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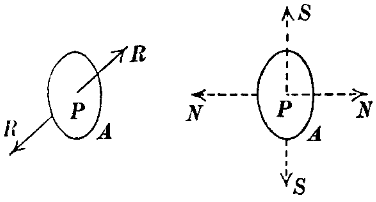
HYDROSTATICS

Chapter I

INTRODUCTION

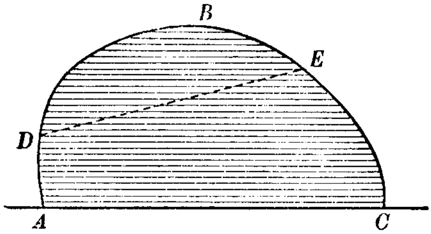
1·1. Hydrostatics is the branch of mathematics which is concerned with the conditions under which systems of forces maintain masses of fluid in a state of relative equilibrium, together with the determinations of the reactions exerted by fluids upon bodies with which they are in contact.

1·11. Shearing stress. At any point P inside a material substance or 'medium' imagine a small plane surface A to be drawn lying wholly within the medium. The surface A may be regarded as dividing or separating the medium at P; and, since action and reaction are equal and opposite, the two portions of the medium on opposite sides of the surface A exert on one another equal and opposite forces R. Without much more data we cannot specify the direction of the force R, but in general it can be resolved into a component N normal to A and a component S tangential to A. If for convenience we take the area of A to be the unit of area, then the force R is the **stress per unit area** across A; the normal component N is the **thrust or tension per unit area** across A according to the sense in which it acts, and the tangential component S is the **shearing stress per unit area** along A.



1·12. A fluid *is a substance which flows or is capable of flowing*. The two most familiar forms of fluid are air and water. The property of these substances which distinguishes them most readily from solids is the ease with which any masses of them can be subdivided. They offer practically no resistance to separation, or, in mathematical language, they do not exert shearing stress.

Let ABC represent a body on a horizontal plane AC, and let DE be a hypothetical oblique plane section. Suppose the body to be a solid in equilibrium. Considering the portion DBE, its weight can be resolved into components in the direction ED and at right angles to ED, and these forces are balanced by the stresses across the plane ED which, as in **1·11**, consist of a shearing stress along DE and a normal thrust. Consequently it is the shearing stress in the plane DE which prevents the



portion DBE from sliding down the plane ED. But if the body were a fluid such as water we know from experience that it will not remain in a heaped up position, and the reason for this is because in equilibrium a fluid cannot exert shearing stress or resist indefinitely any force of that nature.

This fundamental property of a fluid provides another **definition**: viz. *a fluid is a substance which will yield to any continued shearing stress however small.*

This definition embraces substances as widely different as water, treacle and coal tar. To explain the difference between such substances mathematically, it must be observed that when a fluid is *in motion* there are in general frictional forces between its particles; the magnitude of such forces depends on the *viscosity* or 'stickiness' of the fluid. The viscosity manifests itself in fluid motion in the production of shearing stresses which check the free motion of the particles. Thus reverting again to the figure, the external force on DBE tends to produce a shearing stress in the plane DE, and if ABC were a mass of non-viscous fluid such as water it would yield instantly to the externally applied tangential forces and subside on to the horizontal plane; if it were a mass of viscous fluid such as treacle the subsidence would take longer, and if it were a mass of very viscous fluid such as coal tar it would take a very long time to subside. But we include coal tar among the class of fluids because it will yield to any shearing stress *however small* provided that stress acts continuously for a sufficient time. And this property serves to distinguish *viscous fluids* from *plastic solids* such a putty, in that, with the latter class of substances, a stress of a *definite magnitude* is required to produce a deformation; while in the former class it is only necessary to have a certain duration of time for any shearing stress however small to produce a permanent displacement.

It must be remembered that viscosity or 'stickiness' is only effective in producing shearing stresses when a fluid is in motion, and consequently *there are no shearing stresses in a fluid in equilibrium*. This means that if the medium in **1·11** is a fluid in equilibrium, then at every point and however the surface A may be orientated the component S is always zero, and the thrust on any plane surface in the fluid is at right angles to it.

It follows that, in a continuous mass of fluid in equilibrium, *any* portion that we select, no matter how large or small, is in equilibrium under the action of external forces and the normal pressures of the surrounding fluid upon it. And as there are no shearing stresses the resolved parts of these forces in any direction must balance one another.

1·13. A perfect fluid is an ideal substance which, whether at rest or in motion, is incapable of exerting or offering resistance to shearing stress.

1·14. Liquids and gases. Fluids are of two kinds:

(i) **Liquids**, which are incompressible or nearly so.

(ii) **Gases**, which are easily compressible and can be made to expand indefinitely by increasing the volume to which they have access.

Water, for example, is not absolutely incompressible, but it requires such a great pressure to produce a very small relative diminution in volume that for most practical purposes it may be regarded as incompressible.

The distinction between solid, liquid and gas is sometimes stated in this way: a solid of definite mass has size and shape; a definite mass of liquid has size but no shape, and a definite mass of gas has neither size nor shape.

All gases can be converted into liquids by sufficient lowering of temperature and increase of pressure. For each gas there is a *critical temperature* such that for higher temperatures no increase of pressure however great will cause the gas to condense, while for lower temperatures the gas can be condensed by increasing the pressure. At temperatures below the critical temperature the gas is called a *vapour* and when above the critical temperature it is called a *permanent gas*.

1·2. Pressure. We have used the word 'pressure' without defining it. The idea of pressure implies something pressed upon, e.g. a surface subject to a certain force or thrust. When the thrust upon every portion of a plane surface is proportional to the area of the portion there is said to be a *uniform pressure* on the surface; and *the measure of the pressure is the thrust upon a unit of area.*

The **mean pressure** on a given plane surface is the uniform pressure which would produce the same resultant thrust as the actual pressure produces.

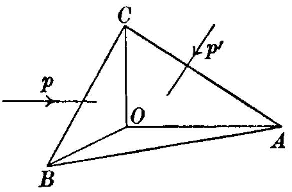
The **pressure at a point** of a surface is the limit of the mean pressure on a small area surrounding the point as the area is diminished indefinitely.

We may define **the pressure at a point of a fluid** explicitly thus: Imagine a small plane surface containing α units of area to be placed at the point (as in **1·11**) and let N denote the normal thrust of the fluid on either side of this surface, then the limit of (N/α) as $\alpha \to 0$ is the pressure at the point.

It will be noticed that this definition tacitly assumes that the result $\lim_{\alpha \to 0} (N/\alpha)$ is independent of the orientation of the small plane surface, and we must now proceed to justify that assumption.

1·21. The pressure at a point in a fluid in equilibrium is the same in every direction.

In the fluid draw a small tetrahedron $OABC$ with three mutually perpendicular faces OBC, OCA, OAB. We suppose that the fluid is subject to an external field of force (such as gravity), the effect of which can be measured as so much force per unit mass of fluid. Then the component parallel to OA of the external forces on the fluid in the tetrahedron may be denoted by



$$X \times \text{mass of fluid in } OABC,$$

or by

$$\rho X \times \text{volume } OABC,$$

where ρ is the mean density of the fluid in $OABC$.

Now let p, p' denote the mean pressures on the surfaces OBC, ABC, so that the actual thrusts on these surfaces are $p \,.\, \triangle OBC$ and $p' \,.\, \triangle ABC$; and let θ denote the angle between OBC and ABC.

By considering the equilibrium of the fluid in the tetrahedron and resolving parallel to OA, we get

$$p \,.\, \triangle OBC - p' \,.\, \triangle ABC \cos\theta + \rho X \,(\text{vol. } OABC) = 0 \quad (1),$$

or

$$(p - p')\, \triangle OBC + \tfrac{1}{3}\rho X \,.\, OA \,.\, \triangle OBC = 0,$$

or

$$p - p' + \tfrac{1}{3}\rho X \,.\, OA = 0 \quad \text{..................(2).}$$

Now let the tetrahedron diminish in size to vanishing point, then p, p' become the pressure at O in two different arbitrarily assigned directions, and since $OA \to 0$, the equation reduces to $p = p'$, which proves the proposition.

1·211. In the case of a fluid in motion, provided that the fluid is the ideal fluid of **1·13**, so that there are no shearing stresses, the foregoing proposition is still true, for in this case instead of the statical equation (1) we shall have a dynamical equation which will only differ from (1) by the addition of a term representing 'mass × acceleration'; i.e. a term like the term in X, but with acceleration instead of X, and this term will give rise to a term in (2) which also tends to zero as the tetrahedron tends to vanishing point.

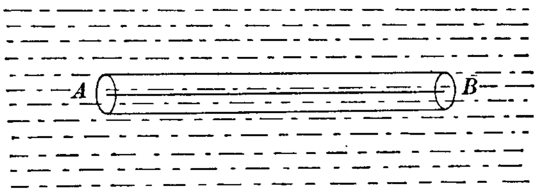
1·22. Dimensions of pressure. Since pressure is force per unit area, its dimensions in terms of the fundamental units of mass, length and time are

$$MLT^{-2}/L^2 = ML^{-1}T^{-2}.$$

1·3. Transmissibility of liquid pressure. *An increase of pressure at any point of a liquid at rest under given external forces is transmitted to every other point of the liquid.*

The essential datum here is that the external forces on any portion of the liquid are prescribed and remain unaffected by the cause which produces the change of pressure.

Let A, B be any two points in the liquid. About the straight line AB describe a thin cylinder with plane ends at right angles



to AB. Then assuming this cylinder to lie wholly in the liquid, the difference of the thrusts on its ends must be equal to the resolved part in the direction AB of the external forces on the liquid in the cylinder. But this is constant, and therefore the difference between the pressures at A and B must be constant, so that any increase of pressure at A must be accompanied by a like increase at B.

When the straight line AB does not lie wholly in the liquid, let the points A, B be joined by a succession of lines AC, CD ... KB which do lie in the liquid; then it can be shewn as above that the pressure difference between each pair of points remains constant.

1·31. Hydraulic or Bramah's press. In outline this apparatus consists of two vertical cylinders communicating with one another by a tube near their bases, and each fitted with a piston, the space below the pistons being filled with water.

If a downward force P be applied to the piston of area A this creates

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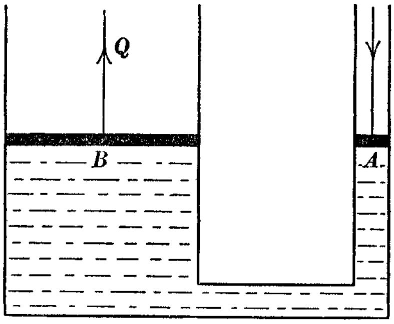
an additional pressure P/A per unit area throughout the liquid, and consequently if the second piston is of area B the resulting upward thrust upon it is represented by $Q = BP/A$; so that

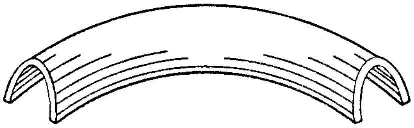
$$Q:P = B:A,$$

or the thrusts on the pistons are proportional to their areas.



Hence by adjusting the ratio of the areas we arrive at the 'hydrostatic paradox', that *any force however small may by its transmission through a liquid be made to support any weight however large.*

The apparatus may be used as a press if the load to be compressed is placed upon the piston B and pressed against a fixed barrier above the piston. In practice cupped leather collars round the pistons are needed to prevent oozing of the water. They are so placed that the water presses the collar against the sides of both the cylinder and the piston. A portion of such a collar is shewn in section in the figure.



1·4. Compression. Let a change of volume take place in a given mass of fluid, and let V, V' denote the initial and final volumes. We may define the **compression** as the ratio of the reduction in volume to the original volume; i.e.

$$\text{compression} = (V - V')/V,$$

or, in the notation of the Calculus,

$$= -dv/v.$$

The **compressibility** of a fluid is the limiting ratio of the compression to the increase of pressure which produces it; or, if P, P' denote initial and final pressures,

$$\text{compressibility} = \lim \left(\frac{V-V'}{V}\right) \Big/ (P'-P)$$
$$= -\frac{1}{v}\frac{dv}{dp}.$$

Similarly we may define the **elasticity** of a fluid as the limiting ratio of the increase of pressure to the compression it produces; i.e.

$$\text{elasticity} = \lim (P'-P) \Big/ \left(\frac{V-V'}{V}\right)$$
$$= -v\frac{dp}{dv}.$$

1·5. Density. The density of a substance is the mass of a unit of volume of the substance; e.g. the number of pounds per cubic foot or the number of grammes per cubic centimetre.

The **specific gravity** of a substance is the ratio of the density of that substance to the density of a standard substance.

The substance usually adopted as the standard is pure water. In rough calculations the mass of a cubic foot of water may be taken as 1000 oz. or 62·5 lb.; but at a temperature of 4° C., i.e. the temperature at which water has its maximum density, the density is approximately 62·425.

In the metric system the density of water is 1, the gramme having been chosen as the mass of 1 cubic centimetre of pure water at 4° C.

If W denotes the weight in absolute units (poundals) of a body of mass M (lb.), then, as in books on Dynamics,

$$W = Mg,$$

where g is the acceleration due to gravity.

But if the body is of density ρ and contains V units of volume, then from the definition of density

$$M = \rho V,$$

so that

$$W = g\rho V \quad \text{.........................(1).}$$

This relation expresses the weight in poundals, when V is measured in cubic feet and ρ is the mass in pounds of 1 cu. ft.

1·51. It will be observed that the dimensions of density in terms of the fundamental units are expressed by the formula $\mathbf{ML}^{-3}$, since it denotes mass per unit volume. On the other hand specific gravity is of no dimensions, but is merely a number representing how many times as heavy as a standard substance a given substance is.

Consequently if w denotes the weight of a unit of volume of the standard substance, then the weight of a unit of volume of a substance of specific gravity S is Sw; and the weight W of V units of volume of the substance of specific gravity S is given by the formula

$$W = VSw \quad \text{.........................(1).}$$

1·52. Specific gravity of mixtures. Let volumes V_1, V_2, V_3 ... of fluids of specific gravities S_1, S_2, S_3 ... be mixed together and let $\bar{S}$ be the specific gravity of the mixture. Then using **1·51** (1) and assuming the volume of the mixture to be the sum of the volumes of its constituents, the weight of the mixture is

$$(V_1 + V_2 + V_3 + \ldots)\,\bar{S}w;$$

but this must be equal to the sum of the weights of the constituents, viz.

$$V_1 S_1 w + V_2 S_2 w + V_3 S_3 w + \ldots.$$

Whence, equating and dropping the factor w, we get

$$\bar{S} = \frac{V_1 S_1 + V_2 S_2 + V_3 S_3 + \ldots}{V_1 + V_2 + V_3 + \ldots} \quad \text{............(1).}$$

If for any reason the mixture has a volume U, not equal to the sum of the volumes of the constituents, the corresponding formula is clearly

$$\bar{S} = (V_1 S_1 + V_2 S_2 + V_3 S_3 + \ldots)/U \quad \text{.........(2).}$$

Secondly, if given weights W_1, W_2, W_3 ... of the fluids are mixed, then from **1·51** (1) the volume of the mixture is $(W_1 + W_2 + W_3 + \ldots)/\bar{S}w$, and, assuming there to be no change in total volume, this must be equal to the sum of the volumes of the constituents, viz.

$$\frac{W_1}{S_1 w} + \frac{W_2}{S_2 w} + \frac{W_3}{S_3 w} + \ldots.$$

Whence we get $$\bar{S}=\frac{W_1+W_2+W_3+\dots}{\frac{W_1}{S_1}+\frac{W_2}{S_2}+\frac{W_3}{S_3}+\dots} \quad \text{...............(3)},$$

as could have been deduced from (1) by applying **1·51** (1).

The formula can also be modified if it is known that a definite change in volume results from the mixing.

1·6. Examples. (i) *Shew that the specific gravity of a mixture of n fluids is greater when equal volumes are taken than when equal weights are taken, assuming no change in volume as the result of mixing.*

Let $\bar{S}, \bar{S}'$ denote the specific gravity of the mixture according as we mix equal volumes or equal weights. Then if $s_1, s_2 \dots s_n$ denote the specific gravities of the constituents, from **1·52** (1) and (3) we have

$$\bar{S}=\frac{1}{n}(s_1+s_2+\dots+s_n) \quad \text{and} \quad \frac{1}{\bar{S}'}=\frac{1}{n}\left(\frac{1}{s_1}+\frac{1}{s_2}+\dots+\frac{1}{s_n}\right).$$

If we now apply the theorem that the arithmetic mean of n unequal positive numbers is greater than their geometric mean, we have

$$\bar{S}>\sqrt[n]{(s_1 s_2 \dots s_n)} \quad \text{and} \quad \frac{1}{\bar{S}'}>\frac{1}{\sqrt[n]{(s_1 s_2 \dots s_n)}}.$$

Whence it follows that $\bar{S}>\bar{S}'$.

(ii) *Pure water is added, drop by drop, to a vessel of volume V filled with a salt solution of specific gravity s, which is allowed to overflow. Find the specific gravity of the solution when a volume v of water has been poured in.*

Let σ denote the specific gravity when a volume v of water has been added, and $\sigma+d\sigma$ the specific gravity when a volume $v+dv$ has been added; i.e. dv may denote the volume of a drop of water.

The addition of a drop at this stage means that a volume dv of specific gravity 1 is added to a volume V of specific gravity σ and forms a total volume $V+dv$ of specific gravity $\sigma+d\sigma$ whereof a drop overflows. Hence, by equating the weight of the mixture to the sum of the weights of the constituents, we get

$$(V+dv)(\sigma+d\sigma)=dv+V\sigma.$$

Neglecting the product of the small quantities $dv, d\sigma$, this equation gives

$$V\,d\sigma+dv\,(\sigma-1)=0$$

or $$\frac{d\sigma}{\sigma-1}+\frac{dv}{V}=0.$$

Hence by integration $$\log(\sigma-1)+\frac{v}{V}=C.$$

But when $v=0$ then $\sigma=s$, so that $C=\log(s-1)$, and

$$\log(\sigma-1)=\log(s-1)-\frac{v}{V}.$$

Therefore $$\sigma=1+(s-1)\,e^{-v/V}.$$