arbitrary twice differentiable function, a new solution is obtained.

Unfortunately one rarely encounters real problems described by such simple equations. Nevertheless, we can draw a few useful conclusions from these examples. For instance, a commonly used method is to seek a transformation from the original variables to new variables in which the equation takes a simpler form. Also, the superposition principle, which enables us to decompose a problem into a set of far simpler problems, is quite general.

1.7 Exercises

1.1 Show that each of the following equations has a solution of the form $u(x, y) = f(ax + by)$ for a proper choice of constants a, b. Find the constants for each example.

(a) $u_x + 3u_y = 0$.
(b) $3u_x - 7u_y = 0$.
(c) $2u_x + \pi u_y = 0$.

1.2 Show that each of the following equations has a solution of the form $u(x, y) = e^{\alpha x+\beta y}$. Find the constants α, β for each example.

(a) $u_x + 3u_y + u = 0$.
(b) $u_{xx} + u_{yy} = 5e^{x-2y}$.
(c) $u_{xxxx} + u_{yyyy} + 2u_{xxyy} = 0$.

1.3 (a) Show that there exists a unique solution for the system

$$\begin{aligned} u_x &= 3x^2y + y, \\ u_y &= x^3 + x, \end{aligned} \tag{1.54}$$

together with the initial condition $u(0, 0) = 0$.
(b) Prove that the system

$$\begin{aligned} u_x &= 2.999999x^2y + y, \\ u_y &= x^3 + x \end{aligned} \tag{1.55}$$

has no solution at all.

1.4 Let $u(x, y) = h(\sqrt{x^2 + y^2})$ be a solution of the minimal surface equation.
(a) Show that $h(r)$ satisfies the ODE

$$rh'' + h'(1 + (h')^2) = 0.$$

(b) What is the general solution to the equation of part (a)?

1.5 Let $p : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Prove that the equation

$$u_t = p(u)u_x \qquad t > 0$$

has a solution satisfying the functional relation $u = f(x + p(u)t)$, where f is a differentiable function. In particular find such solutions for the following equations:

(a) $u_t = ku_x$.
(b) $u_t = uu_x$.
(c) $u_t = u\sin(u)u_x$.

1.6 Solve (1.34), and compute the average time for which the broker holds the stock. Analyze the result in light of the financial interpretation of the parameters (m_1, m_2, k).

1.7 (a) Consider the equation $u_{xx} + 2u_{xy} + u_{yy} = 0$. Write the equation in the coordinates $s = x, t = x - y$.
(b) Find the general solution of the equation.
(c) Consider the equation $u_{xx} - 2u_{xy} + 5u_{yy} = 0$. Write it in the coordinates $s = x + y$, $t = 2x$.