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Vesna Kadelburg , Ben Woolley
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1 Series and induction

In this chapter you will learn how to:

- use the principle of mathematical induction to prove results about sequences, series and differentiation
- use given results for the sums of integers, squares and cubes to find expressions for sums of other series
- use a technique called the method of differences to find an expression for the sum of n terms of a series
- use the expression for the sum of the first n terms to determine whether an infinite series converges and find its limit.

Before you start…

Pure Core Student Book 1, Chapter 6	You should be able to use mathematical induction to prove results about matrices, divisibility and inequalities.	1 Prove that $\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & 3n\\ 0 & 1\end{pmatrix}$.
GCSE	You should be able to use the nth term formula to generate terms of a sequence.	2 A sequence is defined by $u_n = n^2 + 3n - 1$. Find the first three terms.
GCSE	You should be able to simplify expressions by factorising.	3 Simplify $n(n+1)(2n+3) + n(n+1)(n-3)$.
A Level Mathematics Student Book 2, Chapter 4	You should be able to use sigma notation to write a series.	4 Find $\sum_{k=1}^{5} 2^k$.
A Level Mathematics Student Book 2, Chapter 10	You should know how to differentiate using the product rule and chain rule.	5 Given that $y = (2x+1)e^{3x}$, find $\frac{dy}{dx}$.
A Level Mathematics Student Book 2, Chapter 5	You should know how to write an expression in partial fractions.	6 Write $\frac{1}{r(r+1)}$ in partial fractions.

Introduction

In Pure Core Student Book 1, Chapter 6, you learnt about the method of proof by induction, which you can use to prove that observed patterns continue forever. The sorts of patterns you looked at included powers of matrices (for example, prove that $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}^n = \begin{pmatrix}1 & 0\\ n & 1\end{pmatrix}$ for all $n \in \mathbb{N}$), divisibility (for example, prove that $7^n - 3^n$ is divisible by 4 for all $n \in \mathbb{N}$) and inequalities (for example, prove that $2^n > 2n$ for $n \geqslant 3$).

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In this chapter you will revisit these ideas, including examples where you need to conjecture (guess) the pattern first, and then see how to extend them to other contexts, such as sums of series and differentiation.

Finding expressions for sums of series is one of the most difficult problems in mathematics as there is no general method that always works. For example, you know how to find the sum of the first n terms of a geometric series: $5^1 + 5^2 + \ldots + 5^n = \frac{5(5^n - 1)}{4}$. But what about a series such as $1^5 + 2^5 + \ldots + n^5$?

The method of mathematical induction is useful for proving that a conjectured formula for the sum of a series is correct, but it offers no help in finding what the formula might be. Sometimes you can guess the formula by looking at some examples, but most of the time the general expression is far from obvious. In this chapter you will meet the method of differences, which allows you to find the formula in some cases. You will also learn how to derive formulae for sums of more complicated series by combining results you have already derived.

Fast forward

In Chapter 2 you will use induction to prove de Moivre's theorem, a result about powers of complex numbers.

Fast forward

In Chapter 8 you will learn about another type of series called the Maclaurin series.

Section 1: Review of proof by induction

You can use induction to prove statements about a sequence or a pattern, where the statement holds for every natural number n. The proof involves two steps:

1. Prove that the statement is true for some starting value of n (usually, but not always, $n = 1$).
2. Assuming that the statement is true for some k, prove that it is also true for $k + 1$.

Then the principle of mathematical induction states the statement is true for all values of n.

Sometimes you need to conjecture the pattern for yourself before using induction to prove it.

Rewind

You will need to use the product rule for differentiation. This was covered in A Level Mathematics Student Book 2, Chapter 10.

WORKED EXAMPLE 1.1

Let $y = xe^x$.

a Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

b Conjecture an expression for $\frac{d^ny}{dx^n}$ and prove it by induction.

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a $\frac{dy}{dx} = e^x + xe^x = (1+x)e^x$ — Use the product rule to differentiate.

$\frac{d^2y}{dx^2} = e^x + (1+x)e^x = (2+x)e^x$ — The factorised form makes it easier to spot the pattern.

$\frac{d^3y}{dx^3} = e^x + (2+x)e^x = (3+x)e^x$

b Conjecture:

$\frac{d^ny}{dx^n} = (n+x)e^x$

Proof:

When $n = 1$: — Start by showing that the statement is true when $n = 1$.

$\frac{dy}{dx} = (1+x)e^x$

So the statement is true for $n = 1$.

Assume it is true for $n = k$:

$\frac{d^ky}{dx^k} = (x+k)e^x$ — Write down the statement with $n = k$.

When $n = k+1$, — Think about what you are trying to prove. Remember that you cannot use this result! You are working towards: $\frac{d^{k+1}y}{dx^{k+1}} = (x+k+1)e^x$

$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^ky}{dx^k}\right)$ — Relate $\frac{d^{k+1}y}{dx^{k+1}}$ to $\frac{d^ky}{dx^k}$.

$= \frac{d}{dx}((x+k)e^x)$ — Use the result you have assumed for $n = k$.

$= e^x + (x+k)e^x$ — Differentiate using the product rule.

$= e^k(1+x+k)$

$= e^k(x+(k+1))$ — This the the required result for $n = k + 1$.

Hence the result is also true for $n = k+1$.

The result is true for $n = 1$, and if true for $n = k$ it is also true for $n = k+1$. Therefore, the result is true for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction. — Remember to write the conclusion.

Sequences are often given by a term-to-term rule, but you might want to know a formula for the nth term. You might be able to guess the formula by looking at the numbers and then you can use induction to prove that it works for all n.

For example, the term-to-term rule $u_{n+1} = 3u_n + 2, u_1 = 2$ describes a sequence whose first four terms are 2, 8, 26, 80. You might notice that these are all one less than a power of 3, so the formula for the nth term could be $u_n = 3^n - 1$. You can prove by induction that this formula indeed works for all n.

WORKED EXAMPLE 1.2

A sequence is given by $u_1 = 2$ and $u_{n+1} = 3u_n + 2$ for $n \geqslant 1$. Prove that the nth term of the sequence is $u_n = 3^n - 1$.

When $n = 1$:

$u_1 = 2 = 3^1 - 1$ — Show that the result is true for $n = 1$.

So the formula works when $n = 1$.

Assume that the formula works when $n = k$:

$u_k = 3^k - 1$ — Assume that the formula works for some k.

When $n = k + 1$, — Think about what you are trying to prove. You are working towards: $u_{k+1} = 3^{k+1} - 1$

$$u_{k+1} = 3u_k + 2$$
$$= 3(3^k - 1) + 2$$
$$= 3^{k+1} - 1$$

— Use the result you assumed for $n = k$.

So, the formula also works when $n = k + 1$.

The formula works when $n = 1$, and if it works for some $n = k$ then it also works for $n = k + 1$. — Remember to write the conclusion.

Hence, by the principle of mathematical induction, the formula works for all $n \in \mathbb{N}$.

Sometimes each term in the sequence depends on more than one previous term. For example, the term-to-term rule

$$u_{n+2} = 5u_{n+1} - 6u_n \text{ with } u_1 = 5 \text{ and } u_2 = 13$$

produces this sequence:

$$u_1 = 5$$
$$u_2 = 13$$
$$u_3 = 5 \times 13 - 6 \times 5 = 35$$
$$u_4 = 5 \times 35 - 6 \times 13 = 97$$

and so on. You can still use proof by induction, but you need to show that the formula works for two starting values of n.

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WORKED EXAMPLE 1.3

A sequence is given by the recurrence relation $u_1 = 5$ and $u_2 = 13$, $u_{n+2} = 5u_{n+1} - 6u_n$ for $n \geqslant 2$. Prove that the formula for the nth term of the sequence is $u_n = 2^n + 3^n$.

When $n = 1$:

$\text{RHS} = 2^1 + 3^1$
$= 5$
$= u_1$

So, the formula works for $n = 1$.

Check that the formula works for $n = 1$ and $n = 2$.

When $n = 2$:

$\text{RHS} = 2^2 + 3^2$
$= 13$
$= u_2$

So, the formula works for $n = 2$.

Assume the formula works for $n = k$ and $n = k + 1$:

*Assume that the formula works for $n = k$ **and** $n = k + 1$, and prove that it works for $n = k + 2$.*

$u_k = 2^k + 3^k$
$u_{k+1} = 2^{k+1} + 3^{k+1}$

Think about the formula with $n = k$ and $n = k + 1$.

When $n = k + 2$,

Think about what you are trying to prove. You are working towards: $u_{k+2} = 2^{k+2} + 3^{k+2}$

$u_{k+2} = 5u_{k+1} - 6u_k$

Express u_{k+2} in terms of u_k and u_{k+1}.

$= 5(2^{k+1} + 3^{k+1}) - 6(2^k + 3^k)$

Use the results for $n = k$ and $n = k + 1$.

$= 5 \times 2^{k+1} + 5 \times 3^{k+1} - 6 \times 2^k - 6 \times 3^k$

$= (5 \times 2 \times 2^k - 6 \times 2^k) + (5 \times 3 \times 3^k - 6 \times 3^k)$

Look at what you are working towards; group the powers of 2 and the powers of 3.

$= 4 \times 2^k + 9 \times 3^k$
$= 2^2 \times 2^k + 3^2 \times 3^k$
$= 2^{k+2} + 3^{k+2}$

So, the formula also works for $n = k + 2$.

The formula works for $n = 1$ and $n = 2$, and if it works for $n = k$ and $n = k + 1$ then it also works for $n = k + 2$. Therefore, the formula works for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

Write a conclusion.

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EXERCISE 1A

1 Given that $u_{n+1} = 5u_n - 8, u_1 = 3$, prove by induction that $u_n = 5^{n-1} + 2$.

2 A sequence has first term 1 and subsequent terms defined by $u_{n+1} = 3u_n + 1$. Prove by induction that $u_n = \frac{3^n - 1}{2}$.

3 Given that $u_{n+1} = 5u_n + 4, u_1 = 4$,

a find the first four terms of the sequence

b conjecture a formula for the nth term and prove it by induction.

4 Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

a Find $\mathbf{A}^2$, $\mathbf{A}^3$ and $\mathbf{A}^4$.

b Conjecture an expression for $\mathbf{A}^n$ and prove it by induction.

Focus on ...

Powers of matrices have many interesting applications. To explore one of them see Focus on ... Modelling 1.

5 Let $f(n) = 5^n - 1$.

a Find $f(n)$ for $n = 1, 2, 3, 4$.

b Which natural number do all $f(n)$ seem to be multiples of?

c Use mathematical induction to prove your conjecture from part **b**.

6 A sequence is given by the term-to-term rule $u_{n+2} = 5u_{n+1} - 6u_n$ with $u_1 = 1$ and $u_2 = 5$. Prove that the general term of the sequence is $u_n = 3^n - 2^n$.

7 Given that $u_1 = 3, u_2 = 36, u_{n+2} = 6u_{n+1} - 9u_n$, prove by induction that $u_n = (3n-2)3^n$.

8 Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

a Find $\mathbf{A}^2$, $\mathbf{A}^3$ and $\mathbf{A}^4$.

b Conjecture an expression for $\mathbf{A}^n$ and prove it by induction.

9 **a** Suggest which natural number is a factor of all the numbers of the form $9^n - 4^n$.

b Prove your claim by induction.

10 Given that $y = \frac{1}{1-x}$, use induction to prove that $\frac{d^n y}{dx^n} = \frac{n!}{(1-x)^{n+1}}$.

11 Given that $f(x) = \frac{1}{1-3x}$, prove by induction that $f^{(n)}(x) = \frac{3^n n!}{(1-3x)^{n+1}}$.

12 Use mathematical induction to show that $\frac{d^n}{dx^n}(xe^{2x}) = (2^n x + n2^{n-1})e^{2x}$.

13 Prove by induction that $\frac{d^n}{dx^n}(x^2 e^x) = (x^2 + 2nx + n(n-1))e^x$ for $n \geqslant 2$.

14 Given that $y = x\sin x$, use mathematical induction to prove that $\frac{d^{2n}y}{dx^{2n}} = (-1)^n(x\sin x - 2n\cos x)$.

15 The Fibonacci sequence is defined by $u_1 = u_2 = 1, u_n = u_{n-1} + u_{n-2}$ for $n \geqslant 3$. Show that the nth term of the Fibonacci sequence is given by $u_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$.

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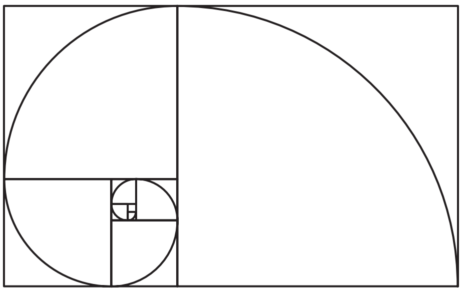
Explore

Leonardo Fibonacci (c. 1170 - c. 1250) was an extremely influential mathematician in the Middle Ages, largely responsible for spreading the number system you use today. He also gave his name to the famous Fibonacci sequence. The formula in question 15 shows the link between the Fibonacci sequence and the golden ratio, a quantity $\frac{1+\sqrt{5}}{2}$ which appears in many surprising places in mathematics.



Section 2: Induction and series

A **series** is a sum of the terms of a sequence. If you add the terms up to a certain point you get a **finite series**, such as $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$. You can also try to form an **infinite series**, for example $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$. Some infinite series, such as the geometric series given here, have a finite sum. In this section you will only look at finite series; you will meet some infinite series in Section 4.

Rewind

You met sequences, series and sigma notation in A Level Mathematics Student Book 2, Chapter 4.

You can use **sigma notation** as a shorter way of writing a series. For example,

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\sum_{k=2}^{5}\frac{1}{k}$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=\sum_{k=1}^{\infty}\frac{1}{2^k}$$

In the A Level Mathematics course you learnt how to find the general formula for the sum of the first n terms of an arithmetic and a geometric series:

$$\sum_{k=1}^{n}(a+(k-1)d)=\frac{n}{2}(2a+(n-1)d)$$

$$\sum_{k=1}^{n}ar^{k-1}=\frac{a(r^n-1)}{r-1}$$

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You also learnt about finite and infinite binomial series, for example:

$$1+4x+6x^2+4x^3+x^4=\sum_{k=0}^{4}{}^4C_k x^k=(1+x)^4$$

$$1-2x+3x^2-4x^3+\ldots=\sum_{k=0}^{\infty}\frac{(-2)(-3)\ldots(-2-(k-1))}{k!}x^k=(1+x)^{-2}$$

In Chapter 8 of this book you will learn about Maclaurin series, which you can use to write a function as an infinite series, for example:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}=\sin x$$

These are some examples of finite and infinite series where it is possible to find an exact expression for the sum. In general, finding an expression for the sum of the first n terms of a series can be surprisingly difficult, if not impossible. For example, it is not possible to express a formula for the sum $\sum_{k=1}^{n}\frac{1}{k^2}=\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}$ in terms of standard functions.

In cases where you manage to guess the formula for the sum of a series, you can then try to prove it by induction. The inductive step relies on making the connection between the sum of the first k terms and the sum of the first $k+1$ terms; this is done simply by adding the next term of the series.

Rewind

Binomial series were covered in A Level Mathematics Student Book 1, Chapter 9, and A Level Mathematics Student Book 2, Chapter 6.

Explore

Although it is not possible to find a general expression for $\sum_{k=1}^{n}\frac{1}{k^2}$, it is possible to find its exact sum to infinity, $\sum_{k=1}^{\infty}\frac{1}{k^2}$. Find out what it is: the result may surprise you!

Key point 1.1

If $S_k=u_1+u_2+\ldots+u_k$ then

$$S_{k+1}=S_k+u_{k+1}$$

WORKED EXAMPLE 1.4

Prove by induction that

$$\sum_{r=1}^{n}r(r+2)=\frac{n(n+1)(2n+7)}{6}\text{ for all } n\in\mathbb{Z}^+.$$

For $n=1$:

Show that the statement is true for the starting value (in this case, $n=1$).

LHS $=1\times3=3$

RHS $=\frac{1(1+1)(2\times1+7)}{6}=\frac{1\times2\times9}{6}=3$

So, the result is true for $n=1$.

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Assume that the result is true for $n = k$: — State the assumption for $n = k$.

$$\sum_{r=1}^{k} r(r+2) = \frac{k(k+1)(2k+7)}{6}$$

Let $n = k+1$: — Consider S_{k+1} and relate it to S_k by using $S_{k+1} = S_k + u_{k+1}$.

$$\sum_{r=1}^{k+1} r(r+2) = \sum_{r=1}^{k} r(r+2) + (k+1)(k+3)$$

$$= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)$$

Substitute in the result for $n = k$ (assumed to be true).

$$= (k+1)\left(\frac{2k^2+7k}{6} + \frac{6k+18}{6}\right)$$

$$= \frac{(k+1)(2k^2+13k+18)}{6}$$

$$= \frac{(k+1)(k+2)(2k+9)}{6}$$

Combine this into one fraction and simplify. It is always a good idea to take out any common factors.

$$= \frac{(k+1)((k+1)+1)(2(k+1)+7)}{6}$$

Show that this is in the required form by separating out $k+1$ in each place it occurs.

So, the result is also true for $n = k+1$.

The result is true for $n = 1$, and if it is true for $n = k$ it is also true for $n = k+1$. Therefore, the result is true for all $n \in \mathbb{Z}^+$, by induction. — Make sure you write a conclusion.

EXERCISE 1B

1. Prove by induction that, for all $n \in \mathbb{Z}^+$:

$$\sum_{r=1}^{n} 2 \times 3^{r-1} = 3^n - 1$$

2. Prove by induction that, for all integers $n > 1$:

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

3. Using mathematical induction prove that, for all positive integers:

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

4. Prove by induction that, for all integers $n > 1$:

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

5. Use mathematical induction to show that, for all integers $n > 1$:

$$\sum_{r=1}^{n} r2^r = 2\left[(n-1)2^n + 1\right]$$

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6 Prove by induction that, for all $n \in \mathbb{Z}^+$:

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

7 Using mathematical induction prove that, for all integers $n > 1$:

$$\sum_{r=1}^{n} r(r!)=(n+1)!-1$$

8 Prove by induction that, for all positive integers:

$$1^2-2^2+3^2-4^2+\ldots+(-1)^{n+1}n^2=(-1)^{n+1}\frac{n(n+1)}{2}$$

9 Prove using mathematical induction that, for all $n \in \mathbb{Z}^+$:

$$(n+1)+(n+2)+(n+3)+\ldots+(2n)=\frac{1}{2}n(3n+1)$$

10 Prove by induction that, for all integers $n > 1$:

$$\sum_{k=1}^{n} k2^k=(n-1)2^{n+1}+2$$

Section 3: Using standard series

You will now look at finding expressions for the sums of series such as $\sum_{k=1}^{n}\left(2n^3-5n\right)$ by combining some standard results.

You can use the following formulae without proof (unless the question explicitly asks you to prove them).

Key point 1.2

Formulae for the sums of integers, squares and cubes:

- $\sum_{r=1}^{n} r=\frac{1}{2}n(n+1)$
- $\sum_{r=1}^{n} r^2=\frac{1}{6}n(n+1)(2n+1)$
- $\sum_{r=1}^{n} r^3=\frac{1}{4}n^2(n+1)^2$

The second and third formulae will be given in your formula book.

Rewind

You proved the second and third formulae in Exercise 1B, Questions 2 and 3.

The first formula in Key Point 1.2 is just a special case of an arithmetic series.

