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Vesna Kadelburg , Ben Woolley
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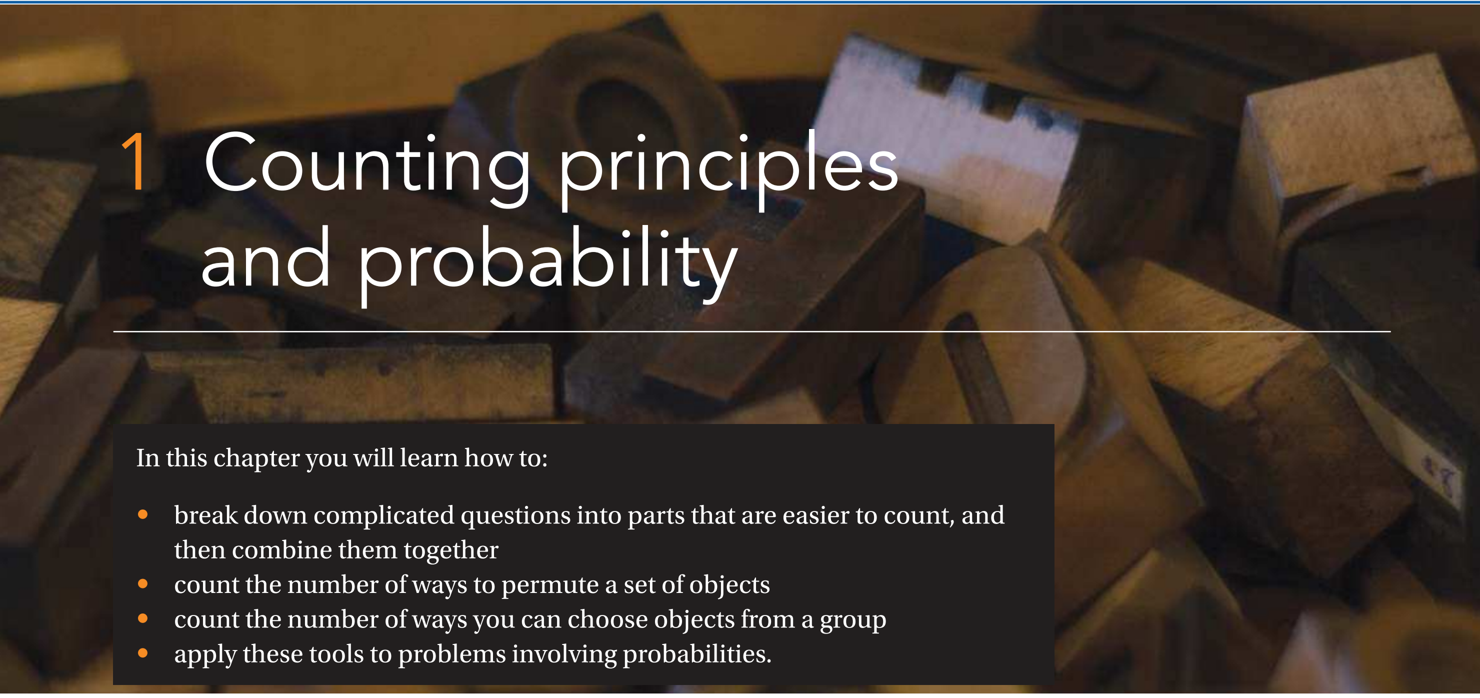


1 Counting principles and probability

In this chapter you will learn how to:

- break down complicated questions into parts that are easier to count, and then combine them together
- count the number of ways to permute a set of objects
- count the number of ways you can choose objects from a group
- apply these tools to problems involving probabilities.

Before you start…

A Level Mathematics Student Book 1, Chapter 9	You should know how to work with the factorial function.	1 Evaluate $\frac{7!}{5!}$
A Level Mathematics Student Book 1, Chapter 17	You should know how to work with basic probability.	2 What is the probability of rolling a prime number on a fair dice?

Making maths count

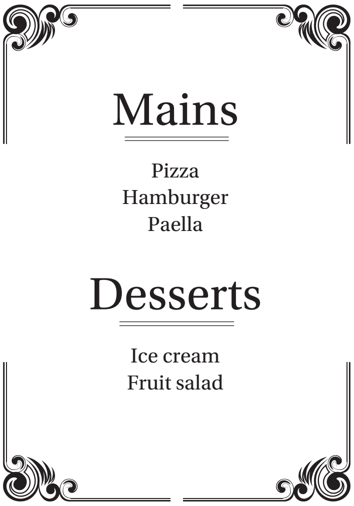
Counting is the one of the first things you learn in Mathematics and at first it seems very simple. If you were asked to count how many people there are in your school, this would not be too tricky. If you were asked how many chess matches need to be played if everyone is to play everyone else, this is a little more complicated. If you were asked how many different football teams could be chosen, you might find that the numbers become far too large to count without using some mathematical techniques.

One of the main uses of this type of counting is in calculating probabilities. You already have tools such as tree diagrams and Venn diagrams, but some problems are easier to solve by counting all the possibilities. The methods can be applied to fields as diverse as games of chance, genetics and cryptography.

Although in the exam all the questions will be set in the context of probability, you need to learn about various counting techniques first, before applying them to probability problems in Section 8.

Section 1: The product principle and the addition principle

Counting very small groups is simple. You need to break down more complicated problems into counting small groups. But how do you combine these together to come up with an answer to the overall problem? The answer lies in the product principle and the addition principle, which can be demonstrated by considering order choices from the menu shown.



Anna would like to order a main course *and* a dessert. She could make six different orders – three choices for the main course and for each choice she makes there, two choices for dessert, so she multiplies the individual possibilities.

Bob would like to order *either* a main course *or* a dessert. He could make five different orders – one of the three main courses or one of the two desserts so he adds the individual possibilities.

You can use the notation $\mathrm{n}(A)$ to mean the number of ways of making a choice about A.

The product principle tells us that when you wish to select one option from A AND one option from B you multiply the individual possibilities together.

Key point 1.1

Product principle: $\mathrm{n}(A \text{ AND } B) = \mathrm{n}(A) \times \mathrm{n}(B)$

The addition principle tells us that when you wish to select one option from A *or* one option from B you add the individual possibilities together.

The addition principle has one caveat. You can only use it if there is no overlap between the choices for A and the choices for B. For example, you *cannot* apply the addition principle to counting the number of ways of getting an odd number or a prime number on a dice. If there is no overlap between the choices for A and for B the two events are mutually exclusive.

Key point 1.2

Addition principle:

$$\mathrm{n}(A \text{ OR } B) = \mathrm{n}(A) + \mathrm{n}(B)$$

if A and B are mutually exclusive.

The hardest part of applying the addition or product principles is breaking the problem down into relevant parts. It is often useful to rewrite questions to emphasise whether what is required is 'AND' or 'OR'.

WORKED EXAMPLE 1.1

An examination has ten questions in section A and four questions in section B. Calculate how many different ways there are to choose questions if you must choose:

a one question from each section

b a question from either section A or section B.

a Choose one question from A (10 ways) AND one from B (4 ways).

Describe the problem accurately.

Number of ways $= 10 \times 4$
$= 40$

'AND' means you should apply the product principle.

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b Choose one question from A (10 ways) OR one from B (4 ways)

Describe the problem accurately.

Number of ways $= 10 + 4$
$= 14$

'OR' means you should apply the addition principle.

In the context of Worked example 1.1 you cannot have a repeat selection of an object. However, there are situations where you might be able to do so.

WORKED EXAMPLE 1.2

In a class there is an award for best mathematician, best sportsperson and nicest person. People can receive more than one award. In how many ways can the awards be distributed if there are twelve people in the class?

Choose one of twelve people for the best mathematician (12 ways)
AND one of the twelve for best sportsperson (12 ways)
AND one of the twelve for nicest person (12 ways).

Describe the problem accurately.

$12 \times 12 \times 12 = 1728$

Apply the product principle.

EXERCISE 1A

1 If there are 10 ways of doing *A*, 3 ways of doing *B* and 19 ways of doing *C*, with *A*, *B* and *C* mutually exclusive, calculate how many ways there are of doing:

a **i** both *A* and *B* **ii** both *B* and *C*

b **i** either *A* or *B* **ii** either *A* or *C*.

2 If there are four ways of doing *A*, seven ways of doing *B* and five ways of doing *C*, with *A*, *B* and *C* mutually exclusive, calculate how many ways there are of doing:

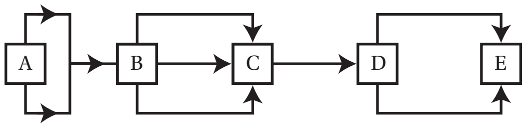
a all of *A*, *B* and *C* **b** exactly one of *A*, *B* or *C*.

3 Show many different paths there are in this diagram:

a from A to C

b from C to E

c from A to E.



4 John is planting out his garden and needs a new rose bush and some dahlias. There are twelve types of rose and four varieties of dahlia in his local nursery. How many possible selections does he have to choose from if he wants exactly one type of rose and one type of dahlia?

5 A lunchtime menu at a restaurant offers five starters, six main courses and three desserts. State how many different choices of meal you can make if you would like:

a a starter, a main course and a dessert

b a main course and either a starter or a dessert

c any two different courses.

6 Five men and three women would like to represent their club in a tennis tournament. In how many ways can one mixed doubles pair be chosen?

7 A Mathematics team consists of one student from each of Years 7, 8, 9 and 10. There are 58 students in Year 7, 68 in Year 8, 61 in Year 9 and 65 in Year 10.

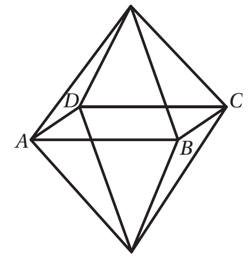
a How many ways are there of picking the team?

Year 10 is split into three classes: 10A (21 students), 10B (23 students) and 10C (21 students).

b If students from 10B cannot participate in the challenge, how many ways are there of picking the team?

8 Student ID codes consist of three letters chosen from A to Z, followed by four digits chosen from 1–9. Repeated characters are permitted. How many possible ID codes are there?

9 A beetle walks along the struts from the bottom to the top of an octahedral sculpture, visiting exactly two of the middle points (A, B, C or D), How many possible routes are there?



10 Professor Small has 15 different ties (7 blue, 3 red and 5 green), 4 waistcoats (red, black, blue and brown) and 12 different shirts (3 each of red, pink, white and blue). He always wears a shirt, a tie and a waistcoat.

a How many different outfits can he use until he has to repeat one?

Professor Small never wears any outfit that combines red with pink.

b How many different outfits can he make with this limitation?

11 State how many different three-digit numbers can be formed using the digits 1, 2, 3, 5, 7:

a at most once only

b as often as desired.

12 State how many ways:

a that four distinguishable toys can be put into three distinguishable boxes

b that three distinguishable toys can be put into five distinguishable boxes.

Section 2: Permutations

A permutation (sometimes called an arrangement) is an ordering of a list of objects. For example, a flag has four horizontal stripes: one each of the colours red, yellow, green and black. If you want to count how many different possible flags you have, you need to find the number of permutations of the colours. You can use this as a generic example to illustrate how to think about permutations.

There are four options for the colour of the top stripe, three options for the colour of the next one (because one of the colours has already been used), two options for the third stripe and one option for the last one. Using the 'product principle', the number of possible options for each are multiplied together, so the total number of permutations is $4 \times 3 \times 2 \times 1 = 24$. The number of ways n different objects can be permuted is equal to the product of all positive integers less than or equal to n; you have already met the notation for this expression: $n!$

Key point 1.3

$$n! = n(n-1)(n-2)\ldots \times 2 \times 1$$

The number of permutations (arrangements) of n distinct objects is $n!$

WORKED EXAMPLE 1.3

A test has twelve questions. How many different arrangements of the questions are possible?

Permute twelve items. — Describe the problem accurately.

Number of permutations $= 12! = 479\,001\,600$. — Use the formula from Key point 1.3.

In most examination questions you will have to combine the idea of permutations with the product and addition principles:

WORKED EXAMPLE 1.4

A seven-digit number is formed by using each of the digits 1 to 7 exactly once. How many such numbers are even?

Pick the final digit to be even (3 ways)
AND
then permute the remaining six digits (6! ways). — Describe the problem accurately.

$3 \times 720 = 2160$ possible even numbers — Apply the product principle.

This example shows the very common situation in counting where you are given a constraint – in this case you have to end with an even digit.

WORKED EXAMPLE 1.5

How many permutations of the letters of the word SQUARE start with three vowels?

Permute the three vowels at the beginning (3! ways)
AND
permute the three consonants at the end (3! ways). — Describe the problem accurately.

Number of ways $= 3! \times 3! = 36$ — Apply the product principle.

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EXERCISE 1B

1 Evaluate:

a i 5! b i $2 \times 4!$ c i $6! - 5!$

ii 6! ii $3 \times 5!$ ii $6! - 4 \times 5!$

2 Evaluate:

a i 8! b i $9 \times 5!$ c i $12! - 10!$

ii 11! ii $9! \times 5$ ii $9! - 7!$

3 Find the number of ways of arranging:

a 6 CDs b 8 photographs c 26 books.

4 a How many ways are there of arranging seven textbooks on a shelf?

b In how many of those permutations is the single biggest textbook in the middle?

5 a How many five-digit numbers can be formed by using each of the digits 1–5 exactly once?

b How many of those numbers are divisible by five?

6 A class of 16 pupils and their teacher are queuing outside a cinema.

a How many different permutations are there?

b How many different permutations are there if the teacher has to stand at the front?

7 A group of nine pupils (five boys and four girls) are lining up for a photograph, with all the girls in the front row and all the boys at the back. How many different permutations are there?

8 How many permutations of the letters of the word 'SQUARE' start with a consonant?

9 a How many six-digit numbers can be made by using each of the digits 1–6 exactly once?

b How many of those numbers are smaller than 300 000?

10 A class of 30 pupils is lining up in 3 rows of 10 for a class photograph. How many different arrangements are possible?

11 A baby has nine different toy animals. Five of them are red and four of them are blue. She arranges them in a line so that, in terms of colour, they are symmetrical. How many different arrangements are possible?

Section 3: Combinations

Suppose that three pupils are to be selected out of a class of eleven to attend a meeting with the head teacher. How many different groups of three can be chosen? In this example you need to choose three pupils out of eleven. They are not to be arranged in any specified order. Therefore, the selection of Ali, Bill and then Cathy is the same as the selection of Bill, Cathy and then Ali. This sort of selection is called a combination. In general the number of ways of choosing r distinct objects out of n is given the symbol $\binom{n}{r}$, ${}_nC_r$ or ${}^nC_r$ said as 'n choose r'.

Rewind

In A Level Mathematics Student Book 1, Chapter 17, you already met this symbol in the context of Pascal's triangle and the binomial expansion.

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Most of the time you can find the value of $\binom{n}{r}$ on your calculator.

However, you can now also justify the formula you met in the context of binomial expansions and binomial probabilities.

In the head teacher examples, there are 11 ways to select the first pupil, 10 ways to select the second and 9 ways to select the third; this makes the total of $11 \times 10 \times 9$ selections. However, there are $3! = 6$ different permutations of any three selected pupils, so you have counted each selection six times. Therefore the number of distinct selections is in fact $\frac{11 \times 10 \times 9}{3!} = 165$.

The expression $\frac{11 \times 10 \times 9}{3!}$ can also be written as $\frac{11!}{8!3!}$.

You may recognise this as the formula for the binomial coefficient $\binom{11}{3}$.

This reasoning can be generalised to count selections from any size group.

Key point 1.4

The formula for the number of ways of choosing r objects from n objects is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

This will appear in your formula book.

Tip

You can also calculate $\binom{n}{r}$ on your calculator, usually via an ${}^nC_r$ option.

Although you do not need to be able to prove this formula, you might find the following proof interesting. It highlights how counting arguments can be used to prove algebraic formulae.

PROOF 1

Total number of arrangements can be thought of as

Pick a set of size r

AND

Permute those r objects

AND

Permute the remaining $n-r$ objects

> Write an expression for the total number of arrangements in terms of selecting a subset and then arranging the selected part and the remaining part.

So

$$n! = \binom{n}{r} \times r! \times (n-r)!$$

> Use the fact that the total number of arrangements is $n!$ and apply the product principle to the previous analysis.

Rearranging:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

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WORKED EXAMPLE 1.6

A group of twelve friends wants to form a team for a five-a-side football tournament.

a In how many different ways can a team of five be chosen?

b Rob and Amir are the only goalkeepers and they cannot play in any other position. If a team has to contain exactly one goalkeeper, how many possible teams are there?

a Choosing 5 players from 12. — Describe the problem accurately.

$$\text{Number of ways} = \binom{12}{5} = 792$$

b Pick a goalkeeper, then fill in the rest of the team. — Decide on a strategy.

Amir is in goal (1 way) — Describe the problem accurately.
AND
choose four other players (? ways)
OR
Rob is in goal (1 way)
AND
choose four other players (? ways).

$$\text{Number of ways with Amir in goal} = \binom{10}{4}$$

With Amir picked there are four remaining slots, but Rob cannot be picked so ten players remain.

$$\text{Number of ways with Rob in goal} = \binom{10}{4}$$

$$\text{Total number of ways} = 1 \times \binom{10}{4} + 1 \times \binom{10}{4} = 210 + 210 = 420$$

Apply the addition principle and the product principle.

EXERCISE 1C

1 Evaluate:

a **i** $\binom{7}{2}$ **ii** $\binom{12}{5}$

b **i** $3 \times \binom{6}{3}$ **ii** $10 \times \binom{6}{5}$

c **i** $\binom{5}{0} \times \binom{9}{5}$ **ii** $\binom{10}{8} \times \binom{3}{1}$

d **i** $\binom{5}{2} + \binom{9}{5}$ **ii** $\binom{6}{0} + \binom{7}{3}$

2 **a** **i** In how many ways can six objects be selected from eight?

ii In how many ways can five objects be selected from nine?

b **i** In how many ways can either three objects be selected from ten or seven objects be selected from twelve?

ii In how many ways can either two objects be selected from five or three objects be selected from four?

c **i** In how many ways can five objects be selected from seven and three objects be selected from eight?

ii In how many ways can six objects be selected from eight and three objects be selected from seven?

3 An exam paper consists of 15 questions. Students can select any 9 questions to answer. How many different selections can be made?

4 Suppose you are revising for seven subjects, and you study three subjects in one evening. You might study a subject on more than one evening.

a In how many ways can you select three subjects to do on Monday evening?

b If you have to revise Maths on Tuesday, in how many ways can you select the subjects to do on Tuesday evening?

5 In the 'Pick 'n' Mix' lottery, players select 7 numbers out of 39. How many different selections are possible?

6 There are 16 boys and 12 girls in a class. Three boys and two girls are needed to take part in the school play. In how many different ways can they be selected?

7 A football team consists of one goalkeeper, four defenders, four midfielders and two forwards. A manager has three goalkeepers, eight defenders, six midfielders and five forwards in the squad. In how many ways can she pick the team?

8 A school is planning some trips over the summer. There are twelve places on the Greece trip, ten places on the China trip and ten places on the Disneyland trip. Each pupil can go on only one trip. If there are 140 pupils in the school, and assuming that they are all able to go on any of the 3 trips, in how many ways can the spaces be allocated?

9 Out of 26 teachers in a school, 4 are needed to accompany a school theatre trip.

a In how many ways can the four teachers be chosen?

b How many selections are possible if Mr Brown and Mrs Brown cannot both go on the trip?

10 A committee of 3 boys and 3 girls is to be selected from a class of 14 boys and 17 girls. State how many ways the committee can be selected if:

a Anna has to be on the committee

b the committee has to include Bill or Emma, but not both.

11 Sam's sweet shop stocks seven different types of 2p sweets and five different types of 5p sweets. If you want at most one of each sweet, state how many different selections of sweets can be made when spending:

a exactly 6p **b** exactly 7p **c** exactly 10p **d** at most 5p.

12 An English examination has two sections. Section A has five questions and section B has four questions. Four questions must be answered.

a How many different ways are there of selecting four questions to answer if there are no restrictions?

b How many different ways are there of selecting four questions if there must be at least one question answered in each section?

13 Ten points are drawn on a sheet of paper so that no three lie in a straight line.

a State how many different triangles can be drawn by connecting three points.

b State how many different quadrilaterals can be drawn by connecting four points.

14 A group of 45 students are to be seated in 3 rows of 15 for a school photograph. Within each row, students must sit in alphabetical order according to name, but there is no restriction determining the row in which a student must sit. How many different seating permutations are possible, assuming no students have identical names?

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Section 4: The exclusion principle

The exclusion principle is a trick for counting what you are interested in by counting what you are not interested in. This typically is needed when counting a situation where a certain property is prohibited.

Key point 1.5

Exclusion principle:

Count what you are *not* interested in and subtract it from the total.

As an example, suppose a five-digit code is formed by using each of the digits 1 to 5 exactly once. If you wanted to count how many such codes do *not* end in '25' you could consider all possible options for the last two digits.

WORKED EXAMPLE 1.7A

How many five-digit codes formed by using each of the digits 1 to 5 exactly once do *not* end in '25'?

Pick the final digit from {1,2,3,4} (4 ways)
AND
permute the remaining four digits (4! ways)
OR
pick the final digit as 5 (1 way)
AND
pick the penultimate digit from {1,3,4} (3 ways)
AND permute the remaining three digits (3! ways).

Describe the problem accurately.

$$(4\times 4!)+(1\times 3\times 3!)=114$$

Use the product and addition principles.

An alternative way to solve the same problem is to use the exclusion principle.

WORKED EXAMPLE 1.7B

How many five-digit codes formed by using each of the digits 1 to 5 exactly once do not end in '25'?

Count permutations of five-digit codes (5! ways)
then EXCLUDE cases where the last two digits are '25' (1 way)
AND
permute the remaining three digits (3! ways))

Describe the problem accurately.

$$5!-1\times 3!=114$$

Use the product and exclusion principles.

A very common use of the exclusion principle occurs when you are asked to count a situation containing an 'at least' or 'at most' restriction.