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1

Surds and Indices

In this chapter we will learn:

- how to manipulate expressions involving surds,
- how to manipulate expressions involving indices.

1.1 Types of Number

Modern Mathematics is built on the back of thousands of years of mathematical thought. Over the centuries, mathematicians saw the need for ever more complicated ideas of number. It is still important nowadays to be aware of the hierarchy of number types, since different mathematical ideas and argumchents can be applied at different levels. We start by setting out the fundamental different types of number that we will encounter:

- The most fundamental numbers are those used for counting: the positive whole numbers $1, 2, 3, 4, \ldots$ These are called the **natural numbers**. The set of all natural numbers is denoted by the special symbol $\mathbb{N}$, so that

$$\mathbb{N} = \left\{1, 2, 3, 4, \ldots\right\}$$

- The natural numbers are sufficient to count (sheep, coins, etc.), and can be used to add, but they are insufficient if we want to be able to subtract (as Alice told the Red Queen, 'nine from eight I can't, you know'). To be able to do subtraction neatly, the number zero and negative whole numbers were introduced, giving us the integers: $\ldots, -3, -1, 0, 1, 2, 3, \ldots$. The set of all the integers is denoted by the special symbol $\mathbb{Z}$ ('Z' for *Zahl*, the German for 'number'), so that

$$\mathbb{Z} = \left\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\right\}$$

  The integers are a very important set of numbers. As well as being able to add and subtract integers, multiplication is possible, as is factorisation into primes. Studying the properties of the integers has generated some of the richest areas of modern mathematics.

- The **rational** numbers are those which can be expressed as fractions of integers in the form $\frac{p}{q}$, where p and q are integers, and $q \neq 0$. The set of rational numbers is denoted by the special symbol $\mathbb{Q}$ ('Q' for *quotient*), so that

$$\mathbb{Q} = \left\{\tfrac{p}{q} \,\middle|\, p, q \in \mathbb{Z}, q \neq 0\right\}$$

For Interest

We are using a standard notation to describe sets here. A set can be written in the form

$$\{x \mid A\}$$

where x is an expression for a number in the set, and A is a condition, or set of conditions, that specify the types of number that are permitted. The vertical bar | (sometimes a colon is used) should be read as 'such that'. Thus the formula for $\mathbb{Q}$ given above can be read as 'the set of numbers $\frac{p}{q}$ such that p and q are integers where q is non-zero'.

- Not all numbers are rational, however. Important numbers, like $\sqrt{2}$ and π, cannot be written as fractions. Numbers that cannot be expressed as fractions are called **irrational** numbers, and the irrational and rational numbers together form the **real** numbers. The collection of all real numbers is denoted by $\mathbb{R}$. Actually what is meant by a number here is quite a difficult question: integers and rationals have a fairly concrete existence which is founded in our experience, but irrational numbers are more elusive. The Pythagorean schools of mathematics in Ancient Greece thought that all numbers should be fractions, and that numbers which were not fractions were irrational in both senses of the word! We will have to be content with thinking that numbers are quantities that can have a place found for them along a number line.
- Eventually, we will want to move off the number line and study numbers that do things that real numbers cannot. In particular, we will want to introduce the square root of -1, denoted i. Numbers of the form $a + ib$, where a, b are real, are called **complex** numbers, and the set of all complex numbers is denoted $\mathbb{C}$.

It is worth observing that:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

Each of our special sets contains all the preceding special sets as a subset. A Venn diagram for these sets would be five concentric circles!

When we cannot express a number as a fraction, we try to express it in decimals. Rational numbers either have decimal expansions which terminate

$$\tfrac{7}{10} = 0.7 \qquad \tfrac{3}{16} = 0.1875 \qquad \tfrac{11}{20} = 0.55$$

or they have recurring decimal expansions, i.e. ones which eventually start repeating in a regular pattern:

$$\tfrac{3}{11} = 0.\dot{2}\dot{7} = 0.27272727\ldots \qquad \tfrac{8}{15} = 0.5\dot{3} = 0.533333\ldots$$

$$\tfrac{7}{17} = 0.\dot{4}117647058823529\dot{9} = 0.41176470588235294117 6\ldots$$

The converse is true: any terminating or recurring decimal describes a rational number. It is therefore easy to write down irrational numbers, by constructing decimals which definitely do not recur:

$$0.101001000100001000001000000100\ldots$$

but it is more interesting to be able to find out whether particular numbers are irrational or not.

Example 1.1.1. *Show that $\sqrt{2}$ i.e. the square root of 2 is irrational.*

Suppose that $\sqrt{2}$ was rational. Then we could write $\sqrt{2} = \frac{a}{b}$ as a fraction. We can assume that the fraction is in its lowest terms, so that the positive integers a, b have no common factor. Squaring the formula for $\sqrt{2}$ and multiplying by b^2 gives

$$a^2 = 2b^2,$$

and hence a^2 is an even integer. But this can only happen when a is even. Thus $a = 2c$ for some integer c. But then $2b^2 = (2c)^2 = 4c^2$, and hence

$$b^2 = 2c^2.$$

But this implies that b^2 is even, and so b is even.

We have come to the conclusion that a and b, which have no common factor, are both even, and hence both divisible by 2. The only way out of this impasse is to deduce that our original idea, that $\sqrt{2}$ was rational, is not true. Thus we deduce that $\sqrt{2}$ is irrational.

This is an example of an important method of argument: *Proof by Contradiction*. If assuming a fact leads to nonsense, we may deduce that our original assumption was incorrect.

Exercise 1A

1. It is easy to 'place' a fraction on the number line. For example, $4\frac{2}{3}$ is two-thirds of the way from 4 to 5. How can we be sure about where to place $\sqrt{2}$? Can we be sure it exists? Were the ancient Greeks right to be worried?
2. Do there exist real numbers which possess two or more different decimal expansions? If so, which?
3. Express $0.\dot{1}2\dot{3}$ and $0.2\dot{2}\dot{7}$ as fractions.
4. If a real number x has a recurring decimal expansion which comprises a sequence of n repeated digits (so that $n = 3$ for $x = 0.1\dot{2}8\dot{5}$), show that $(10^n - 1)x$ has a terminating decimal expansion, and hence that x is rational.

5*. How many remainders are possible when dividing an integer by 17? Show that any fraction with denominator equal to 17 has a recurring decimal expansion. Extend the argument to deal with all rational numbers.

1.2 Surds

Square roots, or surds, were the first examples of irrational numbers to be identified.

For Interest

Irrational numbers were considered *(ab)surd*.

It is important to work with surds without using a calculator. Except to a limited extent (the most common calculators can work with surds $\sqrt{n}$, provided that the integer n is not too big), calculators can only handle the decimal expansion of a surd, and then only to 9 or so decimal places. Using a calculator inevitably means, therefore, that answers obtained will be inexact. They may be very accurate, but they will not be perfect. It is important to be able to work without reference to a calculator when possible. The main properties of surds are these:

Key Fact 1.1 Properties of Surds

- For any $x \geq 0$, the number $\sqrt{x}$ is the **non-negative (positive or zero)** square root of x.
- $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$ for any $x, y \geq 0$.
- $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ for any $x \geq 0$, $y > 0$.

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The result for $\sqrt{xy}$ and $\sqrt{\frac{x}{y}}$ can be seen, because

$$\left(\sqrt{x}\times\sqrt{y}\right)\times\left(\sqrt{x}\times\sqrt{y}\right) = \left(\sqrt{x}\times\sqrt{x}\right)\times\left(\sqrt{y}\times\sqrt{y}\right) = x\times y = xy$$

and so $\sqrt{xy} = \sqrt{x}\times\sqrt{y}$. Moreover, since $\frac{x}{y}\times y = x$, we have

$$\sqrt{\frac{x}{y}}\times\sqrt{y} = \sqrt{x}$$

and hence $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$.

These results can be used in a variety of ways to establish exact identities between surds.

Example 1.2.1. *Simplify the following expressions:*

a) $\sqrt{8}$, *b)* $\sqrt{75}$, *c)* $\sqrt{18}\times\sqrt{2}$, *d)* $\frac{\sqrt{27}}{\sqrt{3}}$,
e) $\sqrt{40}\times\sqrt{2}$, *f)* $\sqrt{28}+\sqrt{63}$, *g)* $\sqrt{5}\times\sqrt{10}$, *h)* $3\sqrt{2}\times 4\sqrt{7}$.

(a) $\sqrt{8} = \sqrt{4\times 2} = \sqrt{4}\times\sqrt{2} = 2\sqrt{2}$,

(b) $\sqrt{75} = \sqrt{25\times 3} = \sqrt{25}\times\sqrt{3} = 5\sqrt{3}$,

(c) $\sqrt{18}\times\sqrt{2} = \sqrt{18\times 2} = \sqrt{36} = 6$,

(d) $\frac{\sqrt{27}}{\sqrt{3}} = \sqrt{\frac{27}{3}} = \sqrt{9} = 3$,

(e) $\sqrt{40}\times\sqrt{2} = \sqrt{40\times 2} = \sqrt{16\times 5} = \sqrt{16}\times\sqrt{5} = 4\sqrt{5}$,

(f) $\sqrt{28}+\sqrt{63} = \sqrt{4}\times\sqrt{7}+\sqrt{9}\times\sqrt{7} = 2\sqrt{7}+3\sqrt{7} = 5\sqrt{7}$,

(g) $\sqrt{5}\times\sqrt{10} = \sqrt{5}\times\left(\sqrt{5}\times\sqrt{2}\right) = \left(\sqrt{5}\times\sqrt{5}\right)\times\sqrt{2} = 5\sqrt{2}$,

(h) $3\sqrt{2}\times 4\sqrt{7} = 12\sqrt{2\times 7} = 12\sqrt{14}$.

Surds can also be used to handle algebraic problems:

Example 1.2.2. *Simplify the following expressions:*

(a) $\sqrt{x^5y^2}$, *(b)* $\sqrt{x^3yz^2}\times\sqrt{xy^2}$, *(c)* $\frac{\sqrt{p^5q}}{\sqrt{p^2q^3}}$

(a) $\sqrt{x^5y^2} = \sqrt{x^4y^2\times x} = x^2y\sqrt{x}$.

(b) $\sqrt{x^3yz^2}\times\sqrt{xy^2} = \sqrt{x^3yz^2\times xy^2} = \sqrt{x^4y^3z^2} = x^2yz\sqrt{y}$.

(c) $\frac{\sqrt{p^5q}}{\sqrt{p^2q^3}} = \sqrt{\frac{p^5q}{p^2q^3}} = \sqrt{\frac{p^3}{q^2}} = \frac{\sqrt{p^3}}{\sqrt{q^2}} = \frac{p\sqrt{p}}{q}$.

Example 1.2.3. *Solve the simultaneous equations*

$$y = \sqrt{x} \qquad y^3 = 2x$$

We see that

$$\begin{aligned} 2x &= y^3 = \sqrt{x}\times\sqrt{x}\times\sqrt{x} = x\sqrt{x} \\ x\sqrt{x}-2x &= 0 \\ x(\sqrt{x}-2) &= 0 \end{aligned}$$

and hence either $x = 0$ or $\sqrt{x} = 2$, so either $x = 0$ or $x = 4$.

Similar rules can be applied to cube and higher roots.

Example 1.2.4. *Simplify the following expressions:*
(*a*) $\sqrt[3]{16}$, (*b*) $\sqrt[3]{12}\times\sqrt[3]{18}$, (*c*) $\sqrt[5]{1215}$.

(a) $\sqrt[3]{16} = \sqrt[3]{8\times 2} = \sqrt[3]{8}\times\sqrt[3]{2} = 2\sqrt[3]{2}$,

(b) $\sqrt[3]{12}\times\sqrt[3]{18} = \sqrt[3]{12\times 18} = \sqrt[3]{216} = 6$,

(c) $\sqrt[5]{1215} = \sqrt[5]{243\times 5} = \sqrt[5]{243}\times\sqrt[5]{5} = 3\sqrt[5]{5}$.

We frequently want to remove a surd from the denominator of a fraction. This is done either by cancelling the same surd in the numerator, or else by 'multiplying top and bottom' by a suitable expression. This process is called **rationalising the denominator**.

Key Fact 1.2 Rationalising the Denominator

- For any $x > 0$,

$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x}}\times\frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}}{x}$$

and so

$$\frac{x}{\sqrt{x}} = \sqrt{x}$$

- For any $y \geq 0$,

$$\frac{1}{x+\sqrt{y}} = \frac{1}{x+\sqrt{y}}\times\frac{x-\sqrt{y}}{x-\sqrt{y}} = \frac{x-\sqrt{y}}{x^2-y}$$

Note the use of the 'Difference of Two Squares' technique to rationalise the denominator when the denominator was $x+\sqrt{y}$. Multiplying by $\frac{x-\sqrt{y}}{x-\sqrt{y}}$ does not change the value of the expression, because this fraction is equal to 1.

Example 1.2.5. *Write in simplified surd form:*
(*a*) $\frac{1}{\sqrt{2}}$, (*b*) $\frac{6}{\sqrt{2}}$, (*c*) $\frac{3\sqrt{2}}{\sqrt{10}}$, (*d*) $\frac{1}{3-\sqrt{2}}$

(a) $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$,

(b) $\frac{6}{\sqrt{2}} = \frac{3\times 2}{\sqrt{2}} = 3\sqrt{2}$,

(c) $\frac{3\sqrt{2}}{\sqrt{10}} = \frac{3\sqrt{2}}{\sqrt{5}\times\sqrt{2}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$,

(d) $\frac{1}{3-\sqrt{2}} = \frac{1}{3-\sqrt{2}}\times\frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{3+\sqrt{2}}{7}$

Using the 'Difference of Two Squares' method to rationalise the denominator, as shown in Example 1.2.5, is a surprisingly useful technique.

Example 1.2.6. *Find a positive integer n such that $\sqrt{n+1}-\sqrt{n} < 10^{-3}$.*

Note that

$$0 < \sqrt{n+1}-\sqrt{n} = \left(\sqrt{n+1}-\sqrt{n}\right)\times\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Now $n+1 > n$, and so $\sqrt{n+1} > \sqrt{n}$, and hence $\sqrt{n+1}+\sqrt{n} > 2\sqrt{n}$. This tells us that

$$0 < \sqrt{n+1}-\sqrt{n} < \frac{1}{2\sqrt{n}},$$

and we see that $\sqrt{n+1}-\sqrt{n} < 10^{-3}$ will be true if $2\sqrt{n} \geq 1000$, and so if $n \geq 500^2 (= 250000)$.

EXERCISE 1B

1. Simplify the following:
   a) $\sqrt{3}\times\sqrt{3}$ b) $\sqrt{8}\times\sqrt{2}$ c) $\sqrt{3}\times\sqrt{12}$ d) $2\sqrt{5}\times 3\sqrt{5}$
   e) $\left(2\sqrt{7}\right)^2$ f) $\sqrt[3]{x}\times\sqrt[3]{x^2y}$ g) $\sqrt[4]{125}\times\sqrt[4]{5}$ h) $\left(2\sqrt[4]{x}\right)^4$
2. Simplify the following (assuming that $x,y>0$):
   a) $\sqrt{18}$ b) $\sqrt{45}$ c) $\sqrt{675}$ d) $\sqrt{x^3y^5}$
   e) $\sqrt{2000}$ f) $\sqrt[3]{250}$ g) $\sqrt[4]{32x^4y^4}$ h) $\sqrt{x^3+2x^2y+xy^2}$.
3. Simplify the following (assuming that $x,y>0$):
   a) $\sqrt{8}+\sqrt{18}$ b) $\sqrt{20}-\sqrt{5}$ c) $2\sqrt{20}+3\sqrt{45}$
   d) $\sqrt{x^3}+\sqrt{xy^2}$ e) $\sqrt{99}+\sqrt{44}-\sqrt{11}$ f) $\sqrt{52}-\sqrt{13}$
   g) $\sqrt{4x^2+4xy+y^2}-\sqrt{x^2+2xy+y^2}$
4. Simplify the following:
   a) $\frac{\sqrt{8}}{\sqrt{2}}$ b) $\frac{\sqrt{40}}{\sqrt{20}}$ c) $\frac{\sqrt{3}}{\sqrt{48}}$ d) $\frac{\sqrt{50}}{\sqrt{200}}$
   e) $\frac{1}{\sqrt{5}}$ f) $\frac{3\sqrt{5}}{\sqrt{3}}$ g) $\frac{4\sqrt{2}}{\sqrt{12}}$ h) $\frac{2\sqrt{18}}{9\sqrt{12}}$
   i) $\frac{1}{2-\sqrt{3}}$ j) $\frac{1}{3\sqrt{5}-5}$ k) $\frac{4\sqrt{3}}{2\sqrt{6}+3\sqrt{2}}$ l) $\frac{12}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$
5. You are given that, correct to 12 decimal places, $\sqrt{26}=5.099019513593$. Find the value of $\sqrt{650}$ correct to 10 decimal places.
6. Solve the simultaneous equations:
   $$7x-(3\sqrt{5})y = 9\sqrt{5} \qquad (2\sqrt{5})x+y = 34$$
7. Assuming that $x>0$, show that $\frac{\sqrt{x}}{\sqrt{x^2+x}+x} = \sqrt{x+1}-x\sqrt{x}$.

8*. Assuming that $x>1$, evaluate
   $$\frac{1}{\sqrt{x+\sqrt{x^2-x}}}-\sqrt{1-\sqrt{1-x^{-1}}}$$

9*. Put the following numbers in ascending order: $7-4\sqrt{3}$, $8-3\sqrt{7}$, $9-4\sqrt{5}$, $10-3\sqrt{11}$.

1.3 Indices

When mathematicians started solving quadratic, cubic and quartic equations, they wrote expressions like xx, xxx and $xxxx$ to denote the repeated product of a variable x with itself (just as abc is the product of a, b and c). It was found to be more economical to use the notations x^2, x^3 and x^4 instead, and so index notation was invented. However, index notation is not just a method of writing expressions conveniently; it introduces a new method of thought about number and algebra without which much of modern mathematics would be impossible.

1.3.1. POSITIVE INDICES

In general the symbol a^m stands for the result of multiplying m copies of a together:

$$a^m = \underbrace{a\times a\times\cdots\times a}_{m \text{ copies}}$$

This operation is described in words as 'a raised to the m^{th} power', or 'a to the power m' or even just 'a to the m'. The number a is called the **base**, and the number m the **index**. For the present, while a can be any number, m must be a positive integer. We shall extend consideration to negative indices below.

Expressions in index notation can be simplified, subject to a few simple rules:

Key Fact 1.3 Rules for Positive Indices

- $a^m \times a^n = a^{m+n}$,
- $a^m \div a^n = a^{m-n}$, if $m > n$,
- $(a^m)^n = a^{mn}$,
- $(ab)^m = a^m b^m$.

- $$a^m \times a^n = \underbrace{a \times a \times \cdots \times a}_{m \text{ copies}} \times \underbrace{a \times a \times \cdots \times a}_{n \text{ copies}} = \underbrace{a \times a \times \cdots \times a}_{m+n \text{ copies}} = a^{m+n}$$
- $$a^m \div a^n = \underbrace{a \times a \times \cdots \times a}_{m \text{ copies}} \div \underbrace{a \times a \times \cdots \times a}_{n \text{ copies}} = \underbrace{a \times a \times \cdots \times a}_{m-n \text{ copies}} = a^{m-n}$$
- $$(a^m)^n = \underbrace{\underbrace{a \times a \times \cdots \times a}_{m \text{ copies}} \times \cdots \times \underbrace{a \times a \times \cdots \times a}_{m \text{ copies}}}_{n \text{ brackets}} = \underbrace{a \times a \times \cdots \times a}_{mn \text{ copies}} = a^{mn}$$
- $$(ab)^m = \underbrace{ab \times ab \times \cdots \times ab}_{m \text{ copies}} = \underbrace{a \times a \times \cdots \times a}_{m \text{ copies}} \times \underbrace{b \times b \times \cdots \times b}_{m \text{ copies}} = a^m \times b^m$$

It is important to remember that, until we meet logarithms, the last of these rules is the only rule that can be applied to powers of different bases.

Example 1.3.1. *Simplify* $(2a^2b)^3 \div 4a^4b$.

Applying the rules,

$$(2a^2b)^3 \div 4a^4b = 2^3(a^2)^3b^3 \div 4a^4b = 8a^6b^3 \div 4a^4b = 2a^2b^2$$

For Interest

A common error is to write $2^3 \times 3^5 = 6^8$, multiplying the bases as well as adding the indices. Avoid it!

1.3.2. Zero and Negative Indices

The previous definition for a^m makes no sense if m is not a positive integer. Nevertheless it is possible to extend the definition of a^m to allow m to be any integer (provided that a is non-zero). If we look at the following table:

n	5	4	3	2	1
$\mathbf{2^n}$	32	16	8	4	2
$\mathbf{3^n}$	243	81	27	9	3

Every time the index n decreases by 1, the value of 2^n halves, and the value of 3^n is a third of its previous value. It is natural to extend the process

n	5	4	3	2	1	0	−1	−2	−3
$\mathbf{2^n}$	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$\mathbf{3^n}$	243	81	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

It seems that 2^0 and 3^0 should both be defined to be 1, while 2^{-m} should be the same as $\frac{1}{2^m}$, and 3^{-m} should be the same as $\frac{1}{3^m}$. This observation can be extended to any non-zero base a, and the resulting extension enables the previous rules for positive integer indices to be extended to general integer indices (and non-zero base).

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Key Fact 1.4 Rules for Integer Indices

- $a^m \times a^n = a^{m+n}$,
- $a^m \div a^n = a^{m-n}$,
- $(a^m)^n = a^{mn}$,
- $(ab)^m = a^m b^m$,
- $a^0 = 1$,
- $a^{-m} = \frac{1}{a^m}$

To show that the rules for positive integer indices can be extended (for non-zero base) to all integer indices, we need to check a number of cases. Here are two of them (m and n are positive integers):

$$a^m \times a^{-n} = a^m \times \frac{1}{a^n} = \frac{1}{a^n \div a^m} = \frac{1}{a^{n-m}} = a^{m-n} \qquad (m < n),$$
$$(a^m)^{-n} = \frac{1}{(a^m)^n} = \frac{1}{a^{mn}} = a^{-mn}.$$

Example 1.3.2. *Simplify the following:*
(a) $4a^2b \times (3ab^{-1})^{-2}$, (b) $4^3 \times 2^{-5}$, (c) $(2xy^2z^3)^2 \div (2x^2y^3z)$

(a) $4a^2b \times (3ab^{-1})^{-2} = 4a^2b \times 3^{-2}a^{-2}b^2 = \frac{4}{9}b^3$,

(b) $4^3 \times 2^{-5} = (2^2)^3 \times 2^{-5} = 2^6 \times 2^{-5} = 2^1 = 2$,

(c) $(2xy^2z^3)^2 \div (2x^2y^3z) = 4x^2y^4z^6 \div 2x^2y^3z = 2yz^5$.

Exercise 1C

1. Simplify the following, writing each answer as a power of 2:
   a) $2^{11} \times (2^5)^3$ b) $(2^3)^2 \times (2^2)^3$ c) 4^3 d) 8^2
   e) $\frac{2^7 \times 2^8}{2^{13}}$ f) $\frac{2^2 \times 2^3}{(2^2)^2}$ g) $4^2 \div 2^4$ h) $2 \times 4^4 \div 8^3$

2. Simplify the following:
   a) $a^2 \times a^3 \times a^7$ b) $c^7 \div c^3$ c) $(e^5)^4$
   d) $5g^5 \times 3g^3$ e) $(2a^2)^3 \times (3a)^2$ f) $(4x^2y)^2 \times (2xy^3)^3$
   g) $(6ac^3)^2 \div (9a^2c^5)$ h) $(49r^3s^2)^2 \div (7rs)^3$ i) $(3h^2)^{-2}$
   j) $(\frac{1}{2}j^{-2})^{-3}$ k) $(3n^{-2})^4 \times (9n)^{-1}$ l) $(2q^{-2})^{-2} \div \left(\frac{4}{q}\right)^2$

3. Solve the following equations:
   a) $3^x = \frac{1}{9}$ b) $5^y = 1$ c) $2^z \times 2^{z-3} = 32$
   d) $7^{3x} \div 7^{x-2} = \frac{1}{49}$ e) $4^y \times 2^y = 8^{120}$ f) $3^t \times 9^{t+3} = 27^2$

4. Write $8^3 \times 4$ as a power of 2.

5. Simplify $\left(\frac{1}{\sqrt{3}}\right)^9$.

6. Solve the equation $\frac{3^{5x+2}}{9^{1-x}} = \frac{27^{4+3x}}{729}$.

7. Why do we not need brackets when considering powers of powers: in other words, why is a^{m^n} equal to $a^{(m^n)}$, and not to $(a^m)^n$?

8*. Which of 3^{4^3} and 4^{3^3} is bigger?

1.3.3. Fractional Indices

Up to now, we have assumed that the rules for indices work for integer indices. What can we deduce if we were to assume that the rules were still true for fractional indices? It would follow that

$$(a^{\frac{1}{2}})^2 = a^{\frac{1}{2}\times 2} = a^1 = a$$

for any positive a. Since $a^{\frac{1}{2}}$ squares to a, we deduce that $a^{\frac{1}{2}}$ is either $\sqrt{a}$ or $-\sqrt{a}$. Just as with surds, we **define** $a^{\frac{1}{2}}$ to be positive, so that $a^{\frac{1}{2}} = \sqrt{a}$. More generally we can show that $(a^{\frac{1}{m}})^m = a$, so that $a^{\frac{1}{m}} = \sqrt[m]{a}$ for any positive integer m and $a > 0$ (while it is possible to take m^{th} roots of negative numbers when m is odd, it is simpler to restrict fractional indices to strictly positive bases). Thus we can define the fractional power of any positive real number.

Key Fact 1.5 Rules for Fractional Indices

- $a^{\frac{1}{m}} = \sqrt[m]{a}$,
- $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$.

With these definitions, it should be noted that the rules for integer indices now hold for all fractional indices (and positive bases).

Example 1.3.3. *Simplify the following:*

(a) $16^{-\frac{3}{4}}$, (b) $(2\frac{1}{4})^{\frac{1}{2}}$, (c) $\frac{(2x^2y^2)^{-\frac{1}{2}}}{(2xy^{-2})^{\frac{3}{2}}}$

(a) $16^{-\frac{3}{4}} = (2^4)^{-\frac{3}{4}} = 2^{-3} = \frac{1}{8}$,

(b) $(2\frac{1}{4})^{\frac{1}{2}} = (\frac{9}{4})^{\frac{1}{2}} = \frac{3}{2}$,

(c) $\frac{(2x^2y^2)^{-\frac{1}{2}}}{(2xy^{-2})^{\frac{3}{2}}} = \frac{2^{-\frac{1}{2}}x^{-1}y^{-1}}{2^{\frac{3}{2}}x^{\frac{3}{2}}y^{-3}} = 2^{-2}x^{-\frac{5}{2}}y^2 = \frac{y^2}{4x^{\frac{5}{2}}}$

Exercise 1D

1. Evaluate the following:

   a) $25^{\frac{1}{2}}$ b) $36^{\frac{1}{2}}$ c) $81^{\frac{1}{4}}$ d) $16^{-\frac{1}{4}}$
   e) $1000^{-\frac{1}{3}}$ f) $27^{\frac{1}{3}}$ g) $64^{\frac{2}{3}}$ h) $125^{-\frac{4}{3}}$
   i) $4^{\frac{3}{2}}$ j) $27^{\frac{4}{3}}$ k) $32^{\frac{3}{5}}$ l) $4^{2\frac{1}{2}}$
   m) $10000^{-\frac{3}{4}}$ n) $(\frac{1}{125})^{-\frac{4}{3}}$ o) $(3\frac{3}{8})^{\frac{2}{3}}$ p) $(2.25)^{-\frac{1}{2}}$

2. Simplify the following expressions:

   a) $a^{\frac{1}{3}} \times a^{\frac{5}{3}}$ b) $3b^{\frac{1}{2}} \times 4b^{-\frac{3}{2}}$ c) $6c^{\frac{1}{4}} \times (4c)^{\frac{1}{2}}$
   d) $(d^2)^{\frac{1}{3}} \div (d^{\frac{1}{3}})^2$ e) $(2x^{\frac{1}{2}})^6 \times (\frac{1}{2}x^{\frac{3}{4}})^4$ f) $(24e)^{\frac{1}{3}} \div (3e)^{\frac{1}{3}}$
   g) $\frac{(5p^2q^4)^{\frac{1}{3}}}{(25pq^2)^{-\frac{1}{3}}}$ h) $(m^3n)^{\frac{1}{4}} \times (8mn^3)^{\frac{1}{3}}$ i) $\frac{(2x^2y^{-1})^{-\frac{1}{4}}}{(8x^{-1}y^2)^{-\frac{1}{2}}}$

3. Solve the following equations:

   a) $x^{\frac{1}{2}} = 8$ b) $x^{\frac{1}{3}} = 3$ c) $x^{\frac{2}{3}} = 4$
   d) $x^{\frac{3}{2}} = 27$ e) $x^{-\frac{3}{2}} = 8$ f) $x^{-\frac{2}{3}} = 9$
   g) $x^{\frac{3}{2}} = x\sqrt{2}$ h) $x^{\frac{3}{2}} = 2\sqrt{x}$ i) $4^x = 32$
   j) $9^y = \frac{1}{27}$ k) $16^z = 2$ l) $100^x = 1000$
   m) $8^z = \frac{1}{128}$ n) $(2^t)^3 \times 4^{t-1} = 16$ o) $\frac{9^y}{27^{2y+1}} = 81$

4*. Which is bigger, $2^{\frac{1}{2}}$ or $3^{\frac{1}{3}}$?

Chapter 1: Summary

- If $x \geq 0$, the square root of x, denoted $\sqrt{x}$, is the non-negative square root of x. Another word for a number which is the square root of another number is surd.
- For $x, y \geq 0$, $\sqrt{xy} = \sqrt{x}\sqrt{y}$.
- In fractions involving surds, the denominator may be rationalised as follows:

$$\frac{1}{\sqrt{x}} = \frac{1}{x}\sqrt{x} \qquad \frac{1}{x+\sqrt{y}} = \frac{x-\sqrt{y}}{x^2-y}$$

- The laws of indices state that

$$a^m \times a^n = a^{m+n} \qquad (a^m)^n = a^{mn}$$
$$a^m \div a^n = a^{m-n} \qquad (ab)^m = a^m b^m$$
$$a^0 = 1 \qquad a^{-n} = \frac{1}{a^n}$$

These identities hold:

  ★ for all $a, b > 0$ and any values of m and n, or

  ★ for all non-zero a, b any integer values of m and n.