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Cambridge University Press
978-1-316-64427-0 — A Level Further Mathematics for OCR A Mechanics Student Book (AS/A Level) with Cambridge Elevate Edition (2 Years)
Jess Barker, Nathan Barker, Michele Conway, Janet Such
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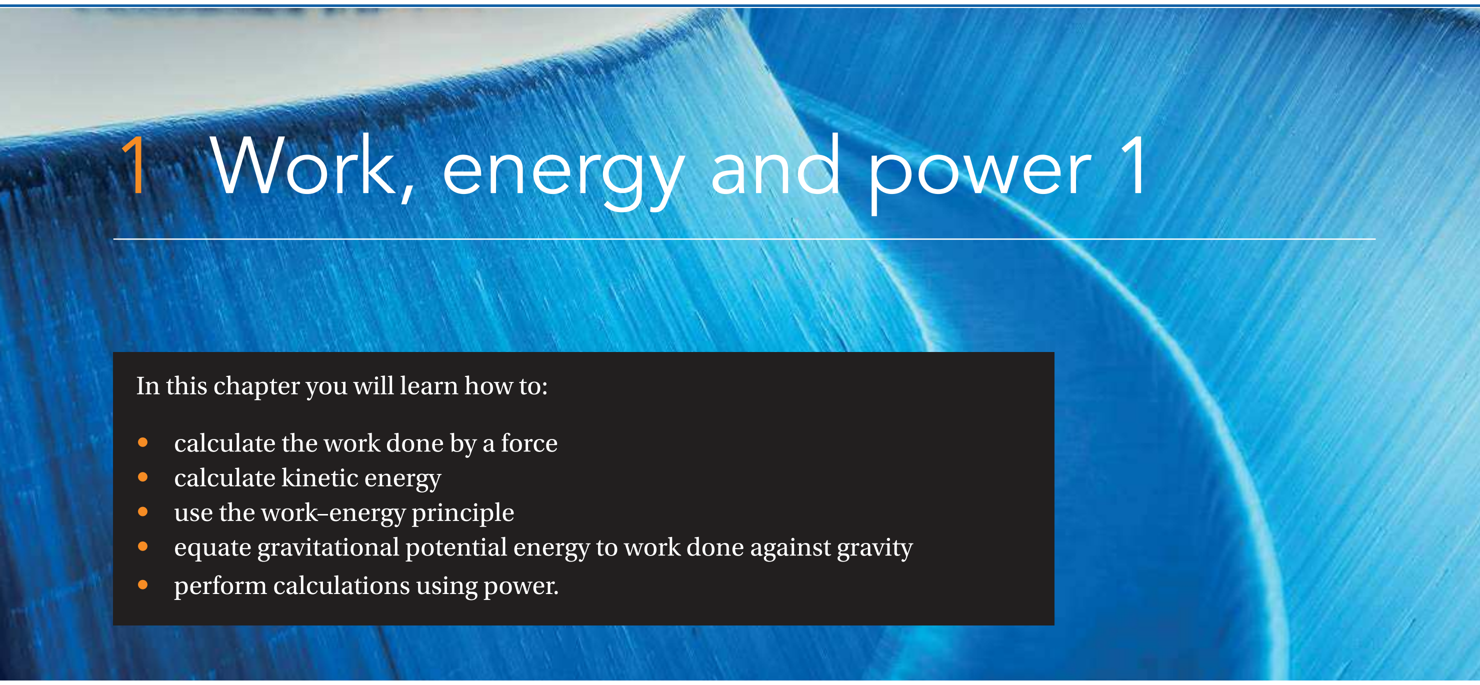

1 Work, energy and power 1

In this chapter you will learn how to:

- calculate the work done by a force
- calculate kinetic energy
- use the work–energy principle
- equate gravitational potential energy to work done against gravity
- perform calculations using power.

Before you start…

GCSE	You should know how to convert units of distance, speed and time.	1 Convert 15 000 metres to kilometres.
A Level Mathematics Student Book 1	You should know how to calculate the weight of an object from its mass, and know the unit of weight.	2 Calculate the weight of a car of mass 1150 kg, stating the unit with your answer.
A Level Mathematics Student Book 1	You should be able to use Newton's second law of motion: $F = ma$	3 A resultant force of 50 N acts on an object of mass 2.5 kg. Calculate the acceleration of the object.
A Level Mathematics Student Book 2	You should be able to resolve a force into components at right angles to each other.	4 A force of 8 N acts on a particle at an angle of 20° to the positive horizontal direction. What are the horizontal and vertical components of the force?

The relationship between work and energy

You have already studied the effect of a force or system of forces in A Level Mathematics.

In this chapter, you will learn the definition of the work done by a force, which is a quantity that is measured in joules, the same units that are used for **energy**. You will learn about propulsive and resistive forces. You will learn about the relationship between work done and two different types of energy: kinetic energy and gravitational potential energy. You will also learn about power, which is the rate of doing work.

Ideas of work, energy and power are crucial in engineering, enabling engineers to design machines to do useful work. Hydroelectric power stations work by converting the work done by falling water, first into kinetic energy as the hydroelectric turbines rotate and then into electricity.

Fast forward

A In Chapter 6, you will learn about elastic potential energy and its conversion to kinetic energy.

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Section 1: The work done by a force

Work is done by a force when the object it is applied to moves. The amount of **work done** is the product of the force and the distance moved in the direction of the force.

Some forces promote movement, while others resist it. For example, when you cycle into a breeze, your pedalling promotes movement but the breeze acts against your movement. Forces that promote movement are called **propulsive forces** and those that resist movement are known as **resistive forces**.

Other propulsive forces include the **tension** in a rope being used to drag an object across the ground and the driving force of a vehicle engine. The driving force of an engine is often described as its **tractive force**. Other resistive forces include friction, vehicle braking and resistance by moving through still air or a liquid.

Key point 1.1

For a force acting in the direction of motion:

work done = force × distance

Work done is measured in joules (J).

1 joule = 1 newton × 1 metre, i.e. 1 J = 1 N m

For example a force of 5 N acting on an object that moves 15 m in the direction of the force does $5 \times 15 = 75$ J of work. Doubling the force to 10 N over the same distance would double the amount of work done to 150 J. Likewise, doubling the distance moved to 30 m with an unchanged force of 5 N would double the amount of work done to 150 J.

WORKED EXAMPLE 1.1

A box is pushed 5 m across a horizontal floor by a horizontal force of 25 N. Calculate the work done by the force.

Work done = force × distance

$= 25 \times 5$ — Use the definition of work done.

$= 125$ J — State units of work done (J) with your answer.

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WORKED EXAMPLE 1.2

A truck driver driving along a horizontal road applies a braking force of 75 kN for 25 m. Calculate the work done by the brakes, giving your answer in kJ.

75 kN = 75 000 N — Convert 75 kN to 75 000 N as you need to work in standard units.

Work done by brakes — Use the definition of work done.

= braking force × distance

= 75 000 × 25

= 1 875 000 J — Change J to kJ.

= 1880 kJ (3 s.f.)

WORKED EXAMPLE 1.3

A 50 kg crate is lifted 12 m by a rope and pulley system. Calculate the work done against gravity.

Work done = force × distance

∴ work done against gravity — Apply the definition of work done to the gravitational force. The force needed to lift the crate is equal to the crate's weight and the distance moved is height gained.

= weight × height gained

Weight of crate = 50 × 9.8

= 490 N — Calculate the weight of the crate, based on the usual approximation for the acceleration due to gravity of 9.8 m s^{-2}.

Work done against gravity

= weight × height gained

= 490 × 12 — Use the definition of work done.

≈ 5880 J

Key point 1.2

When a mass, m, is raised or lowered through a height h:

work done against or by gravity = weight × height = $mg \times h$

Fast forward

In Section 3 you will learn the equivalence of work done against gravity and gravitational potential energy.

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WORKED EXAMPLE 1.4

A competitor of mass 75 kg dives from a 10-metre-high diving board into a pool. Air resistance averages 12 N as he descends 10 m through the air. Resistance from the water then averages 3000 N as he descends 2 m further. Calculate:

a the total work done by gravity as the diver descends 12 metres

b the total work done against air and water resistance during this descent.

a Work done by gravity $= 75g \times 12$

$= 8820$ J

Use *mgh* to calculate the work done by gravity.

b Work done against air resistance $= 12 \times 10$

$= 120$ J

Use force × distance to calculate the work done against each of the resistances.

Work done against water resistance $= 3000 \times 2$

$= 6000$ J

Total work done against resistances $= 6120$ J

WORKED EXAMPLE 1.5

A van of mass 1250 kg travels along a straight road. The driving force of the vehicle engine is 500 N and resistance to motion is 220 N, on average. The van travels 1.5 km from one delivery to the next, descending 8 m in height. Find:

a the work done by the vehicle engine

b the work done by gravity

c the work done against resistance.

a 1.5 km = 1500 m

Convert distance to metres.

Work done by vehicle engine $= 500 \times 1500 = 750\,000$ J

Use force × distance to calculate the work done by the vehicle engine.

b Work done by gravity $= 1250g \times 8$

$= 98\,000$ J

Use *mgh* to calculate the work done by gravity.

c Work done against resistance $= 220 \times 1500$

$= 330\,000$ J

Use force × distance to calculate the work done against resistance.

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EXERCISE 1A

1. A parcel is dragged 5 metres across a horizontal floor by a horizontal rope. The tension in the rope is 12 N. Calculate the work done by the tension in the rope.
2. Susan climbs a vertical rock 32 m high. Susan's mass is 65 kg. Calculate the work done by Susan against gravity.
3. Sunil descends a vertical ladder. His mass is 82 kg and the work done by gravity is 2150 J. Find the height Sunil descends.
4. A ball of mass 100 g is dropped from a window. Calculate the work done by gravity as the ball falls vertically to the ground 6 m below.
5. A puck slides 50 metres across an ice rink, against a resistive force of 2.5 N. Calculate the work done against resistance.
6. A cyclist travelling on horizontal ground applies a driving force of 25 N against a headwind of 10 N and a resistance from friction of 5 N. The cyclist travels 1.2 km. Find:

   a the work done by the cyclist **b** the total work done against wind and friction.
7. A fish basket is raised from the sea floor to a fishing boat at sea level, 18 metres above. The mass of the basket is 15 kg. The resistance to motion from the seawater is 50 N. Calculate the total work done, against gravity and water resistance, in raising the fish basket.
8. A driving force of 400 N does 50 kJ of work moving a van along a horizontal road from A to B. Resistance to motion averages 185 N. Calculate the work done against resistance as the van moves from A to B.

Section 2: Kinetic energy and the work–energy principle

Kinetic energy is the energy an object has because it is moving.

Key point 1.3

An object of mass m moving with speed v has kinetic energy $\frac{1}{2}mv^2$.

If mass is measured in kg and speed is measured in m s^{-1}, kinetic energy is measured in joules.

Tip

If speed is not given in m s^{-1}, you should convert to m s^{-1} before you start the rest of your calculations.

WORKED EXAMPLE 1.6

A particle of mass 1.5 kg is moving with kinetic energy 48 joules. Calculate the speed of the particle.

$$\text{Kinetic energy} = \frac{1}{2}mv^2$$

Use the formula for kinetic energy.

$$48 = \frac{1}{2} \times 1.5 \times v^2$$

$$\therefore v^2 = 64 \quad \text{so} \quad v = 8 \text{ m s}^{-1}$$

Substitute and rearrange to find speed.
As mass was given in kg and kinetic energy in joules, speed is in m s^{-1}. Speed is a positive scalar, so the negative option of the root can be disregarded.

WORKED EXAMPLE 1.7

A cyclist slows down from 25 km h^{-1} to 10 km h^{-1}. The combined mass of the cyclist and her bicycle is 95 kg. Calculate the loss of kinetic energy.

Let u be the starting speed and v be the final speed:

$u = \frac{25}{3.6} = 6.944$ m s^{-1} and

$v = \frac{10}{3.6} = 2.777$ m s^{-1}

> To convert km h^{-1} to m s^{-1} you must multiply by the conversion factor $\frac{1000}{3600}$, which simplifies to division by 3.6.

Loss of kinetic energy $= \frac{1}{2}mu^2 - \frac{1}{2}mv^2$

> Loss of kinetic energy = initial kinetic energy – final kinetic energy

Loss of kinetic energy $= \frac{1}{2} \times 95 \times 6.944^{-2}$

$- \frac{1}{2} \times 95 \times 2.78^2 = 1920$ J (3 s.f.)

WORKED EXAMPLE 1.8

Calculate the increase in kinetic energy when a boat of mass 2 tonnes changes velocity from $3\mathbf{i} + 4\mathbf{j}$ m s^{-1} to $4.5\mathbf{i} + 4.5\mathbf{j}$ m s^{-1}. Give your answer in kJ.

(Starting speed)$^2 = 3^2 + 4^2 = 25$

(Final speed)$^2 = 4.5^2 + 4.5^2 = 40.5$

> Use Pythagoras' theorem to convert the velocity vectors to speeds. You need the square of the *speed*, not the velocity vector, for the kinetic energy formula.

Gain in kinetic energy $= \frac{1}{2}m(v^2 - u^2)$

> You can write $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$ in factorised form.

2 tonnes = 2000 kg

> Convert 2 tonnes to 2000 kg.

Gain in kinetic energy $= \frac{1}{2} \times 2000 \times (40.5 - 25)$

$= 15.5$ kJ

> Divide by 1000 to convert joules to kJ.

The **work–energy principle** is an essential idea in Mechanics that enables us to calculate the work necessary to cause a change in kinetic energy.

Key point 1.4

The net work done by all the forces acting on a particle, including its own weight, is equal to the change in kinetic energy of the particle.

work done $= \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

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WORKED EXAMPLE 1.9

A particle of mass 1.6 kg at rest on a smooth horizontal plane is acted on by a constant horizontal force of 8 N. Find the speed of the particle after it has travelled 5 metres.

Work done $= 8 \times 5 = 40$ J — Work done = force × distance

$40 = \frac{1}{2} \times 1.6 \times v^2$

$v^2 = 50$ — Work–energy principle: since the particle is starting from rest, work done $= \frac{1}{2}mv^2$

$\Rightarrow v = 7.07$ m s^{-1} (3 s.f.)

WORKED EXAMPLE 1.10

Stephen is driving his car along a horizontal road at 55 km h^{-1} when he notices a broken-down vehicle, just off the road, 150 m ahead. Stephen and his car have a mass of 1025 kg and the total resistance to motion is assumed constant at 500 N. Stephen believes he should slow down and that he can slow down sufficiently without applying the brakes. Calculate Stephen's speed, in km h^{-1}, as he reaches the broken-down vehicle, taking account of the resistance to motion.

Assume that Stephen allows the resistance to motion to slow his car down over 150 m. There is no driving or braking force.

Work done against resistance
= resistive force × distance
$= 500 \times 150$
$= 75\,000$ J — Calculate the work done against resistance.

$u = 55$ km h^{-1} $= \frac{55}{3.6} = 15.28$ m s^{-1} — Convert the initial speed u of 55 km h^{-1} to m s^{-1}.

Loss of kinetic energy $= \frac{1}{2} \times m \times (u^2 - v^2)$ — Write down the expression for loss of kinetic energy.

$75\,000 = \frac{1}{2} \times 1025 \times (u^2 - v^2)$ — Work–energy principle: work done against resistance = loss of KE

$u^2 - v^2 = \frac{2 \times 75\,000}{1025} \approx 146.34$

$v^2 = 15.28^2 - 146.3$ — Substitute for u. Rearrange and solve for v.

$v = 9.33$ m s^{-1}

$v = 9.33 \times 3.6$ km h^{-1} — Convert back to km h^{-1}.

$v = 33.6$ km h^{-1} (3 s.f.)

EXERCISE 1B

1 Calculate the kinetic energy of a cyclist and her bicycle having a combined mass of 70 kg, travelling at 12 m s^{-1}. Give your answer in kJ.

2 Calculate the mass of an athlete who is running at 8.5 m s^{-1}, with kinetic energy 3500 J.

3 Calculate the speed of a bus of mass 20 tonnes with kinetic energy 1100 kJ. Give your answer in km h^{-1}.

4 A box of mass 5 kg is pulled from A to B across a smooth horizontal floor by a horizontal force of magnitude 10 N. At point A, the box has speed 1.5 m s^{-1} and at point B the box has speed 2.8 m s^{-1}. Ignoring all other resistive forces, find:

a the increase in kinetic energy of the box

b the work done by the force

c the distance AB.

A 5 Calculate the loss of kinetic energy when a boat of mass 3.5 tonnes reduces in velocity from $(3\mathbf{i} + 4\mathbf{j})$ m s^{-1} to $(2.5\mathbf{i} + 3\mathbf{j})$ m s^{-1}.

6 A car driver brakes on a horizontal road and slows down from 20 m s^{-1} to 12 m s^{-1}. The mass of the car and its occupants is 1150 kg.

a Find the loss in kinetic energy.

b Given that the work done against resistance to motion is 50 kJ, find the work done by the brakes.

7 A child of mass 35 kg descends a smooth slide, after propelling herself from the top at 1.6 m s^{-1}. Ignoring air resistance, calculate her speed at the bottom of the slide, which is 2.1 metres lower down than the top.

8 A bullet of mass 10 grams passes horizontally through a target of thickness 5 cm. The speed of the bullet is reduced from 240 m s^{-1} to 90 m s^{-1}. Calculate the magnitude of the average resistive force exerted on the bullet.

9 A train with mass 100 tonnes is travelling at 108 km h^{-1} on horizontal tracks, when the driver sees a speed reduction sign. The train's speed must be reduced to 75 km h^{-1} over 500 m. Resistance to motion is approximately 8 kN. Calculate the braking force required, in kN.

10 A package of mass 500 grams slides down a parcel chute of length 3.5 metres, starting from rest. The bottom of the chute is 2.2 metres below the top. The speed of the package at the bottom of the chute is 4.5 m s^{-1}. Find the resistance to motion on the chute.

11 Use the equation of motion, $F = ma$, together with the formula, $v^2 = u^2 + 2as$, to derive the relation: $Fs = \frac{1}{2}m(v^2 - u^2)$.

12 Eddy cycles up a hill. His mass, together with his bicycle, is 92 kg. His driving force is 125 N and resistance from friction is 45 N. Eddy travels 350 metres along the road, which rises through a vertical height of 32 metres. His starting speed is 8.2 m s^{-1}. Find his final speed.

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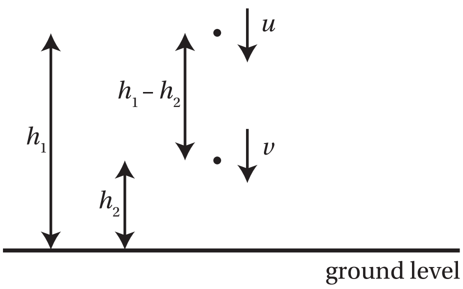
Section 3: Potential energy, mechanical energy and conservation of mechanical energy

Consider an object of mass m falling freely under gravity from height h_1 to height h_2, with starting speed u and final speed v.

Since the only external force acting on the object is gravity, the work–energy principle becomes:



work done by gravity = increase in kinetic energy

$$\Rightarrow mg(h_1 - h_2) = \frac{mv^2}{2} - \frac{mu^2}{2}$$

Rearranging this gives:

$$mgh_1 - mgh_2 = \frac{mv^2}{2} - \frac{mu^2}{2}$$

$$\Rightarrow \quad mgh_1 + \frac{mu^2}{2} = mgh_2 + \frac{mv^2}{2}$$

Each side of this equation is the sum of two terms, one of which is kinetic energy. The other term is gravitational potential energy. **Gravitational potential energy** (GPE) is the energy an object has by virtue of its position. For an object of mass m raised a distance h, the increase in GPE is equal to the product of its weight, mg, and the distance h.

Key point 1.5

Gravitational potential energy (GPE) = mgh

where h is the height above ground (zero) level.

Tip

You can choose any height as your ground (zero) level but it is usually best to choose the lowest height reached by the moving object.

The principle of the conservation of **mechanical energy** states that, if there are no external forces other than gravity doing work on an object during its motion, then the sum of kinetic energy and gravitational potential energy remains constant.

Key point 1.6

If the only force acting on an object is its weight then mechanical energy is conserved:

GPE + KE = $mgh + \frac{1}{2}mv^2$ = constant

where h is the vertical height above the zero level.

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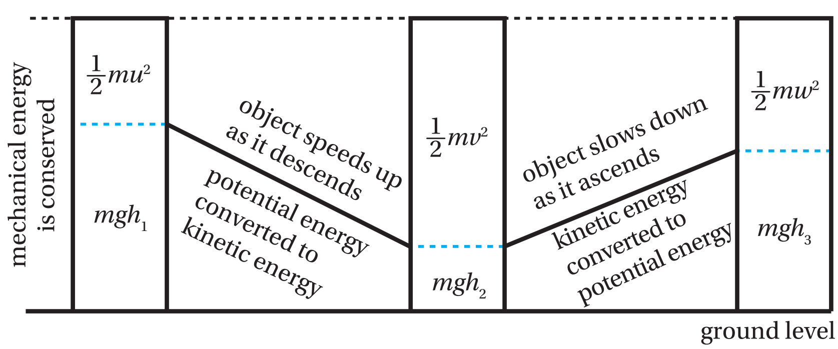
This diagram may help you to understand the formula for conservation of mechanical energy more easily. As an object descends in height it speeds up, so gravitational potential energy is converted into kinetic energy. As an object ascends in height it slows down, so kinetic energy is converted into gravitational potential energy.



WORKED EXAMPLE 1.11

Faisal throws a ball of mass 125 grams vertically upwards from ground level with a speed of 12.5 m s^{-1}. Assuming no external forces apply:

a calculate the speed of the ball after it has risen 5 metres

b calculate the maximum height gained by the ball.

a Starting KE $= \frac{1}{2} \times 0.125 \times 12.5^2$

$= 9.766$ J

Starting KE $= mgh +$ KE at 5 m — Use conservation of mechanical energy over the first 5 m of the ascent.

$9.766 = 0.125g \times 5 +$ KE at 5 m — Take the gravitational potential energy at ground level to be zero.

KE at 5 m $= 3.641$ J — Calculate the kinetic energy of the ball at 5 m.

$\frac{1}{2}mv^2 = 3.641$ — Use the formula for KE with the speed at 5 m equal to v.

$v = \sqrt{\frac{2 \times 3.641}{0.125}}$

$= 7.63$ m s^{-1} (3 s.f.) — Calculate the speed of the ball.

b Starting KE $= mgh_{max}$ — Use conservation of mechanical energy over the whole ascent (final kinetic energy is zero). At the maximum height, all the initial kinetic energy will have been converted into gravitational potential energy.

$9.766 = 0.125g \times h_{max}$

$\Rightarrow h_{max} = 7.97$ metres (3 s.f.) — Calculate the maximum height gained.