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Stephen Ward , Paul Fannon
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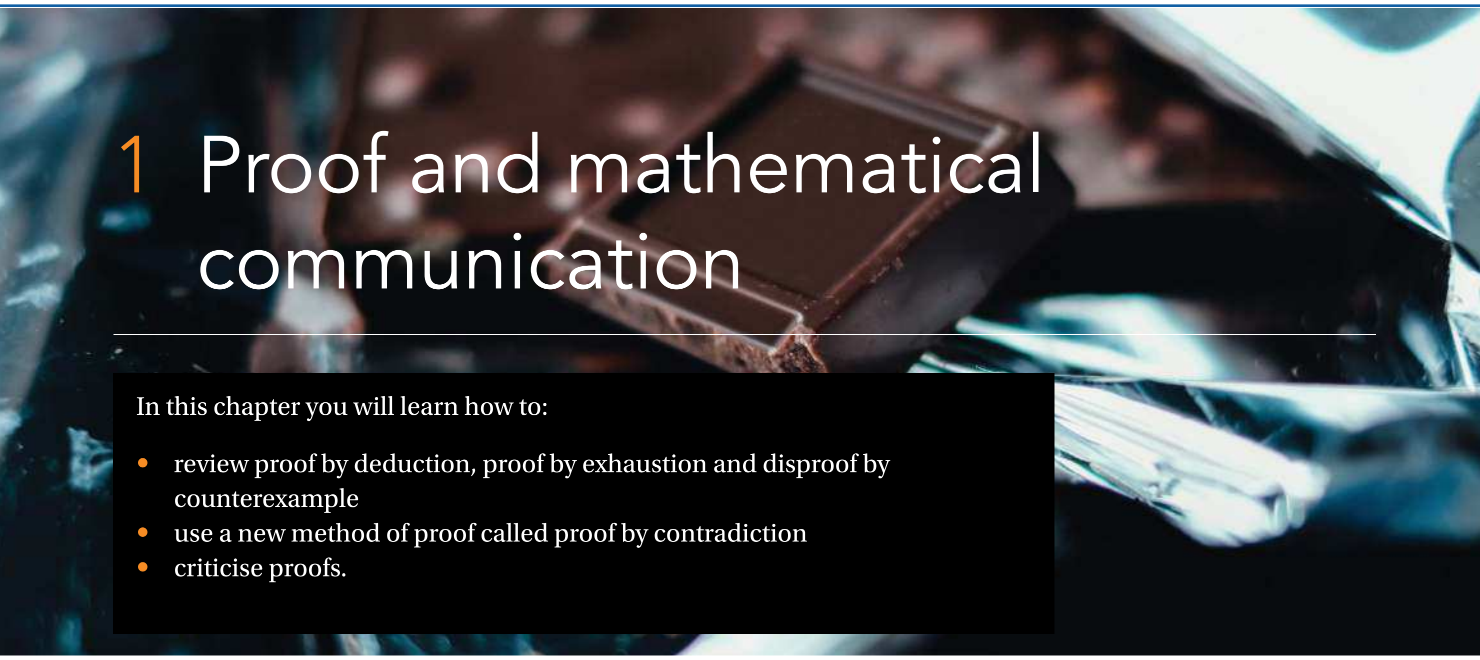

1 Proof and mathematical communication

In this chapter you will learn how to:

- review proof by deduction, proof by exhaustion and disproof by counterexample
- use a new method of proof called proof by contradiction
- criticise proofs.

Before you start…

Student Book 1, Chapter 1	You should be able to use logical connectors.	1 Insert $\Rightarrow$, $\Leftarrow$ or $\Leftrightarrow$ in the places marked a and b: $x^2 - 1 = 8$ a $x^2 = 9$ b $x = 3$
Student Book 1, Chapter 1	You should be able disprove a statement by counterexample.	2 Disprove the statement: 'Apart from 1 there are no other integers that can be written as both n^2 and n^3.'
Student Book 1, Chapter 1	You should be able to prove a statement by deduction.	3 Prove that the sum of any two odd numbers is always even.
Student Book 1, Chapter 1	You should be able to prove a statement by exhaustion.	4 Use proof by exhaustion to prove that 17 is a prime number.

Developing proof

One of the purposes of this chapter is to provide revision of the material from Student Book 1. It draws on all chapters from that book but, in particular, it builds on the fundamental ideas of proof from Chapter 1. This chapter introduces a new and very powerful method of proof that mathematicians often rely on: proof by contradiction.

Section 1: A reminder of methods of proof

In Student Book 1, Chapter 1, you met proof by deduction, proof by exhaustion and disproof by counterexample. The questions in Exercise 1A show how these methods can be used in topics from throughout Student Book 1.

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EXERCISE 1A

1 Use a counterexample to show that this statement is not true:

$\sin 2x = 1 \Rightarrow x = 45°$

2 The velocity of a particle after time t is given by $v = t^2 + 3$. Prove that the particle never returns to its original position.

3 a Prove from first principles that if $y = x^2$ then $\frac{dy}{dx} = 2x$.

b Use a counterexample to show that if $\frac{dy}{dx} = 2x$ then it is not necessarily true that $y = x^2$.

4 Prove that $\binom{n}{1} = n$.

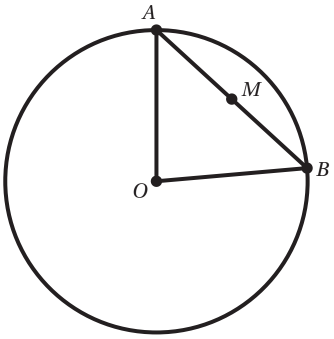
5 Prove by exhaustion that any square number is either a multiple of 4 or one more than a multiple of 4.

6 A set of data has mean A, mode B and median C.

Consider this statement: $B < A \Rightarrow C < A$

Prove this statement or use a counterexample to disprove it.

7 The diagram shows a triangle OAB where A and B lie on the circle with centre O.



M is the midpoint of AB.

θ is the angle AOM.

a Use the cosine rule to prove that $AB = \sqrt{2r^2 - 2r^2 \cos(2\theta)}$.

b Show that $AM = r \sin\theta$.

c Hence prove that $\cos 2\theta = 1 - 2\sin^2\theta$.

8 In quadrilateral $OABC$, O is the origin and **a**, **b**, **c** are the position vectors of points A, B and C.

P is the midpoint of OA, Q is the midpoint of AB, R is the midpoint of BC and S is the midpoint of OC.

a Show that $\overrightarrow{PQ} = \frac{1}{2}\mathbf{b}$.

b Hence prove that $PQRS$ is a parallelogram.

c If $PQRS$ is a rectangle, what can be said about the quadrilateral $OABC$?

9 a Use a counterexample to disprove the statement $\ln(x + y) \equiv \ln x + \ln y$.

b Prove that if $\ln(x + y) \equiv \ln x + \ln y$, then $y = \frac{x}{x-1}$.

c If $y = \frac{x}{x-1}$ does it mean that $\ln(x + y) \equiv \ln x + \ln y$?

10 a Show that $a^3 + 1 \equiv (a+1)(a^2 + ka + 1)$ where k is a constant to be determined.

b Hence prove that $a^3 + 1$ is prime if and only if $a = 1$.

11 Prove algebraically that if $X \sim B(n, p)$ then the sum of the probabilities of the different values X can take is one.

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Section 2: Proof by contradiction

Proof by contradiction starts from the opposite of the statement you are trying to prove, and shows that this results in an impossible conclusion.

WORKED EXAMPLE 1.1

Use proof by contradiction to prove that there are an infinite number of prime numbers.

Assume that there is a largest prime number, P. — Proof by contradiction always starts by assuming the opposite of what you want to prove.

Construct another number N that is the product of all the prime numbers up to and including P. — Now set about trying to find a larger prime than P.

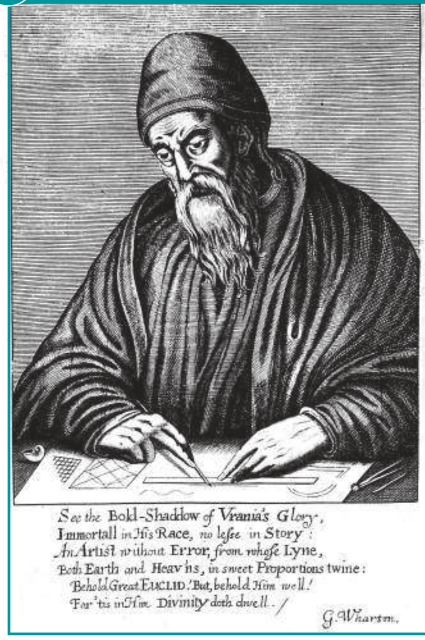
Consider $N+1$. This is one greater than a number divisible by all the primes up to and including P, so it cannot be divisible by any of the primes up to and including P.

Therefore $N+1$ is either itself prime, or is divisible by primes larger than P.

Either way, a prime larger than P has been discovered which contradicts the premise that there is a largest prime number. — Here the contradiction to the original assumption (that there is a largest prime) occurs.

Therefore there are an infinite number of prime numbers.

Did you know?



A variation on the proof in Worked example 1.1 can be found in Euclid's masterpiece *The Elements*, a textbook written in around 300 BCE but still in use in many schools in the first half of the 20th century!

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WORKED EXAMPLE 1.2

Using the fact that if a^2 is even then so is a, prove that $\sqrt{2}$ is irrational.

Assume that $\sqrt{2} = \frac{p}{q}$ where p and q are integers with no common factors.

> Start by assuming the opposite of what you want to prove.
>
> A number is rational if it can be written as a fraction – and this can always be cancelled down until the numerator and denominator share no common factors.

Squaring both sides:

$$2 = \frac{p^2}{q^2}$$

$$\Leftrightarrow p^2 = 2q^2 \qquad (1)$$

This means that p^2 is even so p must also be even.

> Using the fact given.

Therefore $p = 2k$, so $p^2 = 4k^2$.

Substituting into (1):

$$4k^2 = 2q^2$$

$$\Leftrightarrow 2k^2 = q^2$$

This means that q^2 is even, so q must be even.

> Using the given fact again.

This has shown that both p and q are even, so they share a factor of 2. This contradicts the original assertion, so it must be incorrect.

> Here the contradiction (to the fact that p and q share no common factors) arises.

Therefore $\sqrt{2}$ cannot be written as $\frac{p}{q}$.

EXERCISE 1B

1. Prove that if n^2 is even then n is also even.
2. Prove that $\sqrt{3}$ is irrational.
3. Prove that there is an infinite number of even numbers.
4. Prove that the sum of a rational and irrational number is irrational.
5. Prove that if ab is even, with a, b integers, then at least one of them is even.
6. Prove that $\sqrt[3]{2}$ is irrational.
7. Prove that $\log_2 3$ is irrational.

Elevate

See Support Sheet 1 for an example of the same type as Q7 and for further practice questions on proof by contradiction.

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8 Suppose that n is a composite integer. Prove that n has a prime factor less than or equal to $\sqrt{n}$.

9 Prove that if any 25 dates are chosen, some three must be within the same month.

10 Prove that $a^2 - 4b^2$ is never 2 if a and b are whole numbers.

11 **a** Show that if $x = \frac{p}{q}$ is a solution to the equation $x^3 + x + 1 = 0$ then $p^3 + pq^2 + q^3 = 0$.

b Explain why there is no solution to this equation when p is odd and q is odd.

c Prove that there are no rational solutions to $x^3 + x + 1 = 0$.

12 Prove that if a triangle has three sides, a, b and c, such that $a^2 + b^2 = c^2$, then it is a right-angled triangle.

Section 3: Criticising proofs

In Student Book 1, Chapter 1, you were introduced to the notation used in logic:

- $A \Leftrightarrow B$ means that statements A and B are equivalent.
- $A \Rightarrow B$ means if A is true then so is B.
- $A \Leftarrow B$ means if B is true then so is A.

When looking at proof (including solving equations, which is a type of proof!) you have probably looked out for errors in areas such as arithmetic or algebra. You now also need to look for errors in logic.

WORKED EXAMPLE 1.3

Yas was solving the equation $2\log_{10} x = 4$. Find the errors in her working.

Line 1: $2\log_{10} x = 4$

Line 2: $\Leftrightarrow \log_{10}(x^2) = 4$

Line 3: $\Rightarrow x^2 = 10^4 = 10\,000$

Line 4: $\Leftrightarrow x = \pm 1000$

In line 2 the symbol should be $\Rightarrow$: if x is negative line 2 is correct but line 1 is not possible. — This is an error in logic.

In line 3 the symbol should be $\Leftrightarrow$: if $x^2 = 10^4$ then $\log_{10}(x^2) = 4$. — This is an error in logic.

In line 4 the square root of 10 000 should be 100. — This is an arithmetic error.

Because some of the implications go only one way, the final solution might not work in the original equation. It should be checked. — This is an error in logic.

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EXERCISE 1C

1. Lambert was asked to solve the equation $x = \sqrt{3x+4}$. Here is his working.

   Line 1: $x = \sqrt{3x+4}$

   Line 2: $\Leftrightarrow\ x^2 = 3x+4$

   Line 3: $\Leftrightarrow\ x^2 - 3x - 4 = 0$

   Line 4: $\Leftrightarrow\ (x-4)(x+1) = 0$

   Line 5: $\Leftrightarrow\ x = 4$ or $x = -1$

   a By checking his solutions, find the correct solution.

   b In which line of working is his mistake?

2. Craig was asked to solve the equation $x^2 = 3x$. Here is his solution.

   $$x^2 = 3x$$
   $$\Leftrightarrow\quad x = 3$$

   a Show that $x = 0$ is also a solution to the original equation.

   b What logical symbol should Craig have used in the second line?

3. Freja was asked to solve the equation $x - \frac{1}{x-3} = 1 + \frac{5-2x}{x-3}$. Here is her working.

   Line 1: $x - \frac{1}{x-3} = 1 + \frac{5-2x}{x-3}$

   Line 2: $\Leftrightarrow\ x - 1 = \frac{6-2x}{x-3}$

   Line 3: $\Leftrightarrow\ (x-1)(x-3) = 6-2x$

   Line 4: $\Leftrightarrow\ x^2 - 4x + 3 = 6 - 2x$

   Line 5: $\Leftrightarrow\ x^2 - 2x - 3 = 0$

   Line 6: $\Leftrightarrow\ (x-3)(x+1) = 0$

   Line 7: $\Leftrightarrow\ x = 3$ or $x = -1$

   a By checking her solutions find the correct solution.

   b In which line of working is her mistake?

4. Jamie was asked to solve $\log_2(-x) + \log_2(2-x) = 3$.

   Here is her working.

   Line 1: $\log_2(-x) + \log_2(2-x) = 3$

   Line 2: $\Leftrightarrow\ \log_2(-x(2-x)) = 3$

   Line 3: $\Leftrightarrow\ \log_2(x^2 - 2x) = 3$

   Line 4: $\Leftrightarrow\ x^2 - 2x = 2^3$

   Line 5: $\Leftrightarrow\ x^2 - 2x - 8 = 0$

   Line 6: $\Leftrightarrow\ (x-4)(x+2) = 0$

   Line 7: $\Leftrightarrow\ x = 4$ or $x = -2$

   In which line of working did Jamie make a mistake?

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5 Andrew was asked to prove the statement: 'The function $y = x^3 - 3x$ has a minimum at $x = 1$.'

Here is his working.

Line 1: $\frac{dy}{dx} = 3x^2 - 3 = 0$

Line 2: $x^2 = 1$

Line 3: $x = 1$

Line 4: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

Line 5: $= \frac{d}{dx}(0)$

Line 6: $= 0$

Line 7: So it is a minimum.

Describe the errors in this proof.

6 Ann was asked to answer the question: '$x + q$ is a factor of $x^3 + px + q$. Find the remaining factor.'

If $x + q$ is a factor of $x^3 + px + q$ then:

$$x^3 + px + q \equiv (x + q)(x^2 + bx + 1)$$
$$\equiv x^3 + x^2(b + q) + x(bq + 1) + q$$

Comparing coefficients of x^2: $0 = b + q \Rightarrow b = -q$

Therefore the remaining factor is $x^2 - qx + 1$.

Brian claims that Ann's solution isn't complete. Explain why he is correct, and give a full solution.

7 Find the error in this proof that $\sqrt{16}$ is irrational.

Line 1: Assume that $\sqrt{16} = \frac{p}{q}$ where p and q are integers with no common factors.

Line 2: Squaring both sides gives $16 = \frac{p^2}{q^2}$.

Line 3: So $p^2 = 16q^2$. (1)

Line 4: This means that p^2 is even so p must also be even.

Line 5: You can then write that $p = 2k$, so $p^2 = 4k^2$.

Line 6: Substituting this into equation (1) gives $4k^2 = 16q^2$.

Line 7: So $k^2 = 4q^2$.

Line 8: This means that q^2 is even, so q must be even.

Line 9: But you have shown that both p and q are even, so they share a factor of 2.

Line 10: This contradicts the original assertion, so $\sqrt{16}$ cannot be written as $\frac{p}{q}$.

Checklist of learning and understanding

- You should be able to apply counterexamples, proof by exhaustion and proof by deduction to material from Student Book 1.
- Proof by contradiction is a method of proof that works by showing that assuming the opposite of the required statement leads to an impossible situation.
- When criticising proofs, look out for flaws in logic as well as mistakes in algebra or arithmetic.

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Mixed practice 1

1. Which symbol should be used to replace '?' in this working?

   $x^2 = 8x$

   ? $\quad x = 8$

   A $\Rightarrow$ **B** $\Leftrightarrow$ **C** $\Leftarrow$ **D** $\equiv$

2. Prove that $n^2 - n$ is always even.

3. Prove that $\sqrt{5}$ is irrational.

4. Prove that there is an infinite number of square numbers.

5. Find the error(s) in this working to solve $\tan x = 2\sin x$ for $0° \leqslant x < 360°$.

   Line 1: $\frac{\sin x}{\cos x} = 2\sin x$

   Line 2: $\Leftrightarrow$ $\sin x = 2\sin x \cos x$

   Line 3: $\Leftrightarrow$ $1 = 2\cos x$

   Line 4: $\Leftrightarrow$ $0.5 = \cos x$

   Line 5: $\Leftrightarrow$ $x = 60°$

6. Fermat says that if x is prime then $2x+1$ is prime.

   Which value of x provides a counterexample to this statement?

   A 9 **B** 11 **C** 13 **D** 15

7. Prove that $\log_2 5$ is irrational.

8. $OABC$ is a parallelogram with O at the origin and **a**, **b**, **c** are the position vectors of points A, B and C.

   P is the midpoint of BC and Q is the point on OB such that $OQ : QB$ is 2 : 1.

   Prove that AQP is a straight line.

9. Prove that if a and b are whole numbers $a^2 - b^2$ is either odd or a multiple of 4.

10. **a** Prove that if $f(x)$ is a polynomial of finite order, with integer coefficients, and n is an integer, then $f(n)$ is an integer.

    b Use a counterexample to show that this statement is not always correct.

    If $f(x)$ is a polynomial, where $f(n)$ is an integer whenever n is an integer, then $f(x)$ must have integer coefficients.

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11 Consider this working to solve $x+\frac{4x}{x-2}=\frac{8}{x-2}$.

Line 1: $\Leftrightarrow\ x(x-2)+4x=8$

Line 2: $\Leftrightarrow\ x^2-2x+4x=8$

Line 3: $\Leftrightarrow\ x^2+2x=8$

Line 4: $\Leftrightarrow\ (x-1)^2=9$

Line 5: $\Rightarrow\ x-1=3$

Line 6: $\Leftrightarrow\ x=4$

a In which lines are there mistakes?

b Rewrite the solution correctly, making appropriate use of logical connectors.

12 This proof is trying to demonstrate that there are an arbitrary number of consecutive composite (non-prime) numbers.

Line 1: Consider $n!$ for $n \geqslant r \geqslant 1$.

Line 2: $n!+r$ is divisible by r.

Line 3: So the numbers $n!+1, n!+2, \ldots, n!+n$ are not prime.

Line 4: Therefore, this is a list of n consecutive composite numbers.

Which is the first line to contain an error?

13 Prove that if a, b and c are integers such that $a^2+b^2=c^2$, then either a or b is even.

14 **a** By considering a right-angled triangle, prove that if A is an acute angle then

$$\tan(90° - A) = \frac{1}{\tan A}$$

b Hence prove that $\tan 10° \times \tan 20° \times \tan 30° \ldots \times \tan 80°$ is a rational number.

15 Prove that for values of x between 90° and 180° $\sin x - \cos x \geqslant 1$.

Elevate

See Extension Sheet 1 to complete the details of a couple of famous proofs.