THE FOURTH DIMENSION

INTRODUCTION

To the general reader, the name of the fourth dimension brings reminiscences of *Flatland* and *The Time Machine*. On hearing that to the mathematician the extension from three dimensions to four or five is trivial, he thinks he is being told that a study of mathematics, if reasonably intense, creates physical faculties or powers of visualisation with which the uninitiated are not endowed. Learning that Minkowski and Einstein combine space and time into a single continuum, he tries to believe in the existence of a state of mind in which the *sensations* of space and time are confused, and naturally he fails.

The position of students of mathematical physics, and of all but a fortunate few of the students of pure mathematics, is little better. Accustomed to regard a Cartesian frame of axes as a scaffolding erected in the real space around them, they can attach no *meaning* to a fourth coordinate, but having used complex electromotive forces with success in the theory of alternating currents, and having treated a symbol of differentiation as a detachable algebraic variable even to the extent of resolving operators into partial fractions for the solution of differential equations, these students are prepared to give pragmatical sanction to the most fantastic language.

The pure mathematician makes no attempt to *imagine* a space of four dimensions; he lays no claim to visualising a world that is inconceivable to other men. Only he finds that certain notions in *algebra* are discussed most readily in terms adopted from geometry and given a meaning entirely algebraic, and since it is to the mathematician alone that algebraic problems are of concern in themselves, fear lest the man in the street should mistake the very subject of a mathematical conversation he might overhear has not prevented the mathematician from using the vocabulary he finds best suited to his own needs. Now it has happened that the talk of a few mathematicians has suddenly become of universal and absorbing interest, and a dictionary explaining the meanings they are in the habit of giving to some familiar words is required.

It is this dictionary that I have tried to write, and I have written it in the simplest terms I could find, in the hope that it will prove intelligible to anyone familiar with elementary trigonometry and with the solution of simultaneous linear equations in algebra; for this reason, I have *not* treated the point as indefinable, I have supposed the numbers used always to be real, and I have avoided Frege-Russell definitions. The reader's first feeling will be one of disillusion. Are Einstein and Eddington talking not about a new heaven and a new earth but about linear algebraic equations? To discuss the question is beyond the province of a lexicographer.

Perhaps even the mathematical student, if he can overcome a reasonable irritation at the restrictions, from his point of view arbitrary, to four dimensions and to real numbers, and at the absence of certain obvious forms of abbreviation, may derive some help from the pamphlet. The possibility of constructing an abstract 'space' is always assumed, but the details of the construction, even for two or three dimensions only, are either taken for granted or disguised as theorems on matrices or on linear equations. The idea of direction and the measurement of angles in a constructed plane demand careful consideration. The nature of pure rotation in four dimensions is by no means obvious; on the contrary, rotation is the most difficult of the elementary notions used in the theory of relativity, and with an account of rotation our formal work comes to a natural end.

1. POINTS

1·1. The fundamental idea in the work before us is that of *an ordered group of four numbers*; $(2, -1, 3, 0)$, $(\frac{1}{3}, \frac{2}{5}, -\frac{1}{2}, -\frac{1}{4})$, $(-\sqrt{2}, 1, 1, \sqrt[3]{5})$ are particular groups of the kind contemplated, and (ξ, η, ζ, τ) is a form that can be used to represent any such group, the group becoming definite when actual numerical values are assigned to ξ, η, ζ, τ.

By calling the groups *ordered* we mean that the group (a, b, c, d) is regarded as different from a group such as (a, d, c, b) obtained by interchanging two of its constituent numbers, unless the numbers interchanged happen to be equal.

Each group is to be treated as one composite individual, to be combined or compared with other groups according to rules that are laid down. There is nothing peculiarly mathematical in dealing in this way with a structure that can *if necessary* be analysed, or in having a name for a group as a whole; on the contrary, not only collective nouns such as 'team' and 'nation' but common nouns such as 'book' and 'tree' stand for objects that it is convenient to regard as wholes although it is easy to distinguish parts of which they are formed.

An ordered group of four numbers may be called a *tetrad*. This name evokes no preconceptions and suggests no analogies, and the use of an unfamiliar word emphasises that theorems asserted *must* be founded on definitions given. But the definitions that are natural lead to theorems that *can be made* verbally similar to theorems in geometry by the introduction of terms derived from geometry. The vocabulary of geometry begins with the word 'point', and the mathematician finds it worth while to give to the ordered group of four numbers the *name* of 'point in four-dimensional space'. To him four-dimensional space *is* simply the totality of tetrads under another name. The conception of such a totality is no more abstruse than that of 'all the pairs of numbers whose squares differ by 5'; the name given to it must not be allowed to mislead.

In this pamphlet we shall use the word 'point' simply as an abbreviation for 'point in four-dimensional space', for we are concerned with no other application of the word.

1·2. The four constituent numbers that determine a point are called the *absolute coordinates* of the point. In the examples of points given in the first paragraph, the first absolute coordinate of the first point is 2, the fourth absolute coordinate of the second point is $-\frac{1}{4}$, and the third point has both its second absolute coordinate and its third equal to 1. Often it is convenient to use single letters to denote points and to denote the absolute coordinates of a point P by ξ_P, η_P, ζ_P, τ_P; for example, to write

1·21 $$\xi_A = -\tfrac{7}{3}, \quad \eta_A = \tfrac{1}{2}, \quad \zeta_A = 0, \quad \tau_A = -\tfrac{4}{5}$$

is one way of expressing that A is to be used for a time as a symbol for the particular tetrad $(-\frac{7}{3}, \frac{1}{2}, 0, -\frac{4}{5})$.

There is one point whose absolute coordinates are all zero; this point is called the absolute origin and is denoted by O.

1·3. The symbol of equality is used of points only to express actual identity. The points

$$(a, b, c, d), \ (e, f, g, h)$$

1—2

are identical if and only if the four conditions

1·31 $a=e, \quad b=f, \quad c=g, \quad d=h$

are all satisfied, and it is therefore natural to use

1·32 $(a, b, c, d)=(e, f, g, h)$

as a compact substitute for the set of equations 1·31. If the points are denoted by P, Q, their identity is expressed simply by $P=Q$.

2. STEPS AND VECTORS

2·1. The ordered pair of points (P, Q) is described as the *step* PQ, and is said to have P for its beginning and Q for its end. The absolute coordinates of Q can be derived from the absolute coordinates of P by the addition of the numbers $\xi_Q-\xi_P$, $\eta_Q-\eta_P$, $\zeta_Q-\zeta_P$, $\tau_Q-\tau_P$; these numbers form a tetrad, that is, a point, but our language would acquire an unfamiliar tone if we were to speak of being directed from one point to another point by means of a third *point*, and therefore we avail ourselves for the tetrad in this connection of the alternative name of *vector*, which is introduced for use whenever the name of point is undesirable. There is no difference in definition between the point (a, b, c, d) and the vector (a, b, c, d), but the point is said rather to *represent* the vector than to *be* the vector.

Single letters used to denote tetrads that are being regarded as vectors will be in Clarendon type. The vector $\mathbf{r}$ is the tetrad $(\xi_\mathbf{r}, \eta_\mathbf{r}, \zeta_\mathbf{r}, \tau_\mathbf{r})$; the constituent numbers of the tetrad are called the *absolute components* of the vector. If P, Q are points, the vector $(\xi_Q-\xi_P, \eta_Q-\eta_P, \zeta_Q-\zeta_P, \tau_Q-\tau_P)$, which is called the vector of the step PQ, is sometimes denoted by $Q-P$. Given a point P and a vector $\mathbf{r}$, there is one and only one step from P which has the vector $\mathbf{r}$; this is the step from P to $(\xi_P+\xi_\mathbf{r}, \eta_P+\eta_\mathbf{r}, \zeta_P+\zeta_\mathbf{r}, \tau_P+\tau_\mathbf{r})$, and the point which is the end of the step is said to represent $\mathbf{r}$ with respect to P. The point whose absolute coordinates are the absolute components of $\mathbf{r}$ represents the vector $\mathbf{r}$ with respect to the absolute origin O.

2·2. Two steps are said to be *congruent* if they have the same vector. The congruence of the steps PQ, RS is expressed symbolically in the form

2·21 $Q-P=S-R.$

Since the set of equations

2·22 $$\xi_Q - \xi_P = \xi_S - \xi_R, \quad \eta_Q - \eta_P = \eta_S - \eta_R,$$
$$\zeta_Q - \zeta_P = \zeta_S - \zeta_R, \quad \tau_Q - \tau_P = \tau_S - \tau_R$$

is equivalent to

2·23 $$\xi_R - \xi_P = \xi_S - \xi_Q, \quad \eta_R - \eta_P = \eta_S - \eta_Q,$$
$$\zeta_R - \zeta_P = \zeta_S - \zeta_Q, \quad \tau_R - \tau_P = \tau_S - \tau_Q,$$

congruence of PQ with RS is equivalent to congruence of PR with QS. Moreover, the symbolical expression of the congruence in the latter form is

2·24 $$R - P = S - Q,$$

and comparison of this with 2·21 shews that the straightforward use of the notation will not mislead.

Ex. i*. The pair of points $\{(2, -1, 4, 1), (4, 3, -2, 1)\}$ is a step whose vector is the tetrad $(2, 4, -6, 0)$; the step from the point $(-4, 1, 3, 3)$ with the same vector ends at the point $(-2, 5, -3, 3)$.

2·3. By the product of the vector $\mathbf{r}$ by the number k, positive, zero, or negative, is meant the vector $(k\xi_{\mathbf{r}}, k\eta_{\mathbf{r}}, k\zeta_{\mathbf{r}}, k\tau_{\mathbf{r}})$; this vector is denoted by $k\mathbf{r}$.

The sum of any finite number of vectors is definable as the vector each of whose absolute components is the sum of the absolute components of the individual vectors. That is to say, the expression

2·31 $$(\xi_1, \eta_1, \zeta_1, \tau_1) + (\xi_2, \eta_2, \zeta_2, \tau_2) + \dots + (\xi_n, \eta_n, \zeta_n, \tau_n)$$

is defined to denote the vector

2·32
$$(\xi_1 + \xi_2 + \dots + \xi_n, \eta_1 + \eta_2 + \dots + \eta_n, \zeta_1 + \zeta_2 + \dots + \zeta_n, \tau_1 + \tau_2 + \dots + \tau_n).$$

Since a change in the order of the vectors in 2·31 only deranges the terms in the numerical sums which give the constituents in 2·32, any such change is without effect on the constituents, and therefore is without effect on the vector itself: the sum of a number of vectors does not depend on the order in which the vectors are taken.

If starting from any point Q_0 we form a chain of points

* To be brief and clear, the examples are constructed for the most part with whole numbers; it must be remembered that no restriction of this kind is imposed by the definitions.

$Q_0 Q_1 Q_2 \ldots Q_n$ such that the successive steps $Q_0 Q_1, Q_1 Q_2, \ldots, Q_{n-1} Q_n$ have given vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, then

2·33 $\xi_{Q_1} - \xi_{Q_0} = \xi_{\mathbf{r}_1}, \ \xi_{Q_2} - \xi_{Q_1} = \xi_{\mathbf{r}_2}, \ \ldots, \ \xi_{Q_n} - \xi_{Q_{n-1}} = \xi_{\mathbf{r}_n},$

and by addition

2·34 $$\xi_{Q_n} - \xi_{Q_0} = \xi_{\mathbf{r}_1} + \xi_{\mathbf{r}_2} + \ldots + \xi_{\mathbf{r}_n};$$

similar results follow for the other coordinates, and therefore $\mathbf{r}_1 + \mathbf{r}_2 + \ldots + \mathbf{r}_n$ is the vector of the step $Q_0 Q_n$. It is this proposition which gives importance to the vector $\mathbf{r}_1 + \mathbf{r}_2 + \ldots + \mathbf{r}_n$, but to define the sum by means of the chain of steps would render it necessary to prove the sum independent not only of the order in which the vectors were taken but also of the point Q_0 from which the chain was begun. It is to be noticed that the relations between the points and the vectors can be written in the form

2·35 $$Q_1 - Q_0 = \mathbf{r}_1, \ Q_2 - Q_1 = \mathbf{r}_2, \ \ldots, \ Q_n - Q_{n-1} = \mathbf{r}_n,$$

and that the conclusion that would be reached by adding the equations as if the symbols were algebraic is the correct conclusion

2·36 $$Q_n - Q_0 = \mathbf{r}_1 + \mathbf{r}_2 + \ldots + \mathbf{r}_n.$$

Ex. ii. The step from $(4, 3, -2, 1)$ with the vector $(0, 1, 4, -2)$ ends at $(4, 4, 2, -1)$. If we add this vector to the vector of the step $\{(2, -1, 4, 1), (4, 3, -2, 1)\}$, we have the vector $(2, 5, -2, -2)$, which is the vector of the step straight from $(2, -1, 4, 1)$ to $(4, 4, 2, -1)$. The same addition can be performed by regarding $(2, 4, -6, 0)$, $(0, 1, 4, -2)$ as the vectors of

$$\{(-4, 1, 3, 3), (-2, 5, -3, 3)\}, \quad \{(-2, 5, -3, 3), (-2, 6, 1, 1)\}$$

and $(2, 5, -2, -2)$ as the vector of $\{(-4, 1, 3, 3), (-2, 6, 1, 1)\}$.

2·4. Subtraction may be defined in terms of addition or by means of the absolute components. The equation

2·41 $$\mathbf{r} - \mathbf{s} = \mathbf{t}$$

is equivalent to

2·42 $$\mathbf{r} = \mathbf{s} + \mathbf{t}$$

and also to the set of equations

2·43 $\xi_{\mathbf{r}} - \xi_{\mathbf{s}} = \xi_{\mathbf{t}}, \ \eta_{\mathbf{r}} - \eta_{\mathbf{s}} = \eta_{\mathbf{t}}, \ \zeta_{\mathbf{r}} - \zeta_{\mathbf{s}} = \zeta_{\mathbf{t}}, \ \tau_{\mathbf{r}} - \tau_{\mathbf{s}} = \tau_{\mathbf{t}}.$

If $-\mathbf{r}$ is defined to mean the product of $\mathbf{r}$ by -1, which is the vector $(-\xi_{\mathbf{r}}, -\eta_{\mathbf{r}}, -\zeta_{\mathbf{r}}, -\tau_{\mathbf{r}})$, subtraction of $\mathbf{r}$ is equivalent to addition of $-\mathbf{r}$.

If any vector is subtracted from itself, the result is the vector (0, 0, 0, 0), which is called the zero vector. This is the vector of any step in which the end coincides with the beginning. Just as it is often advantageous to express a linear relation

$$p_1x_1 + p_2x_2 + \ldots + p_mx_m = q_1y_1 + q_2y_2 + \ldots + q_ny_n$$

between algebraic variables in the form

$$p_1x_1 + p_2x_2 + \ldots + p_mx_m - q_1y_1 - q_2y_2 - \ldots - q_ny_n = 0,$$

so the linear relation

2·44 $$p_1\mathbf{r}_1 + p_2\mathbf{r}_2 + \ldots + p_m\mathbf{r}_m = q_1\mathbf{s}_1 + q_2\mathbf{s}_2 + \ldots + q_n\mathbf{s}_n$$

between vectors is usefully taken in the form

2·45

$$p_1\mathbf{r}_1 + p_2\mathbf{r}_2 + \ldots + p_m\mathbf{r}_m - q_1\mathbf{s}_1 - q_2\mathbf{s}_2 - \ldots - q_n\mathbf{s}_n = (0, 0, 0, 0).$$

For brevity, the zero vector (0, 0, 0, 0) is denoted simply by **0**. We have, for any vector $\mathbf{r}$,

2·46 $$\mathbf{r} + \mathbf{0} = \mathbf{r}, \quad 0 \times \mathbf{r} = \mathbf{0},$$

and for any number k,

2·47 $$k \times \mathbf{0} = \mathbf{0}.$$

The vector **0** is the vector that is represented by the origin.

A vector that is not the zero vector is said to be a *proper* vector. The equation

2·48 $$k\mathbf{r} = \mathbf{0}$$

implies that either k is zero or $\mathbf{r}$ is the zero vector.

Ex. iii. There is nothing elusive about a zero step. The step

$$\{(3, \tfrac{1}{2}, 0, -\tfrac{1}{3}), (3, \tfrac{1}{2}, 0, -\tfrac{1}{3})\}$$

is no more subtle than any other group of eight numbers regarded as a pair of tetrads.

3. VECLINES, VECPLANES, AND VECSPACES

3·1. If $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ are given vectors and $x_1, x_2, \ldots, x_n$ are unknown numbers, the equation

3·11 $$x_1\mathbf{r}_1 + x_2\mathbf{r}_2 + \ldots + x_n\mathbf{r}_n = \mathbf{0}$$

is equivalent to the set of four simultaneous algebraic equations

3·12 $$\begin{cases} x_1\xi_1 + x_2\xi_2 + \ldots + x_n\xi_n = 0, \\ x_1\eta_1 + x_2\eta_2 + \ldots + x_n\eta_n = 0, \\ x_1\zeta_1 + x_2\zeta_2 + \ldots + x_n\zeta_n = 0, \\ x_1\tau_1 + x_2\tau_2 + \ldots + x_n\tau_n = 0, \end{cases}$$

where ξ_m, η_m, ζ_m, τ_m are the absolute components of $\mathbf{r}_m$. The equations in this set are of course all satisfied if $x_1, x_2, \ldots, x_n$ are all zero; since only the ratios of $x_1, x_2, \ldots, x_n$ to one another are involved in the equations, we infer from 3·12 that a solution of 3·11 with $x_1, x_2, \ldots, x_n$ not all zero is certainly possible if n is greater than 4 but is not possible if n is not greater than 4 unless the vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ are related to each other in some special way. We say that the vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ are *linearly related* if there are numbers $x_1, x_2, \ldots, x_n$ not all zero which render 3·11 an identity; if all the vectors are proper, *two* or more of the numbers must be different from zero. The relation 3·11 is not distinguished from the relation obtained by multiplying throughout by a constant other than zero, and the deduction we make from 3·12 can be expressed in the form that between any five or more vectors there is at least one effective linear relation, the word 'effective' conveying the condition that the coefficients are not all zero.

For the theory of systems of linear algebraic equations, the reader is referred to books on algebra. Perhaps the best account for the English student is in Bôcher's *Introduction to Higher Algebra* (The Macmillan Co., 1907), a masterpiece of exposition.

3·2. Two vectors $\mathbf{a}, \mathbf{b}$ are said to be *collinear* if there are numbers a, b not both zero such that

3·21 $$a\mathbf{a} + b\mathbf{b} = \mathbf{0}.$$

If $\mathbf{a}$ is the zero vector, we can satisfy 3·21 by taking b to be zero, whatever the vector $\mathbf{b}$: the zero vector is collinear with every vector. If $\mathbf{a}$ is not $\mathbf{0}$, then b is not zero, and 3·21 is equivalent to

3·22 $$\mathbf{b} = (-a/b)\,\mathbf{a}:$$

the vectors collinear with a proper vector are the multiples of that vector.

The vectors collinear with a proper vector $\mathbf{a}$ compose a class of vectors that will be called the *vecline** built on $\mathbf{a}$. If $\mathbf{r}$ is any member of this class, there is by hypothesis one number $x_{\mathbf{r}}$ which is such that

3·23 $$\mathbf{r} = x_{\mathbf{r}}\mathbf{a},$$

and this number is unique, for the two equations

$$\mathbf{r} = x_{\mathbf{r}}'\mathbf{a},\ \ \mathbf{r} = x_{\mathbf{r}}''\mathbf{a}$$

* To the best of my belief, this word and others on the same model that will be used later are new.

imply $$(x_{\mathbf{r}}' - x_{\mathbf{r}}'')\,\mathbf{a} = \mathbf{0}$$
and therefore $$x_{\mathbf{r}}' = x_{\mathbf{r}}'',$$
since **a** is proper.

If **f** is a proper vector in the vecline built on **a**, then since every multiple of **f** is a multiple of **a**, every vector that belongs to the vecline built on **f** belongs to the vecline built on **a**; but if **f** is proper and equal to $x_{\mathbf{f}}\mathbf{a}$, the coefficient $x_{\mathbf{f}}$ is not zero and **a** is expressible as $(1/x_{\mathbf{f}})\,\mathbf{f}$; hence **a** is a proper vector in the vecline built on **f**, and every vector that belongs to the vecline built on **a** belongs to the vecline built on **f**. It follows that the vecline built on **f** coincides with the vecline built on **a**, that is, that no two veclines have a proper vector in common; in other words, the vecline built on a proper vector **a** is the only vecline of which **a** is a member, and this vecline can therefore be described simply as the vecline that contains **a**. But every vecline contains the zero vector.

Ex. iv. The vecline containing $(2, 1, -2, -1)$ is the class of all tetrads of the form $(2x, x, -2x, -x)$. The vector $(-3, -\frac{3}{2}, 3, \frac{3}{2})$, which is the product of the original vector by $-\frac{3}{2}$, belongs to the vecline. The vector $(5, \frac{5}{2}, 15, -\frac{5}{2})$ does not, for the *only* multiple of $(2, 1, -2, -1)$ whose first constituent is 5 is the product by $\frac{5}{2}$, and this differs from $(5, \frac{5}{2}, 15, -\frac{5}{2})$ in the third constituent. The vector $x\,(2, 1, -2, -1)$ is expressible as $-\frac{2}{3}x\,(-3, -\frac{3}{2}, 3, \frac{3}{2})$, and conversely the vector $X\,(-3, -\frac{3}{2}, 3, \frac{3}{2})$ is identical with $-\frac{3}{2}X\,(2, 1, -2, -1)$; that is, every vector in the vecline built on either of the vectors $(2, 1, -2, -1)$, $(-3, -\frac{3}{2}, 3, \frac{3}{2})$ belongs to the vecline built on the other of these two vectors.

3·3. Three vectors **a**, **b**, **c** are said to be *coplanar* if there are numbers a, b, c not all zero such that

3·31 $$a\mathbf{a} + b\mathbf{b} + c\mathbf{c} = \mathbf{0}.$$

If **a** and **b** are collinear, there are values of a and b not both zero such that $a\mathbf{a} + b\mathbf{b}$ is the zero vector, and we can satisfy 3·31 by taking a and b to have such values and c to be zero, whatever the vector **c**: if two vectors are collinear, every vector is coplanar with them. If **a** and **b** are not collinear, 3·31 can not be satisfied if c is zero, and is therefore equivalent to

3·32 $$\mathbf{c} = (-a/c)\,\mathbf{a} + (-b/c)\,\mathbf{b}.$$

The vectors coplanar with two vectors **a**, **b** that are not themselves collinear compose a class that will be called temporarily the *vecplane* built on **a** and **b**. If **r** belongs to this vecplane, there is a pair of numbers $(x_{\mathbf{r}}, y_{\mathbf{r}})$ such that

3·33 $$\mathbf{r} = x_{\mathbf{r}}\mathbf{a} + y_{\mathbf{r}}\mathbf{b},$$

and this pair of numbers is unique, for the two equations

$$\mathbf{r} = x_{\mathbf{r}}'\mathbf{a} + y_{\mathbf{r}}'\mathbf{b}, \quad \mathbf{r} = x_{\mathbf{r}}''\mathbf{a} + y_{\mathbf{r}}''\mathbf{b}$$

imply $$(x_{\mathbf{r}}' - x_{\mathbf{r}}'')\,\mathbf{a} + (y_{\mathbf{r}}' - y_{\mathbf{r}}'')\,\mathbf{b} = \mathbf{0},$$

and to suppose this last equation, which is of the form of 3·21, satisfied except by $$x_{\mathbf{r}}' = x_{\mathbf{r}}'', \quad y_{\mathbf{r}}' = y_{\mathbf{r}}''$$

would be to suppose **a** and **b** collinear.

Let **f** be a vector in the vecplane built on **a** and **b** but not collinear with **b**. Then there is a relation

3·34 $$\mathbf{f} = x_{\mathbf{f}}\mathbf{a} + y_{\mathbf{f}}\mathbf{b},$$

and therefore any vector **r** which is expressible as $p\mathbf{f} + q\mathbf{b}$ is expressible as $(px_{\mathbf{f}})\,\mathbf{a} + (py_{\mathbf{f}} + q)\,\mathbf{b}$; that is, every vector that belongs to the vecplane built on **f** and **b** belongs to the vecplane built on **a** and **b**. But since **f** is not a multiple of **b**, the coefficient $x_{\mathbf{f}}$ in 3·34 is not zero, and 3·34 is equivalent to

3·35 $$\mathbf{a} = (1/x_{\mathbf{f}})\,\mathbf{f} + (-\,y_{\mathbf{f}}/x_{\mathbf{f}})\,\mathbf{b};$$

hence **a** is a vector in the vecplane built on **f** and **b**, and therefore every vector that belongs to the vecplane built on **a** and **b** belongs to the vecplane built on **f** and **b**. Combining the two results, we conclude that the vecplane built on **f** and **b** coincides with the vecplane built on **a** and **b**. If further **g** is any vector in the vecplane built on **f** and **b** but not collinear with **f**, a repetition of the argument shews that the vecplane built on **f** and **g** coincides with the vecplane built on **f** and **b** and coincides therefore with the original vecplane. The proof fails if **f** is collinear with **b**, but in that case **f** is not collinear with **a**, and the vecplanes built on **a** and **b** and on **f** and **g** can both be compared with the vecplane built on **a** and **f**. Thus if **f** and **g** are any two vectors that belong to the vecplane built on **a** and **b** and are not themselves collinear, the vecplane built on **f** and **g** is identical with the vecplane built on **a** and **b**. In other words, the vecplane built on two vectors that are not collinear is the only vecplane that contains them both. The vecplane containing two vectors **a**, **b** is sometimes called simply the vecplane **ab**.

Obviously a vecplane that contains a proper vector **a** includes the whole of the vecline that contains **a**. The conclusion of the last paragraph can therefore be expressed in the form that if two