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Stephen Ward , Paul Fannon
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1 Discrete random variables

In this chapter you will learn how to:

- predict the mean, mode, median and variance of a discrete random variable
- understand how a linear transformation of a variable changes the mean and variance
- prove and use the formulae for expectation and variance of a special distribution called the uniform distribution
- recognise when it is appropriate to use a uniform distribution.

A If you are following the A Level course, you will also learn how to:

- calculate the mean of a discrete random variable after a non-linear transformation.

Before you start…

A Level Mathematics Student Book 1, Chapter 21	You should know how to use the rules of probability.	1 Two events A and B are independent. If $\mathrm{P}(A) = 0.4$ and $\mathrm{P}(B) = 0.3$, find $\mathrm{P}(A \text{ AND } B)$.
A Level Mathematics Student Book 1, Chapter 21	You should know how to find probabilities of discrete random variables.	2 $\mathrm{P}(X = x) = kx$ for $x = 1, 2, 3$. Find the value of k.
A Level Mathematics Student Book 1, Chapter 20	You should know how to find the mean, variance and standard deviation of data, including familiarity with formulae involving sigma notation.	3 Find the variance of 2, 5 and 8.
A Level Further Mathematics Student Book 1, Chapter 11	You should know how to calculate sums of powers of n.	4 Find and simplify an expression for $\sum_{1}^{n} r(r-1)$.

What are discrete random variables?

A **random variable** is a variable that can change every time it is observed – such as the outcome when you roll a dice. A **discrete random variable** can only take certain values. In A Level Mathematics Student Book 1, Chapter 21, you covered the **probability distributions** of discrete random variables – a table or rule giving a list of all possible outcomes along with their probabilities.

Many real-life situations follow probability distributions – such as the velocity of a molecule in a waterfall or the amount of tax paid by an individual. It is extremely difficult to make a prediction about a single observation, but it turns out that you can predict remarkably accurately the overall behaviour of many millions of observations. In this chapter you will see how you can predict the mean and variance of a discrete random variable.

> **Tip**
>
> Discrete variables don't have to take integer values. However, the possible distinct values can be listed, though the list may be infinite. For example:
> If X is the standard UK shoe size of a random adult member of the public, X takes values 2, 2.5, 3, 3.5 up to 15.5 and is a discrete random variable.
> If Y is the exact foot length of a random adult member of the public (in cm), Y takes values in the interval [20, 35] and is a continuous random variable.

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Section 1: Average and spread of a discrete random variable

The most commonly used measure of the average of a random variable is the **expectation**. It is a value representing the mean result if the variable were to be measured an infinite number of times.

Key point 1.1

The expectation of a discrete random variable X is written $\mathrm{E}(X)$ and calculated as

$$\mathrm{E}(X) = \sum x_i p_i$$

where x_i is each possible value that X can take and p_i is the associated probability.

Tip

The expectation of a random variable does not need to be a value that the variable can actually take.

Tip

The subscript i in the formula in Key point 1.1 is just a counter referring to each possible value and its associated probability.

You do not need to be able to prove this result, but you might find it helpful to see this proof.

PROOF 1

The mean of n pieces of discrete data is $\bar{x} = \frac{1}{n}\sum f_i x_i$ — Start from the definition of the mean.

$= \sum\left(\frac{f_i}{n}\right)x_i$ — Since $\frac{1}{n}$ is constant you can take it into the sum.

If n is large, $\frac{f_i}{n}$ will tend towards the probability of x_i happening, therefore $\bar{x} = \mu = \sum x_i p_i$. — When the sample size tends to infinity, the sample mean $\bar{x}$ becomes the true population mean, μ.

WORKED EXAMPLE 1.1

The random variable X has a probability distribution as shown in the table. Calculate $\mathrm{E}(X)$.

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$

$\mathrm{E}(X) = 1\times\frac{1}{10}+2\times\frac{1}{4}+3\times\frac{1}{10}+4\times\frac{1}{4}+5\times\frac{1}{5}+6\times\frac{1}{10}$ — Use the values from the distribution in the formula in Key point 1.1.

$= \frac{7}{2}$

As well as knowing the expected average, you may also be interested in how far away from the average you can expect an outcome to be. The **variance**, σ^2, of a random variable is a value representing the degree of variation that would be seen if the variable were to be repeatedly measured an infinite number of times. It is a measure of how spread out the variable is.

Fast forward

You will see in Section 2 how to find expectations of other functions of X.

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Key point 1.2

The variance of a discrete random variable X is written $\text{Var}(X)$ and calculated as $\text{Var}(X) = \text{E}(X^2) - \text{E}(X)^2$ where $\text{E}(X^2) = \Sigma\, x_i^2 p_i$

The quantity $\Sigma\, x_i^2 p_i$ is the expected value of X^2, read as 'the mean of the squares'. This variance formula is often read as 'the mean of the squares minus the square of the mean'.

WORKED EXAMPLE 1.2

Calculate $\text{Var}(X)$ for the probability distribution in Worked example 1.1.

From Worked example 1.1:
$\text{E}(X) = 3.5$

$$\text{E}(X^2) = 1^2 \times \frac{1}{10} + 2^2 \times \frac{1}{4} + 3^2 \times \frac{1}{10} + 4^2 \times \frac{1}{4} + 5^2 \times \frac{1}{5} + 6^2 \times \frac{1}{10} = 14.6$$

Use the values from the distribution in the formula in Key point 1.2.

$$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2 = 14.6 - 12.25 = 2.35$$

Two other less commonly used measures of average are the **mode** and the **median**. For data, the mode is the most common result and this extends to variables.

Key point 1.3

The mode of a discrete random variable X is the value of X associated with the largest probability.

For data, the median is the value that has half the data values below it and half above it. You can interpret this in terms of probabilities.

Key point 1.4

The median, M, of a discrete random variable X is any value that has

$$\text{P}(X \leqslant M) \geqslant 0.5 \text{ and } \text{P}(X \geqslant M) \geqslant 0.5$$

If there are two possible values, you have to find their mean.

Did you know?

Standard deviation – the square root of variance – is a much more meaningful representation of the spread of a variable. So why is variance used at all? The answer is purely to do with mathematical elegance. It turns out that the algebra of variance is far neater than the algebra of standard deviations.

Tip

Many calculators can simplify this process. You normally have to treat the values of the random variable as data and the probabilities as the frequency.

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When there are two possible values and you have to take their mean, the median will take a value different from any observed value of the random variable.

WORKED EXAMPLE 1.3

For the distribution in Worked example 1.1 find:

a the mode

b the median.

a The largest probability is $\frac{1}{4}$ so there are two modes: 2 and 4.

b

x	1	2	3	4	5	6
$\mathrm{P}(X=x)$	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$
$\mathrm{P}(X\leqslant x)$	0.1	0.35	0.45	0.7	0.9	1

You can create a table of $\mathrm{P}(X \leqslant x)$.

So the median is 4.

Look for the first value that has a value of $\mathrm{P}(X \leqslant x)$ greater than or equal to 0.5. You could also check that $\mathrm{P}(X \geqslant x) \geqslant 0.5$ but this is not necessary here.

A probability distribution can also be described by a function.

WORKED EXAMPLE 1.4

W is a random variable that can take values –0.5, 1.5, 2.5 and k where $k > 0$.

$$\mathrm{P}(W = w) = \frac{w^2}{29}$$

a Find the value of k.

b Find the expected mean of W.

c Find the standard deviation of W.

a $\frac{(-0.5)^2}{29} + \frac{1.5^2}{29} + \frac{2.5^2}{29} + \frac{k^2}{29} = 1$

Use the fact that the total of all the probabilities must be 1.

$$0.25 + 2.25 + 6.25 + k^2 = 29$$

$$k^2 = 20.25$$

$$k = 4.5 \text{ since } k > 0.$$

b $\mathrm{E}(W) = -0.5 \times \frac{(-0.5)^2}{29} + 1.5 \times \frac{1.5^2}{29} + 2.5 \times \frac{2.5^2}{29} + 4.5 \times \frac{4.5}{29} = 3.79$ (3 s.f.)

Use Key point 1.1.

c $\mathrm{E}(W^2) = (-0.5)^2 \times \frac{(-0.5)^2}{29} + 1.5^2 \times \frac{1.5^2}{29} + 2.5^2 \times \frac{2.5^2}{29} + 4.5^2 \times \frac{4.5}{29} = 15.7$ (3 s.f.)

To find the standard deviation you first need to find the variance, which means you need to find $\mathrm{E}(W^2)$ and use Key point 1.2.

Continues on next page

$\text{Var}(W) = 15.7 - 3.79^2 \approx 1.28$

So $\sigma \approx \sqrt{1.28} = 1.13$ (3 s.f.)

Although you only write down three significant figures in the working, make sure you use the full accuracy from your calculator to find the final answer.

WORK IT OUT 1.1

Find the variance of X, the random variable defined by this distribution.

x	0	1	2
$P(X = x)$	0.2	0.3	0.5

Which is the correct solution? Identify the errors made in the incorrect solutions.

A	$E(X) = \frac{0+1+2}{3} = 1$ $E(X^2) = \frac{0^2+1^2+2^2}{3} = \frac{5}{3}$ $\text{Var}(X) = \frac{5}{3} - 1^2 = \frac{2}{3}$
B	$E(X) = 1 \times 0.3 + 2 \times 0.5 = 1.3$ $E(X^2) = 1 \times 0.3 + 4 \times 0.5 = 2.3$ $\text{Var}(X) = 2.3 - 1.3^2 = 0.61$
C	$E(X) = 0 \times 0.2 + 1 \times 0.3 + 2 \times 0.5 = 1.3$ $E(X^2) = 0^2 \times 0.2^2 + 1^2 \times 0.3^2 + 2^2 \times 0.5^2 = 1.09$ $\text{Var}(X) = 1.09 - 1.3^2 = -0.6$

EXERCISE 1A

1 Calculate the expectation, mode, median, variance and standard deviation of each of these discrete random variables.

a i

x	1	2	3	4
$P(X = x)$	0.4	0.3	0.2	0.1

ii

x	8	9	10	11
$P(X = x)$	0.4	0.3	0.2	0.1

b i

x	10	20	30	40
$P(X = x)$	0.4	0.3	0.2	0.1

ii

x	80	90	100	110
$P(X = x)$	0.4	0.3	0.2	0.1

c i

w	0.1	0.2	0.3	0.4
$P(W = w)$	0.4	0.1	0.25	0.25

ii

v	0.1	0.2	0.3	0.4
$P(V = v)$	0.5	0.3	0.1	0.1

d i $P(X = x) = \frac{x^2}{14}$, $x = 1, 2, 3$

ii $P(X = x) = \frac{1}{x}$, $x = 2, 3, 6$

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2 A discrete random variable X is given by $P(X=x)=k(x+1)$ for $x=2,3,4,5,6$.

a Show that $k=0.04$.

b Find $E(X)$.

3 A discrete random variable V has the probability distribution shown and $E(V)=5.1$.

v	1	2	5	8	p
$P(V=v)$	0.2	0.3	0.1	0.1	q

a Find the values of p and q.

b Find the median of V.

4 A discrete random variable X has its probability given by

$P(X=x)=k(x+3)$, where $x=0,1,2,3$.

a Show that $k=\frac{1}{18}$.

b Find the exact value of $E(X)$.

5 The probability distribution of a discrete random variable X is defined by

$P(X=x)=kx(4-x)$, $x=1,2,3$.

a Find the value of k.

b Find $E(X)$.

c Find the standard deviation of X.

6 A fair six-sided dice, with sides numbered 1, 1, 2, 2, 2, 5, is thrown. Find the mean and variance of the score.

7 The table shows the probability distribution of a discrete random variable X.

x	0	1	2	3
$P(X=x)$	0.1	p	q	0.2

a Given that $E(X)=1.5$, find the values of p and of q.

b Find the standard deviation of X.

8 A biased dice with four faces is used in a game. A player pays 5 counters to roll the dice. The table shows the possible scores on the dice, the probability of each score and the number of counters the player receives in return for each score.

Score	1	2	3	4
Probability	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{20}$
Number of counters player receives	4	5	15	n

Find the value of n in order for the player to get an expected profit of 3.25 counters per roll.

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9 Two fair dice labelled with the values 1 to 6 are thrown. The random variable D is the difference between the larger and the smaller score, or zero if they are the same.

a Copy and complete this table to show the probability distribution of D.

d	0	1	2	3	4	5
$P(D=d)$	$\frac{1}{6}$	$\frac{5}{18}$				

b Find $E(D)$.

c Find $Var(D)$.

d Find the median of D.

e Find $P(D > E(D))$.

10 **a** In a game a player pays an entrance fee of £n. He then selects one number from 1, 2, 3 or 4 and rolls three fair four-sided dice, numbered 1 to 4. If his chosen number appears on all three dice he wins four times the entrance fee. If his number appears on exactly two of the dice he wins three times the entrance fee. If his number appears on exactly one dice he wins £1. If his number does not appear on any of the dice he wins nothing.

Copy and complete the probability table.

Profit (£)	$-n$			
Probability	$\frac{27}{64}$			

b The game organiser wants to make a profit over many plays of the game. Given that he must charge a whole number of pence, what is the minimum amount the organiser must charge?

11 Viewers are asked to rate a new film on a three-point scale. Their marks are modelled by the random variable S as shown.

s	1	2	3
$P(S=s)$	0.3	a	b

a The mean, median and mode of S are all equal. Find the variance of S.

b Two independent viewers of the film are both asked their opinion.

i What is the probability that their total score is more than 4?

ii Show that the expectation of their total score is 4.

12 The number of books borrowed by each person who visits a library is modelled by the random variable B.

b	0	1	2	3	4
$P(B=b)$	0.2	0.3	0.3	0.1	0.1

a Find the mean of B.

b Show that the expectation of B is larger than the median of B.

c Show that the standard deviation of B is less than the median of B.

d 10 people visited the library during an audit period. The numbers of books they borrowed are independent of each other. Find:

i the probability that exactly three people borrow no books

ii the expected number of people who borrow no books.

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Section 2: Expectation and variance of transformations of discrete random variables

Linear transformations

You might have noticed a link between parts **a** and **b** of question 1 in Exercise 1A. The distributions were very similar but in part **b** all the x-values were multiplied by 10. All the averages and the standard deviations were also multiplied by 10 but the variances were multiplied by 100. This is an example of a **transformation**.

The most common type of transformation is a **linear transformation**. This is where the new variable (Y) is found from the old variable (X) by multiplying by a constant and/or adding on a constant. You might do this, for example, if you change the units of measurement. This kind of change is also known as 'linear coding'.

If you know the original mean and variance and how the data were transformed, you can use a shortcut to find the mean and variance of the new data.

Key point 1.5

If X is a random variable and Y is a new random variable such that $Y = aX + b$, then:

$$\text{E}(Y) = a\text{E}(X) + b$$

$$\text{Var}(Y) = a^2\text{Var}(X)$$

Fast forward

You will prove Key point 1.5 after you have developed a little more theory.

This means that the standard deviation of Y, σ_Y, is $|a|\sigma_X$. This makes sense as multiplying the data by a does change how spread out they are, but adding on b does not change the spread.

WORKED EXAMPLE 1.5

A random variable X has expectation 7 and variance 100. Y is a transformation of X given by $Y = 100 - 2X$. Find:

a the expectation of Y

b the standard deviation of Y.

a $\text{E}(Y) = 100 - 2 \times \text{E}(X)$
$= 100 - 2 \times 7$
$= 86$

This is just a direct application of Key point 1.5.

b $\text{Var}(Y) = (-2)^2\,\text{Var}(X)$
$= 4 \times 100$
$= 400$
$\sigma_Y = \sqrt{400} = 20$

To find the standard deviation you first need to find the variance of Y, using Key point 1.5.

Common error

It is easy to get confused with the minus sign in the transformations in Worked example 1.5. Remember that both variances and standard deviations are always positive.

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Non-linear transformations

You can also apply non-linear transformations to X, such as X^2, $\sin X$ or $\frac{1}{2X+3}$. When you do this there is no shortcut to finding the mean and variance of the transformed variable. You need to adapt Key point 1.1.

Consider the discrete random variable X = outcome on a fair six-sided dice. If $Y = X^2$, you can construct the probability distribution for Y:

x	1	2	3	4	5	6
y	1	4	9	16	25	36
P	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The probability of Y being 9 is just the same as the probability of X being 3. So $\text{E}(Y) = 1\times\frac{1}{6} + 4\times\frac{1}{6} + 9\times\frac{1}{6} + 16\times\frac{1}{6} + 25\times\frac{1}{6} + 36\times\frac{1}{6}$, i.e. it is $\sum x_i^2 p_i$.

Key point 1.6

If X is a discrete random variable with expectation $\text{E}(X)$ and g is a function applied to X, then

$$\text{E}(\text{g}(X)) = \sum \text{g}(x_i)p_i$$

WORKED EXAMPLE 1.6

The discrete random variable X has the distribution shown in the table.

x	30	45	60
$\text{P}(X = x)$	0.25	0.5	0.25

If $Y = \sin X°$, find:

a $\text{E}(Y)$ **b** $\text{Var}(Y)$.

a $\text{E}(Y) = \sin 30° \times 0.25 + \sin 45° \times 0.5 + \sin 60° \times 0.25 = 0.695$ (3 s.f.)

Apply Key point 1.6

b $\text{E}(Y^2) = \sin^2 30° \times 0.25 + \sin^2 45° \times 0.5 + \sin^2 60° \times 0.25 = 0.5$

To find $\text{Var}(Y)$ you need $\text{E}(Y^2)$ which is $\text{E}(\sin^2 X)$.

$\text{Var}(Y) \approx 0.5 - 0.695^2 = 0.0169$ (3 s.f.)

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You can use Key point 1.6 to prove Key point 1.5.

PROOF 2

Let $Y = aX + b$

Then: $\mathrm{E}(Y) = \Sigma(ax_i + b)p_i$

$= a\Sigma x_i p_i + b\Sigma p_i$

$= a\mathrm{E}(X) + b$

Apply Key point 1.6 to the function $g(x) = ax + b$.

You can separate out a sum into its different terms, taking out constant factors.

Use the fact that $\Sigma p_i = 1$ for any probability distribution and the definition of expectation from Key point 1.1. You have now established the first part of Key point 1.6.

Considering $\mathrm{E}(Y^2)$ to get to the variance:

$\mathrm{E}(Y^2) = \Sigma(ax_i + b)^2 p_i$

$= \Sigma(a^2 x_i^2 + 2abx_i + b^2)p_i$

$= a^2\Sigma x_i^2 p_i + 2ab\Sigma x_i p_i + b^2\Sigma p_i$

$= a^2\mathrm{E}(X^2) + 2ab\mathrm{E}(X) + b^2$

Apply Key point 1.6 to the function $g(x) = (ax + b)^2$ and expand the brackets.

You can separate out a sum into its different terms, taking out constant factors.

Use the fact that $\Sigma p_i = 1$ for any probability distribution and the definitions of $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$.

Using the definition of variance from Key point 1.2:

$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \mathrm{E}(Y)^2$

$= a^2\mathrm{E}(X^2) + 2ab\mathrm{E}(X) + b^2 - (a\mathrm{E}(X) + b)^2$

$= a^2\mathrm{E}(X^2) + 2ab\mathrm{E}(X) + b^2 - a^2\mathrm{E}(X)^2 - 2ab\mathrm{E}(X) - b^2$

$= a^2\mathrm{E}(X^2) - a^2\mathrm{E}(X)^2$

$= a^2(\mathrm{E}(X^2) - \mathrm{E}(X)^2)$

$= a^2\mathrm{Var}(X)$

Expand the brackets and lots of terms cancel!

Taking out a factor of a^2 leaves the expression for $\mathrm{Var}(X)$ from Key point 1.2. This completes the proof.

EXERCISE 1B

1. $\mathrm{E}(X) = 9$ and $\mathrm{Var}(X) = 25$. Find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$ if:

a	**i** $Y = 3X$	**ii** $Y = 4X$
b	**i** $Y = X - 1$	**ii** $Y = X - 2$
c	**i** $Y = 2X - 1$	**ii** $Y = 3X - 5$
d	**i** $Y = 10 - 3X$	**ii** $Y = 8 - 2X$
e	**i** $Y = \dfrac{X-1}{4}$	**ii** $Y = \dfrac{X+5}{10}$.

