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LINEAR ALGEBRA FOR EVERYONE

GILBERT STRANG
Massachusetts Institute of Technology

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Gilbert Strang's page : **math.mit.edu/∼gs**
For orders : **math.mit.edu/weborder.php**
LAFEeveryone@gmail.com

The website for this book (with Solution Manual) is **math.mit.edu/everyone**
2019 book : **Linear Algebra and Learning from Data** (**math.mit.edu/learningfromdata**)
2023 book : **Introduction to Linear Algebra, 6th Edition** (**math.mit.edu/linearalgebra**)
2016 book : **Introduction to Linear Algebra, 5th Edition** (**math.mit.edu/linearalgebra**)
2014 book : **Differential Equations and Linear Algebra** (**math.mit.edu/dela**)

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Those videos (including 18.06SC and 18.065) are also on **www.youtube.com/mitocw**
18.06 Linear Algebra 18.06SC with problem solutions 18.065 Learning from Data

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Preface

This is a linear algebra textbook with a new start. Chapter 1 begins as usual with vectors. We see their linear combinations and dot products. Then the new ideas come with matrices. Let me illustrate those ideas right away by an example.

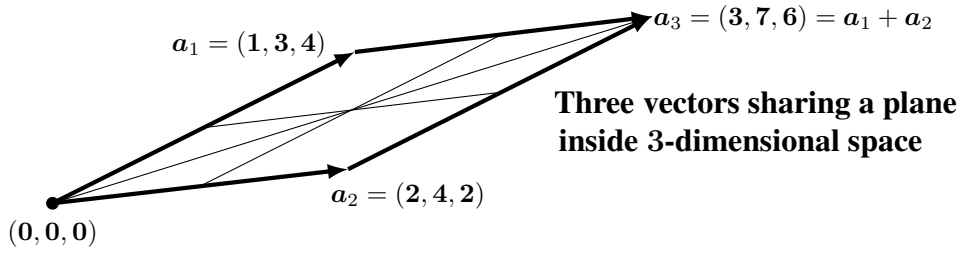
Suppose we are given a 3 by 3 matrix A with columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$:

$$\boldsymbol{A} = \begin{bmatrix} & & \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 7 \\ 4 & 2 & 6 \end{bmatrix}.$$

Those columns are three-dimensional vectors. The first vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ connect the center point $(0, 0, 0)$ to the points $(1, 3, 4)$ and $(2, 4, 2)$. The picture shows those points in 3-dimensional space (xyz space). The key to this matrix is the third vector $\boldsymbol{a}_3$ going to the point $(3, 7, 6)$.

When I look at those vectors, I see something exceptional. Adding columns 1 and 2 produces column 3. In other words $\boldsymbol{a}_1 + \boldsymbol{a}_2 = \boldsymbol{a}_3$. In a 3-dimensional picture, $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ go from the center point $(0, 0, 0)$ to the points $(1, 3, 4)$ and $(2, 4, 2)$. The picture shows how to add those vectors. It is normal that all combinations of two vectors will fill up a plane. (The plane is actually infinite, we just drew the part between the vectors.) What is really exceptional is that **the third point $\boldsymbol{a_3} = (3, 7, 6)$ lies on this plane of $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$.**

Most points don't lie on that plane. Most vectors $\boldsymbol{a}_3$ are *not* combinations of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. Most 3 by 3 matrices have *independent* columns. Then the matrix will be invertible. But these three columns are *dependent* because they lie on the same plane : $\boldsymbol{a}_1 + \boldsymbol{a}_2 = \boldsymbol{a}_3$.



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That picture reveals the most important fact about those three vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$. But we need the right language to describe it. Our goal is to see the idea, and to get better and better at expressing it. Here are three steps in a good direction.

1	**Idea in words**	Column 3 is the sum of column 1 and column 2
2	**Idea in symbols**	$\boldsymbol{a}_3 = \boldsymbol{a}_1 + \boldsymbol{a}_2$
3	**Matrix times vector**	$\begin{bmatrix} \\ \boldsymbol{a}_3 \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \boldsymbol{a}_1 \quad \boldsymbol{a}_2 \\ \\ \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$

Step 3 shows how a matrix C multiplies a vector $\boldsymbol{x}$. The columns $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ in C multiply the numbers $x_1 = 1$ and $x_2 = 1$ in $\boldsymbol{x}$. *The output $C\boldsymbol{x}$ is a combination of the columns*. Here that column combination is $\boldsymbol{a}_1 + \boldsymbol{a}_2$.

One more crucial step allows several combinations at once. This is the way forward. We cannot take this step with pictures or words. **A matrix multiplies a matrix**.

4 Matrix times matrix $\boldsymbol{A} = \boldsymbol{CR}$ is $\begin{bmatrix} \\ \boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3 \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \boldsymbol{a}_1 \quad \boldsymbol{a}_2 \\ \\ \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix}$

Those three columns of $\boldsymbol{R}$ give three combinations in $\boldsymbol{A}$ of the two columns in $\boldsymbol{C}$

Column 1: $\mathbf{1}\boldsymbol{a}_1 + \mathbf{0}\boldsymbol{a}_2 = \boldsymbol{a}_1$ **Column 2**: $\mathbf{0}\boldsymbol{a}_1 + \mathbf{1}\boldsymbol{a}_2 = \boldsymbol{a}_2$

Column 3: $\mathbf{1}\boldsymbol{a}_1 + \mathbf{1}\boldsymbol{a}_2 = (\mathbf{1}, \mathbf{3}, \mathbf{4}) + (\mathbf{2}, \mathbf{4}, \mathbf{2}) = (\mathbf{3}, \mathbf{7}, \mathbf{6}) = \boldsymbol{a}_3$

That matrix multiplication $\boldsymbol{A} = \boldsymbol{CR}$ displays major information:

A has dependent columns : the combination of columns $1 + 2 - 3$ gives $(0, 0, 0)$

A also has dependent rows: some combination of its rows will give $(0, 0, 0)$!

The "*column space*" of this A is only a plane and not the whole 3D space

The "*row space*" of this A is also a plane and not the whole 3D space

The square matrix A has no inverse. Its determinant is zero. It is unusual.

Subspaces of Vector Spaces

In Chapters 1 to 4, the organizing ideas are **vector spaces**. The columns of A are in m-dimensional space $\mathbf{R}^m$. The rows are in n-dimensional space $\mathbf{R}^n$. But the action is in **four subspaces inside $\mathbf{R}^m$ and $\mathbf{R}^n$**. The matrix A multiplies $\boldsymbol{x}_{\text{row}}$ in its *row space* and $\boldsymbol{x}_{\text{null}}$ in its *nullspace*. The output is $A\boldsymbol{x}_{\text{row}} = \boldsymbol{b}$ in its *column space* because $A\boldsymbol{x}_{\text{null}} = \textit{zero}$. Then the complete solution to $A\boldsymbol{x} = \boldsymbol{b}$ has a row space part and a nullspace part:

$$\boldsymbol{x} = \boldsymbol{x}_{\textbf{row}} + \text{ any } \boldsymbol{x}_{\textbf{null}} \qquad A\boldsymbol{x} = A\boldsymbol{x}_{\textbf{row}} + A\boldsymbol{x}_{\textbf{null}} = \boldsymbol{b} + \mathbf{0} = \boldsymbol{b}$$

Every linear equation $A\boldsymbol{x} = \boldsymbol{b}$ is solved this way: particular $\boldsymbol{x}_{\text{row}}$ + any homogeneous $\boldsymbol{x}_{\text{null}}$.

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The Plan for the Book

That example is part of a new start. I believe it is a better start (for the reader and the course). By working with specific matrices to introduce the algebra, the subject unfolds. Chapter 1 develops the matrix equation $A = C$ times R. C takes independent vectors like $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ from the column space of A. R takes independent vectors from the row space of A. Those two "vector spaces" are at the center of linear algebra. We meet them properly in Chapter 3. But you will know about independence from examples in Chapter 1.

The big step is to factor A into C times R. Matrix multiplication is a crucial operation, and Chapter 1 ends with four different ways to do it—seen on the back cover of the book. This sets the path to all the great factorizations of linear algebra:

$\boldsymbol{A = LU}$	Chapter 2 solves n equations $A\boldsymbol{x} = \boldsymbol{b}$ in n unknowns: A is square
$\boldsymbol{A = CR}$	Chapter 3 reduces to r independent columns and r independent rows
$\boldsymbol{A = QR}$	Chapter 4 changes the columns of A into perpendicular columns of Q
$\boldsymbol{S = Q\Lambda Q^{\mathrm{T}}}$	Chapter 6 has **eigenvectors** in $\boldsymbol{Q}$. The **eigenvalues** of $\boldsymbol{S}$ are in $\boldsymbol{\Lambda}$
$\boldsymbol{A = U\Sigma V^{\mathrm{T}}}$	Chapter 7 has **singular vectors** in $\boldsymbol{U}$ and $\boldsymbol{V}$ and **singular values** in $\boldsymbol{\Sigma}$

The columns of A are in m-dimensional space $\mathbf{R}^m$, the rows are in $\mathbf{R}^n$. The m by n matrix multiplies vectors $\boldsymbol{x}$ in $\mathbf{R}^n$ to produce $A\boldsymbol{x}$ in $\mathbf{R}^m$. But the real action of A is seen in the four fundamental subspaces.

Chapter 2 only allows one solution $\boldsymbol{x}_{\mathbf{row}}$. The matrix A is square and invertible. Chapter 3 finds every solution to $A\boldsymbol{x} = \boldsymbol{b}$ by adding every $\boldsymbol{x}_{\mathbf{null}}$. Chapter 4 deals with equations that don't have a solution (because $\boldsymbol{b}$ has a piece from the mysterious fourth subspace). I hope you will like the "big picture of linear algebra" on page 124: **all four subspaces**.

Those five factorizations are a perfect way to organize and remember linear algebra. The eigenvalues in the matrix $\boldsymbol{\Lambda}$ and the singular values in $\boldsymbol{\Sigma}$ come from S and A in a beautiful way—but not a simple way. Those numbers in $S\boldsymbol{x} = \lambda\boldsymbol{x}$ and $A\boldsymbol{v} = \sigma\boldsymbol{u}$ see deeply into the symmetric matrix S and the m by n matrix A. Often S appears in engineering and physics. Often A is a matrix of data. And data is now coming from everywhere.

Please don't miss Tim Baumann's page 272 on compressing photographs by the SVD.

Chapter 5 explains the amazing formulas produced by determinants. Amazing but unfortunately difficult to compute! We solve equations $A\boldsymbol{x} = \boldsymbol{b}$ before (not after) we find the determinant of A. Those equations ask us to produce $\boldsymbol{b}$ from the columns of A.

$$\boldsymbol{Ax} = \boldsymbol{x_1}\,(\textbf{column 1}) + \boldsymbol{x_2}\,(\textbf{column 2}) + \cdots + \boldsymbol{x_n}\,(\textbf{column } \boldsymbol{n}) = \textbf{ right side } \boldsymbol{b}.$$

In principle, determinants lead to eigenvalues (Chapter 6) and singular values (Chapter 7). In practice, we look for other ways to find and use those important numbers. And yet determinants tell us about geometry too—like the volume of a tilted box in n dimensions.

A short course can go directly from dimensions in 3.5 to eigenvalues for 2×2 matrices.

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Final Chapter : Learning from Data

We added Chapter 8 on **data science**. The data often comes in a rectangular matrix A. Each row measures n properties of a sample. That number n is large, and the number of samples is often very large. Out of this big matrix, applied linear algebra has to find out what is important. The goal is to produce understanding that leads to a decision.

In machine learning the output is a nonlinear function of the input. Deep learning aims to find that function from the training data. It produces (often with giant computations) a learning function $\boldsymbol{F(x, v)}$. The input vectors $\boldsymbol{v}$ contain the features of each training sample (many $\boldsymbol{v}$'s from many samples). The vector $\boldsymbol{x}$ contains the weights assigned to those features. And hidden inside F is a nonlinear function that mixes the inputs and the weights. A favorite is the simple function ReLU $(y) =$ maximum of 0 and y : a *ramp function*.

Optimizing the weights to learn from the training data $\boldsymbol{v}$ is the big computation. When that is well done, we can input new samples that the system has never seen. The success of deep learning is that $F(\boldsymbol{x}, \boldsymbol{v})$ is often close to the correct output. The system has identified an image, or translated a sentence, or chosen a winning move.

This is *piecewise linear algebra*, a mixture of weight matrices and ReLU. It is included with no expectation of testing students. This is a chapter to use later, in any way you want. You could experiment with the website **playground.tensorflow.org**.

Machine learning has become important and powerful—based on linear algebra and calculus (optimizing the weights) and on statistics (controlling the mean and variance). *But it is not required or even expected to be part of the course. What I hope is that the faster start allows you to reach eigenvalues and singular values*—those are true highlights of this subject. Chapters 2 to 7 confirm the jump of intuition near the end of Chapter 1 :

> If all columns of A lie on an r-dimensional plane, then all rows of A also lie on a (usually different) r-dimensional plane. That fact has far-reaching consequences.

This is a textbook for a normal linear algebra class—to explain the key ideas of this beautiful subject to *everyone*. As clearly as I can. Thank you.

Gilbert Strang

After writing this introduction, I looked back at the opening example. That 3 by 3 matrix A has dependent columns (column $1+$ column 2 $=$ column 3). By linear algebra, A must also have dependent rows. The numbers to show this are $\boldsymbol{-5}$ and $\boldsymbol{3}$:

$$\boldsymbol{-5}\,(\text{row } 1) + \boldsymbol{3}\,(\text{row } 2) = \text{row } 3 \text{ of } A \qquad \boldsymbol{-5}\,(\mathbf{1},\mathbf{2},\mathbf{3}) + \boldsymbol{3}\,(\mathbf{3},\mathbf{4},\mathbf{7}) = (\mathbf{4},\mathbf{2},\mathbf{6}).$$

The number of independent rows equals the number of independent columns. *Wonderful* !

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Websites for Linear Algebra

The dedicated website for this textbook is **math.mit.edu/everyone**. Several key sections of the book can be downloaded. That site also has brief solutions to the problem sets. (For homework , the instructor may ask for more detailed solutions.) Every class will find a balance between learning the essential ideas of this subject and practicing the steps on small matrices.

The most beautiful website for linear algebra is **3Blue1Brown.com**, created by Grant Sanderson. He chose that name based on an unusual genetic feature of his eyes.

For blackboard lectures you could go to the OpenCourseWare site **ocw.mit.edu** created by MIT. The videos for Math 18.06 and Math 18.06SC and Math 18.065 have had millions of viewers, very often accessed through YouTube. 18.06 is MIT's large linear algebra course. The 18.06SC videos include problem solutions by the instructors. 18.065 is the newer course that leads to Chapter 8 of this book and to **math.mit.edu/learningfromdata**.

Important to add : The "*new start*" in this book was first tested in 18.065—which begins with a substantial review of linear algebra. Two more online materials were added in 2020 to the 18.06 site :

> A 70-minute video outlines the new start in teaching and learning linear algebra. That "2020 vision" has guided Sections 1.3 and 1.4 of this new book. I am convinced that working with the columns of actual matrices is a direct route to understanding linear independence and column spaces and matrix multiplication. The column-row factorization $\boldsymbol{A} = \boldsymbol{CR}$ is at the heart of solving linear equations $\boldsymbol{Ax} = \boldsymbol{b}$.

Properly used, the Web has become truly valuable to students and all readers. It provides a different way to explain linear algebra, and it is alive ! Please use the video lectures to see the flow of the course. And please use the book to capture that flow and hold it and practice it and understand it.

This book begins with independent vectors from the column space and row space : $\boldsymbol{A} = \boldsymbol{CR}$. The book ends with *orthogonal* vectors from those spaces : the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s in the SVD with $A\boldsymbol{v} = \sigma\boldsymbol{u}$. In between this we have the central ideas of linear algebra.

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Computational Linear Algebra

We need to solve large problems quickly and accurately. Linear algebra is often the key to computations in engineering and science and management and finance. This page aims to provide an update on hard power and soft power—computers and math.

The 500 fastest computers in the world are listed on **top500.org/news**. The leaders are mostly paid for and controlled by governments, and we can give their locations: Kobe, Oak Ridge, Livermore, Wuxi, Guangzhou. The next group includes Italy, Switzerland, Germany, France, Korea, Australia, Taiwan, UAE, UK, Russia, Spain, Saudi Arabia, Finland, Norway, and Brazil. The scientific interest is in their speed and special processors. A good source of information is **top500.org/resources/frequently-asked-questions**.

The benchmark tests are linear algebra problems—factoring a big matrix into L times U (lower triangular times upper triangular) and then solving a system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$. The actual test is in the High Performance LINPACK benchmark. Those problems are at the heart of Chapters 2 and 3, now speeded up by parallel processing.

The top machine achieves 415 petaflops $= 415$ times 10^{15} double precision floating point operations per second. This is with extremely careful coding for special hardware. The important point is that ordinary computers have also seen a tremendous increase in speed. And there are comparable savings from better algorithms.

Numerical Linear Algebra

This is the subject of major research: *fast algorithms for matrix computations*. The first was thousands of years ago with "elimination". For that idea and any new one, part of the test is to count the steps: in this case $n^3/3$ for n linear equations with n unknowns. To compute eigenvalues and now singular values, big progress brought that also to cn^3. Favorite textbooks among many good ones are

- L. N. Trefethen and David Bau, *Numerical Linear Algebra*, SIAM (1997)
- Gene Golub and Charles van Loan, *Matrix Computations*, Johns Hopkins (2013).

For the mathematics of linear algebra (not focused on computation) we mention

- Roger Horn and Charles Johnson, *Matrix Analysis*, Cambridge (2013)
- R. Bhatia, *Matrix Analysis*, Springer and *Positive Definite Matrices*, Princeton

Randomized Numerical Linear Algebra

The usual algorithms meet a barrier when the problem becomes very large. We can hardly store and read all the data. Even matrix multiplication AB becomes impossible. Eigenvalues hit their own barrier at matrix size $n = 10^4$ or 10^5.

A solution has been found if the matrix has enough structure. Random samples tell us about the matrix! This is a deep and beautiful subject—see a masterpiece by P. G. Martinsson and J. A. Tropp in *Acta Numerica* (2020) and arXiv: 2002.01387.

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Gratitude for Help

This book was written during the months of lockdown for the coronavirus. Life was limited but time for writing was nearly unlimited. Difficult for our society but perfect for an author. The idea for a new and more active start had just gone into a new video for Math 18.06 on OpenCourseWare. (The matrix multiplication $A = CR$ in Section 1.4 is part of the idea, with independent columns of A going into C.) Developing that idea into this textbook has been exciting every morning.

The time was right but help was needed. It came in the best possible way. My good friend Ashley C. Fernandes in Mumbai received handwritten pages every day. Then he returned LaTeX pages overnight. Those pages went back and forth many times. Working with Ashley has made *Linear Algebra for Everyone* possible; this is our eighth book. I am truly grateful for these happy months.

My greatest gratitude is to my wife Jill and our sons David and John and Robert.

This book is dedicated to them.

An important part of linear algebra is learning to speak the language ! Two key words are **"vector space"**. That is a collection of vectors $\boldsymbol{v}$—all with the same number of components. Then they can be added and subtracted and multiplied by constants—and all those results must belong to the vector space too ! Here are examples in 3 dimensions : **The $\boldsymbol{x}$-$\boldsymbol{y}$ plane or any plane through $(0, 0, 0)$ or any straight line through $(0, 0, 0)$. Or just the single point $(0, 0, 0)$. The plane, line, and point have dimensions $2, 1, 0$.**

Suppose we start with a matrix A : m rows and n columns. It leads to **"Four Fundamental Subspaces"**. The column space of A contains all combinations of the columns. The row space of A contains all combinations of the rows. The nullspaces of A and A^{T} contain all the solutions x and y to $A\boldsymbol{x} = \boldsymbol{0}$ and $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$.

And I mention an amazing fact : The column space and the row space have the **same dimension**. The matrix has r independent columns and r independent rows. You will see that the dimension r is a critical number in linear algebra.

The subject of linear algebra is understandable but it is not trivial !

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Dictionary of Matrices

A good way to tell you what this book contains is to name the matrices you will meet. This wide variety of matrices is a special feature of linear algebra.

Chapter 1	*Chapter* 2	*Chapter* 3
Identity matrix I	Elimination matrix E	Echelon matrix R
Column basis C	Lower triangular L	Free matrix F
Row basis R	Upper triangular U	Special solutions S
Rank 1 matrix $\boldsymbol{uv}^{\mathrm{T}}$	Inverse matrix A^{-1}	Mixing matrix $M = W^{-1}$
	Transpose matrix A^{T}	Incidence matrix A
	Permutation P	Laplacian matrix L
	Fourier matrix F	Pseudoinverse A^{+}

Chapter 4	*Chapter* 5
Orthogonal matrix Q	Cofactor matrix C
Projection matrix P	Change of basis B
Least squares $A^{\mathrm{T}}A$	Tilted box E
Upper triangular R	House matrix H
Reflection matrix H	

Chapter 6	*Chapter* 7	*Chapter* 8
Symmetric matrix S	Singular values Σ	Weight matrix A
Eigenvectors X	Left singular vectors U	Convolution C
Eigenvalues Λ	Right singular vectors V	Jacobian matrix J
Fibonacci matrix F	Compressed matrix A_k	Hessian matrix H
Jordan matrix J	Sample covariance AA^{T}	Covariance matrix V
Similar matrix BAB^{-1}	Hilbert matrix H	Shift matrix S
Exponential e^{At}		

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