

**More Information** 

# 1 Proof and mathematical communication



In this chapter you will learn how to:

- use terms such as identity and equation to describe mathematical objects
- disprove a mathematical idea using a counter example
- use deduction and exhaustion to prove a mathematical idea.

## Before you start...

GCSE	You should know how to use the definition of the square root function.	1 What is $\sqrt{9}$ ?
GCSE	You should know how to manipulate algebraic expressions.	2 Factorise $4x^2 - 1$ .
GCSE	You should know basic angle facts.	<ul><li>3 a What is the sum of the angles in a triangle?</li><li>b What is the sum of the exterior angles of any polygon?</li></ul>
GCSE	You should be able to define rational and irrational numbers.	4 Which of these numbers are irrational? $\pi$ , 0.3, 0.5, $\sqrt{2}$
GCSE	You should be able to work with function notation.	5 If $f(x) = 2x^2 - 3$ find $f(3)$ .

# Why is proof important?

One thing for which mathematicians are valued is their ability to communicate their ideas precisely and to make very convincing arguments, called proofs. In this chapter we will look at the language used by mathematicians and some of the ways they prove their ideas.

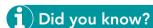
# Section 1: Mathematical structures and arguments

We can represent mathematical ideas in many different ways such as tables, diagrams, graphs or words. One of the most fundamental representations is an **equation**: a mathematical statement involving an '=' sign. For example:

$$x^2 - 1 = 8$$

An equation is only true for some values of x (or perhaps none); in this case it is true for  $x=\pm 3$ .

Another similar mathematical structure is called an **identity**. An identity is a relation which is true for **all** values of the unknown. It is given the  $\equiv$  symbol.



The first recorded use of the equals sign occurs in Robert Recorde's 1557 book The Whetstone of Witte. He explains that he used two parallel lines 'because no two things can be more equal'.



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For example,  $x^2 - 1 \equiv (x - 1)(x + 1)$ .

Two statements connected by the identity symbol are called **congruent expressions**. They are equal for **all** values of the variable(s).

There are some rules which only apply to identities. For example, if two polynomials are identically equal then their coefficients must be the same.



A **polynomial** is a function that is a sum of terms containing non-negative (positive or zero) integer powers of x.

A **coefficient** is the constant in front of (multiplying) a variable. For example, in the quadratic  $2x^2-3x$ , 2 is the coefficient of  $x^2$  and  $x^2$  and  $x^2$  is the coefficient of x.

The word integer just means 'whole number'.



Whenever you are simplifying an expression, technically you should use an identity symbol (=). However, it is common in mathematics to use an equals sign (=) instead.

## Fast forward

In Chapter 3 you will see that you often have to write quadratics in the form shown here.

#### **WORKED EXAMPLE 1.1**

$$2x^2 + 12x - 3 \equiv a(x+p)^2 + q$$

Find the value of a, p and q.

$$2x^2 + 12x - 3 \equiv a(x+p)^2 + q$$
 Multiply out the brackets to allow coefficients to be compared.

$$\equiv ax^2 + 2apx + ap^2$$

Coefficient of  $x^2$ : 2 = a Compare coefficients. The two expressions are equal for all values of x, so all the coefficients must be equal.

Coefficient of x: 12 = 2ap

Constant term:  $-3 = ap^2 + q$ 

$$4p = 12$$
 Substitute  $a = 2$  into the second equation.

p=3

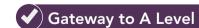
$$2 \times 3^2 + q = -3$$
 Substitute  $a = 2$ ,  $p = 3$  into the third equation.

q = -21

Another common structure in mathematics is called a **function**. This is just a rule for changing an input into an output.

For example, if you have a function called f that transforms the number x into  $\sqrt{x^2 + 9}$  you would write that as  $f(x) = \sqrt{x^2 + 9}$ .

So 
$$f(4) = \sqrt{4^2 + 9} = \sqrt{25} = 5$$
 and  $f(y) = \sqrt{y^2 + 9}$ .



See Gateway to A Level section A for a reminder of expanding brackets.



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#### 1 Proof and mathematical communication

## Implication and equivalence

You can manipulate both equations and inequalities by doing the same thing to both sides. You often structure your solution by writing lines of working underneath each other. In more formal work, you can emphasise the logic of the argument by using special symbols.

# 🔑 Key point 1.1

 The symbol ⇒ means that a subsequent equation follows from the previous one.

 $P \Rightarrow Q$  means 'P implies Q' or 'if P is true then Q is true' or 'P is sufficient for Q'.

• The symbol ← means that the previous statement follows from the subsequent statement.

 $P \leftarrow Q$  means 'P is implied by Q' or 'if Q is true then P is true' or 'P is necessary for Q'

The symbol 

means that a subsequent equation is equivalent to the previous one.

 $P \Leftrightarrow Q$  means 'P is equivalent to Q' or 'Q is true if and only if P is true.'

This can also be written in the shorthand 'Q iff P'.

You will also sometimes see the symbol ∴ for 'therefore,' which means we are drawing a conclusion from previous lines of working.

#### **WORKED EXAMPLE 1.2**

Insert either  $a \Rightarrow$  or  $a \Leftrightarrow$  symbol on each line of working:

**a** 
$$2x + 1 = 9$$

$$2x = 8$$

$$x=4$$

$$\mathbf{b} \quad x = 4$$

$$x^2 = 16$$

**a** 
$$2x + 1 = 9$$

 $\Leftrightarrow$  2x = 8 These statements are equivalent: the logic flows both ways.

 $\Leftrightarrow x = 4$ 

Again 2x = 8 and x = 4 are equivalent.

**b** x = 4

 $\Rightarrow x^2 = 16$ 

x = 4 implies that  $x^2 = 16$  but the reverse is not true as  $x^2 = 16$  implies  $x = \pm 4$  (not just x = 4).

When solving equations, it is important to know whether or not each line of working is equivalent to the previous one. If it is, then you can be sure that you have found the complete set of solutions. For example:

$$x^2 - x = 2$$

$$\Leftrightarrow x^2 - x - 2 = 0$$

$$\Leftrightarrow$$
  $(x-2)(x+1)=0$ 

$$\Leftrightarrow x = 2 \text{ or } -1$$



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From this chain of equivalences you can be certain that both 2 and -1 are solutions of the equation, and that there are no others.

However, if some of the lines are only connected by implications, it is possible to find 'solutions' which don't actually work, or to miss some of



# Gateway to A Level

See Gateway to A Level sections B and C for a reminder of solving quadratic equations by factorising.

#### **WORKED EXAMPLE 1.3**

A student is attempting to solve the equation  $\sqrt{x+6} = x$ .

**a** Find the error with the following:

$$\sqrt{x+6} = x$$

$$x + 6 = x^2$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2)=0$$

$$x = 3 \text{ or } x = -2$$

**b** Solve the equation correctly.

$$\mathbf{a} \quad \sqrt{x+6} = x$$

$$\Rightarrow x + 6 = x^2$$

Look at each line in turn to see whether  $a \Leftrightarrow symbol$  is

They are not equivalent since:

$$x + 6 = x^2$$

$$\Rightarrow \pm \sqrt{x+6} = x$$

The first line implies the second, but the second does not

This leads to one incorrect solution, coming from  $-\sqrt{x+6} = x$ . All subsequent lines are equivalent, so one of the solutions is correct.

imply the first, so they are not equivalent.

**b** Check both "solutions":

$$x=3$$
:  $\sqrt{(3+6)}=3$ , so this is a solution.

$$x=-2$$
:  $\sqrt{(-2+6)}=2 \neq -2$ , so this is not a solution.

$$\therefore$$
 The correct solution is  $x = 3$ .

You need to check both solutions: substitute the *x* values into the LHS and the RHS and check whether they are equal.

# ) Tip

Squaring an equation is a common way of introducing incorrect solutions since it prevents lines of working being equivalent.

In practice it is often easier not to worry about whether every line is equivalent, but to be aware that the 'solutions' you get need to be checked by substituting them back into the original equation. Any that are not correct can then just be deleted.

Dividing by zero can remove solutions in the same way that squaring can introduce them.

# Tip

LHS and RHS are standard abbreviations for the left-hand side and right-hand side of an equation.

# **▶**|) Fast forward

You will also see this problem arise when you solve equations involving logarithms in Chapter 7.

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Insert an appropriate  $\Rightarrow$ ,  $\Leftarrow$  or a  $\Leftrightarrow$  symbol in the space marked  $\square$ . Hence explain why the solution is incomplete.

 $x^2 = 6x$ 

Dividing by *x*:

x=6

is a ←

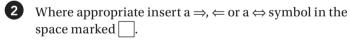
If x = 6 then  $x^2 = 6x$ , but the reverse is not always true...

So x = 6 may not be the only solution – in this case there is also the possibility that x = 0.

#### **EXERCISE 1A**

1 If  $f(x) = x^2 + 1$ , find and simplify where possible:

- a i f(2)
- **ii** f(3)
- **b i** 2f(1) + f(4)
- **ii** f(5) f(3)
- $\mathbf{c}$  i  $\mathbf{f}(x)$
- ii f(y)
- d i f(2x)
- ii f(3y)
- **e i** f(x+1)
- ii f(x-2)



- **a** i Shape P is a rectangle. Shape P is a square.
  - ii Shape Q is a quadrilateral.  $\square$  Shape Q is a rhombus.
- **b** i n is even. n is a whole number.
  - ii n is a prime number. n is a whole number.
- c i A triangle has two equal sides. A triangle has two equal angles.
  - ii Two circles have the same area \_\_\_ Two circles have the same radius.
- **d** i  $x^2 2x 3 = 0$  x = 3
  - ii  $x^2 2x + 1 = 0$  x = 1
- e i Sam can do 10 press-ups. Sam can do 100 press-ups.
  - ii Niamh is over 21. Niamh is over 18.
- **f** i Neither *A* nor *B* is true.  $\square$  *A* is false and *B* is false.
  - ii A and B are not both true.  $\square A$  and B are both not true.

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Fast forward
I ast for war

A You will learn more about functions in Student Book 2, Chapter 5.



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- **g** i Chris is a boy. Chris is a footballer.
  - ii Shape X is a right-angled triangle.  $\square$  Shape X is an isosceles triangle.
- 3  $x^3 4x^2 3x + 18 = (x + a)(x b)^2$  for all x. Find the values of a and b.
- 4  $x^4 + 8x^3 + 2x + 16 = (x^3 + a)(x + b)$  for all x. Find the values of a and b.
- What is the flaw in the following working? Question: For x = 4, find the value of 2x + 2Working:  $2 \times 4 = 8 + 2 = 10$
- Where is the flaw in the following argument?
  Suppose 1 = 3

Subtract 2: -1 = 1Square: 1 = 1

Therefore the first line is true.

- 7 Consider the equation  $\sqrt{x^2+9} = 3x-7$ .
  - a Add appropriate symbols ( $\Leftrightarrow$  or  $\Rightarrow$ ) to each line of working in the solution shown.

$$\sqrt{x^2 + 9} = 3x - 7$$

- Square:  $x^2 + 9 = 9x^2 42x + 49$
- Subtract  $x^2 + 9$ :  $0 = 8x^2 42x + 40$
- Divide by 2:  $0 = 4x^2 21x + 20$
- Factorise: 0 = (4x 5)(x 4) $x = \frac{5}{4} \text{ or } x = 4$
- **b** Hence explain the flaw in the solution shown.
- 8 a Insert appropriate symbols  $(\Rightarrow, \Leftarrow \text{ or } \Leftrightarrow)$  in the spaces marked  $\square$ .  $\frac{1}{x^2} = \frac{2}{x}$   $\square x = 2x^2$   $\square 0 = 2x^2 x$   $\square 0 = x(2x 1)$   $\square x = 0 \text{ or } x = \frac{1}{2}$ 
  - **b** Hence explain the error in the working.
- 9 a Insert appropriate symbols ( $\Rightarrow$ ,  $\Leftarrow$  or  $\Leftarrow$ ) in the spaces marked  $\square$ .  $x^2 + 3x = 4x + 12$   $\square x(x+3) = 4(x+3)$   $\square x = 4$ 
  - **b** Hence explain the error in the working.

# i Did you know?

In all of these examples, we are assuming that each statement is either true or false. The study of this type of logic is called Boolean algebra.



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Cambridge University Press 978-1-316-64428-7 — A Level Mathematics for OCR Student Book 1 (AS/Year 1) Edited by Vesna Kadelburg , Ben Woolley Excerpt

1 Proof and mathematical communication

Do you agree with the following statement?
Either A or B is true ⇔ A and B are not both true.

Where is the flaw in the following argument? Suppose two numbers *a* and *b* are equal.

$$a = b$$

Multiply by a:  $a^2 = ab$ 

Subtract  $b^2$ :  $a^2 - b^2 = ab - b^2$ 

Factorise: (a-b)(a+b) = b(a-b)

Cancel (a-b): (a+b)=b

Use the fact that a = b: 2b = b

Divide by b: 2 = 1

# i Did you know?

The word 'or' can be ambiguous. In formal logical we use the terms OR and XOR to have two different meanings.

## Section 2: Inequality notation

You know from your previous study that solving a linear inequality is just like solving an equation, as long as you don't multiply or divide by a negative number. Your answer is written as an inequality.

For example:

$$2x-5 \geqslant 9$$

 $\Leftrightarrow 2x \geqslant 14$ 

 $\Leftrightarrow x \geqslant 7$ 

This solution can be written in **set notation**:  $\{x : x \ge 7\}$ .

This is read as 'x such that x is greater than or equal to 7'.

It can also be written in **interval notation**:  $[7, \infty)$ . This means that the solution lies in the interval from 7 (included) to infinity (not included).

It is assumed that x is a real number ( $x \in \mathbb{R}$ ) unless stated otherwise. So the interval  $[7, \infty)$  includes all the real numbers greater than or equal to 7, and the interval  $(-\infty, 7)$  all the real numbers smaller than 7. Note that infinity  $(\infty)$  is not a real number, so can never be included in the interval.

# Gateway to A Level

For a reminder and more practice of solving linear inequalities, see Gateway to A Level section D.



The ∈ symbol in set notation means 'is in the set...' or 'belongs to the set...'

# P Key point 1.2

- $x \in (a, b)$  means a < x < b
- $x \in [a, b]$  means  $a \le x \le b$
- $x \in [a, b)$  means  $a \le x < b$
- $x \in (a, b]$  means  $a < x \le b$



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Two different intervals can be combined using notation taken from set theory:

# $^{\prime})$ Key point 1.3

 $x \in A \cup B$  means that x can be in either A or B (or both).  $A \cup B$  is called the **union** of *A* and *B*.

 $x \in A \cap B$  means that x is in both A and B.  $A \cap B$  is called the **intersection** of A and B.

If there are no solutions to the inequality we can write  $x \in \emptyset$ , where  $\emptyset$  is the symbol for the empty set.

#### **WORKED EXAMPLE 1.5**

Write the following in set notation:

a 
$$1 \leqslant x < 7$$

**b** 
$$x > 1$$
 or  $x < -2$ 

c 
$$x < 2$$
 and  $x > 4$ 

**b** 
$$\{x:x>1\}\cup\{x:x<-2\}$$
 wis greater than 1 or less than -2 so you need the union.

No values of 
$$x$$
 are both smaller than 2 and greater than 4.

#### **EXERCISE 1B**

1 Write the following inequalities in set notation and interval notation.

**a i** 
$$x > 7$$

ii 
$$x < 6$$

**b i** 
$$x \le 10$$

ii 
$$x \geqslant 5$$

**c i** 
$$0 < x \le 1$$

ii 
$$5 < x < 7$$

d i 
$$x > 5$$
 or  $x \le 0$ 

ii 
$$x \geqslant 10$$
 or  $x < 2$ 

2 Write the following statements as inequalities in *x*.

**a** i 
$$x \in [1, 4)$$

ii 
$$x \in (2, 8]$$

**b** i 
$$x \in [1, 3]$$

ii 
$$x \in (2, 4)$$

c i 
$$x \in (-\infty, 5)$$

ii 
$$x \in [12, \infty)$$

**d** i 
$$\{x: 0 < x < 10\} \cap \{x: x \ge 8\}$$
 ii  $\{x: 1 < x < 4\} \cap \{x: x \ge 3\}$ 

ii 
$$\{x: 1 < x < 4\} \cap \{x: x \ge 3\}$$

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#### Proof and mathematical communication

- Represent the following intervals on a number line.
  - a i (1, 5]

- ii [-2, 1]
- **b** i  $(-\infty, 4) \cap (-2, \infty)$
- ii  $(-\infty, 0) \cap [-5, \infty)$
- **c** i  $(-\infty, 3] \cap (0, 3)$
- ii  $[-2, 5] \cap (2, 5)$
- **d** i  $[2,5) \cup (7,\infty)$
- ii  $(-\infty, -3] \cup (0, 5]$
- Write the following statements using intervals combined using set notation. Rewrite each as a single interval if possible.
  - **a** i  $1 \le x \le 3$  and x > 2
- **b** ii  $x \le -1$  and 1 < x < 3
- c i  $x \le -2$  or  $x \ge 2$
- **d** ii  $1 \le x < 5$  or  $x \ge 7$
- Solve the following inequalities and express your solution using interval notation.
  - **a i**  $\frac{x}{2} + 1 \leqslant \frac{3x}{5}$
- ii  $2x-1 > \frac{4x}{3} + 3$
- **b i**  $-5 \le \frac{x}{3} 1 < 3$
- **ii**  $4 < \frac{3x}{2} 2 \leqslant 7$
- **c** i  $\frac{3x-2}{5} > \frac{x+1}{2}$
- ii  $\frac{x-3}{7} \ge \frac{2x+1}{3}$

# Section 3: Disproof by counter example

It is usually not possible to prove that something is always true by looking at examples. However, it is possible to use examples to prove that something is **not** always true. This is called a **counter example**.

#### **WORKED EXAMPLE 1.6**

Disprove by counter example that  $(x + 1)^2 = x^2 + 1$  for all x.

When x = 2:

LHS:  $(2+1)^2 = 9$ 

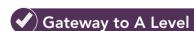
RHS:  $2^2 + 1 = 5$ 

So x = 2 is a counter example.

When searching for a counter example, try different types of numbers.

#### **EXERCISE 1C**

- Disprove the statement  $\sqrt{x^2 + 9} \equiv x + 3$ .
- Use a counter example to prove that  $2x \not\equiv 2 \sin x$ .
- Use a counter example to prove that  $\sqrt{x^2}$  is not always x.
- Prove that the product of two prime numbers is not always odd.



For a reminder of rational and irrational numbers, see Gateway to A Level section E.



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- 5 Prove that the number of factors of a number is not always even.
- 6 Prove that the sum of two irrational numbers is not always irrational.
- 7 Use a counter example to disprove the following statement:

$$x < 3 \Rightarrow x^2 < 9$$

- 8 A student claims that  $n^2 + n + 41$  takes prime values for all positive integers. Use a counter example to disprove this claim.
- Oo two lines that never meet have to be parallel?

# Section 4: Proof by deduction

Proving a result is usually much harder than disproving it. You need to start with what is given in the question and form a series of logical steps to reach the required conclusion.

Algebra is a useful tool that allows you to express ideas in general terms. You will often need to use algebraic expressions for even and odd numbers. For example, it is common to express:

- an even number as 2n, for some integer n
- an odd number as 2n + 1, for some integer n.

# Focus on ...

The Focus on ... sections in this book show you proofs of some important results you will meet in this course.

#### **WORKED EXAMPLE 1.7**

Prove that the product of an even and an odd number is always even.

Let the even number be 2n, for some integer n. Define

Define a general even number.

Let the odd number be 2m + 1, for some integer m.

And define a general odd number.

Note that you mustn't use 2n + 1 as that would be the next integer up from 2n, which would be a specific odd number, not a general one.

2n(2m+1) = 2(2mn+n)= 2k for some integer k.

Aim to write the product in the form 2k to show that it is even.

So this is even.

Make a conclusion.



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#### 1 Proof and mathematical communication

#### **WORKED EXAMPLE 1.8**

Prove that the difference between the squares of consecutive odd numbers is always a multiple of 8.

Let the smaller odd number be 2n-1

Let the larger odd number be 2n+1

Define two consecutive odd numbers. This time you **do** want *n* in both, as the two numbers are related.

$$(2n+1)^{2} - (2n-1)^{2} = (4n^{2} + 4n + 1) - (4n^{2} - 4n + 1)$$

$$= 4n + 4n$$

$$= 8n$$

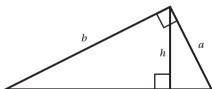
Square each, and subtract the smaller from the larger.

So this is a multiple of 8.

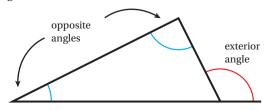
Make a conclusion.

#### **EXERCISE 1D**

- 1 Prove that if n is odd then  $n^2$  is also odd.
- 2 Prove that the sum of an even number and an odd number is odd.
- 3 Prove that the sum of any three consecutive integers is always a multiple of three.
- 4 Prove that:
  - a the sum of two consecutive multiples of 5 is always odd
  - **b** the product of two consecutive multiples of 5 is always even.
- Prove that the height, h, in the following diagram is given by  $h = \frac{ab}{\sqrt{a^2 + b^2}}$ .



- 6 Prove that the sum of the interior angles of a hexagon is 720°.
- 7 Prove that if a number leaves a remainder 2 when it is divided by 3 then its square leaves a remainder 1 when divided by 3.
- 8 **a** Expand  $(x+2)^2$ .
  - **b** Prove the statement:  $y = x^2 + 4x + 10 \Rightarrow y > 0$ .
- Prove that an exterior angle in a triangle is the sum of the two opposite angles.





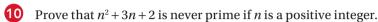
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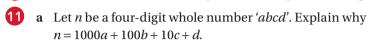
For a further example on algebraic proof and more practice questions, see Support sheet 1.



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- **b** Prove that n is divisible by 9 if and only if a + b + c + d is a multiple of 9.
- **c** Prove that *n* is divisible by 11 if and only if a b + c d is divisible by 11.
- By considering  $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ , prove that an irrational number raised to an irrational power can be rational.

## Section 5: Proof by exhaustion

You should be aware that simply considering some examples does not constitute a mathematical proof. However, in some situations it is possible to check **all** possibilities and this can lead to a valid proof. This is called **proof by exhaustion**.

#### **WORKED EXAMPLE 1.9**

Prove that 89 is a prime number.

We only need to check prime numbers smaller than the square root of 89, since any factor above this would have to be paired with a factor below the square root.

A prime number does not have any prime factors smaller than itself.

89 is not divisible by 2, 3, 5 or 7. Therefore it must be a prime number.

#### **WORKED EXAMPLE 1.10**

A whole number is squared and divided by 3. Prove that the remainder can only be 0 or 1.

You cannot check all whole numbers, but you can split them into three groups when considering division by 3: those that give no remainder, those that give remainder 1 and those that give remainder 2. You can then check what squaring does to numbers from each group.

Let n be a whole number.

Now use algebra to write each type of number.

Then n is either a multiple of 3 (n=3k)

or one more than a multiple of 3 (n=3k+1)

or two more than a multiple of 3 (n=3k+2)

Continues on next page ...



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If n = 3k then  $n^2 = (3k)^2 = 9k^2$ 

Now check what happens when you square each type of number.

which is a multiple of 3.

If n = 3k + 1 then

$$n^2 = (3k+1)^2 = 9k^2 + 6k + 1$$

$$=3(3k^2+2k)+1$$

which is one more than a multiple of 3.

If n = 3k + 2 then

$$n^2 = (3k+2)^2 = 9k^2 + 12k + 4$$

$$=3(3k^2+4k+1)+1$$

which is one more than a multiple of 3.

So there is either no remainder or the remainder is 1.

You have now checked each possible whole number, so you can write the conclusion.

# $(\mathcal{O})$ Explore

Proof by exhaustion can only be used when there is a relatively small number of possibilities to check. The use of computers has made it possible to apply this method to a wider variety of problems (although some mathematicians question whether we can always trust a computer check). Find out about the Four Colour Theorem – one of the most famous problems that has been solved in this way.

#### **EXERCISE 1E**

- 1 Prove that 11 is a prime number.
- 2 Prove that 83 is a prime number.
- 3 Prove that all regular polygons with fewer than 7 sides have angles with a whole number of degrees.
- 4 Prove that no square number less than 100 ends in a 7.
- Let f(x) be the function that gives the number of factors of x. For example, f(10) = 4 because it has factors 1, 2, 5 and 10.

Prove that for any single digit positive number  $f(n) \le n$ .

Prove that  $n^2 + 2$  is not divisible by 4 for integers between 1 and 5 inclusive.

# ✓ Gateway to A Level

See Gateway to A Level section F for a reminder of function notation.

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#### A Level Mathematics for OCR A Student Book 1

- 7 Prove that  $n^2 + n$  is always even if  $n \in \mathbb{Z}$ .
- 8 Prove that when the square of a whole number is divided by 5, the remainder is either 0, 1 or 4.
- Prove that  $2x^3 + 3x^2 + x$  is always divisible by 6 if x is an integer.
- The modulus function, |x|, is defined as x if x is positive and -x if x is negative so, for example, |-2| = 2 and |5| = 5. |0| is defined to be 0. Prove the triangle inequality:  $|a + b| \le |a| + |b|$ .

## Fast forward

You will learn more about the modulus function in Student Book 2.

## lacksquare Checklist of learning and understanding

- You can express mathematical ideas using descriptions such as diagrams, equations and identities.
- You can communicate a mathematical argument using a series of equations or identities put together in a logical order.

These can be connected using implication symbols:  $\Rightarrow$  or  $\Leftrightarrow$ .

- The symbol ⇒ means that a subsequent statement follows from the previous one.
- The symbol ⇔ means that a subsequent statement is equivalent to the previous one.
- An identity is a relation that is true for **all** values of the unknown. It is given the  $\equiv$  symbol.
- You can represent solutions of inequalities using set notation or interval notation.
  - In interval notation, the square bracket [ or ] means that the endpoint is included, and the round bracket ( or ) means that the endpoint is **not** included.
- One counter example is sufficient to prove that a statement is not always true.
- An algebraic proof is often required to show that a statement is always true.
- Proof by exhaustion involves checking all possibilities. This can only be done if there is a small number of options, or the options can be split up into a small number of cases.



**More Information** 

Cambridge University Press 978-1-316-64428-7 — A Level Mathematics for OCR Student Book 1 (AS/Year 1) Edited by Vesna Kadelburg , Ben Woolley Excerpt

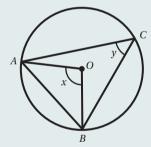
1 Proof and mathematical communication

## Mixed practice 1

- 1 Prove that the product of any two odd numbers is always odd.
- Prove that if n is even then  $n^2$  is divisible by 4.
- 3 Prove that if  $\frac{a}{b} = \frac{c}{d}$  it does not follow that a = c and b = d.
- Prove the following statement or disprove it with a counter example:

'The sum of two numbers is always larger than their difference.'

- 5 Prove that the product of two rational numbers is always rational.
- 6 Prove that the sum of the interior angles in an n-sided shape is  $(180n 360)^{\circ}$ .
- Given that  $x^3 + y^3 \equiv (x + y)(ax^2 + bxy + cy^2)$  find the values of a, b and c.
- 8 Prove the following statement:  $n \text{ is odd} \Rightarrow n^2 + 4n + 3 \text{ is a multiple of 4}$
- Prove that the angle from a chord to the centre of a circle is twice the angle to a point on the circumference in the major sector.



- Prove that all cube numbers are either multiples of 9 or within one of a multiple of 9.
- Prove the following statements, or disprove them with a counter example:
  - **a** ab is an integer  $\Leftrightarrow a$  is an integer and b is an integer
  - **b** *a* is irrational and *b* is irrational  $\Leftrightarrow$  *ab* is irrational.
- Prove that the product of any three consecutive positive integers is a multiple of 6.
- Prove that the difference between the squares of any two odd numbers is a multiple of 8.
- **14** a Prove that  $n^2 79n + 1601$  is not always prime when n is a positive whole number.
  - **b** Prove that  $n^2 1$  is never prime when n is a whole number greater than 2.
- 15  $x = a^2 b^2$  where a and b are both whole numbers. Prove that x is either odd or a multiple of 4.



For questions on another principle used in proof, see Extension sheet 1.