CHAPTER 1

NUMBER

1·1. Introduction.

The foundation upon which the whole structure of the subject of Mathematical Analysis rests is the *theory of real numbers.*

Accordingly an obvious starting-point for our study of this subject is the series of "natural numbers"

$$1, 2, 3, \ldots, n, \ldots.$$

These numbers are so familiar that it seems quite reasonable to assume that the concept of "number" is one of our primitive, or even intuitive notions, and that consequently it does not require *definition.* In fact most of the existing text-books do begin by accepting the natural numbers as "known," and therefore as not requiring definition. The reader who is satisfied with this point of view will be saved a good deal of preliminary difficulty by omitting much of the work with which the present chapter is concerned.

The problem with which we are faced is to decide what we are to accept as "*given.*" It would certainly be the *easiest* way out of the difficulty to accept the natural numbers as "given"; and without going outside what is usually understood to be "mathematics," this is the only reasonable starting-point which can be made.

There are two main reasons why we shall not accept the concept of number as a primitive concept which does not require definition. One reason is that by doing so we might be in danger of leading the reader to think that the foundations of the subject cannot be based upon anything more fundamental than "number" as a primitive undefinable concept. The second reason is that our preliminary investigation of a logical definition of number is the natural introduction to one of the best methods of defining an irrational number.

It is often stated that the foundations of Real Variable Theory have not yet reached an entirely satisfactory position, and to a certain extent this may be true; but no reader who wishes to make a systematic study of modern Analysis ought to remain in entire ignorance of the field of study which has been opened up by Frege, Bertrand Russell and others* in reducing to logic those arithmetical notions which had previously been shewn by Peano to be sufficient for mathematics. To consider this question fully would make this chapter unduly lengthy; accordingly it has been thought sufficient to indicate the main essential ideas, and leave the reader to consult other treatises for a more detailed explanation†.

No science is entirely self-contained; each borrows the strength of its ultimate foundations from something outside itself, such as experience, or logic or metaphysics. This is the case with Pure Mathematics, and the definition of number which is given in this chapter is a *logical* definition; hence some knowledge of that field of study which has come to be known as "mathematical philosophy" is unavoidable.

Although the natural numbers seem to represent what is easiest and most familiar in mathematics, very few people would be prepared with a *definition* of what is meant by "number" or "1" or "3," and a much greater difficulty arises when we consider how to define "0."

It may be remarked that 0 is a recent addition to the series of natural numbers, and the Greeks and Romans had no such digit.

All the essential ideas involved in the logical definition of number will be explained as simply and as untechnically as possible; but however carefully we attempt to avoid difficult and unfamiliar phraseology, this chapter will unavoidably appear difficult, and perhaps artificial to the beginner. As we have already remarked, however, we must deal with these unfamiliar ideas, unless we are content to shirk all the difficulties by the unjustifiable assumption that we already understand what is meant by "number."

Historically, the progress of mathematics has been *constructive* in the direction of rapidly increasing complexity. In the case of number, with which we are now concerned, the natural numbers

$$1, 2, 3, \ldots, n, \ldots$$

* Frege's *Grundlagen der Arithmetik* (1884) gave the first correct logical definition of number, but the book attracted little attention, and its contents remained practically unknown until they were rediscovered by Russell in 1901.

† The reader may profitably consult B. Russell's *Introduction to Mathematical Philosophy*, and J. E. Littlewood's *Elements of the Theory of Real Functions* (1926).

were the first to be considered, and their earliest use was in an ordinal sense, when they were employed for the purpose of *counting*. Our familiarity with the use of the natural numbers for counting is one of the chief obstacles to be removed before we are able to give a satisfactory definition of a cardinal number. The commonest every-day use of numbers (for the purpose of counting) is just the aspect of number which is least helpful for this purpose, and the definition of a *cardinal number* must not involve the use of counting. The importance of the distinction between cardinal and ordinal numbers will be emphasised as we proceed; and the reader will see later that counting, although so familiar, is logically a very complex operation. All that need be said at the moment is that counting employs the natural numbers in an ordinal sense, and the *logical* definition of "order" and "ordinal number" is by no means easy.

The impossibility of defining a cardinal number by the process of enumeration is very obvious when viewed psychologically. "Counting," it is said, "consists of successive acts of attention; the result of such a succession is a number." In other words, "the number *seven* is the result of seven acts of attention." This makes the vicious circle obvious.

The introduction of fractions into arithmetic was the next step, these arising naturally in connection with the problem of *measurement*; and their introduction was comparatively easy. On the contrary the negative numbers caused a great deal of trouble. For some time negative numbers were called absurd and fictitious, and the fact that the product of $-a$ and $-b$ could give a positive number ab was for a long time a difficulty to many minds*. The subsequent introduction of irrational numbers, such as $\sqrt{2}$, $\sqrt[3]{6}$, π and e, did not excite much comment. In actual calculations approximate rational values were used, and it seemed quite natural to subject them to the same laws as rational numbers. Irrational numbers arose first in connection with geometry, with the discovery by the early Greek geometers that there is no fraction of which the square is 2, a result which naturally emerges out of the problem of determining the length of the diagonal of a

* In the latter half of the eighteenth century, Maseres (1731–1824) and Frend (1757–1841) published works on Algebra and Trigonometry in which the use of negative numbers was disallowed, although Descartes had used them freely more than a hundred years before.

unit square. With the invention of algebra the same question arose in the solution of equations, but here it took a wider form involving also complex numbers, which will be discussed later.

Although irrational numbers were discovered as early as the time of Pythagoras, no real advance towards constructing a *rigorous* theory of irrational numbers was made until the time of Weierstrass (1815–97) and Dedekind (1831–1916). It may be remarked that if we agree to accept the natural numbers as fundamental, and thereby avoid the necessity of considering any mathematical philosophy, even then some rigorous theory of irrational numbers is necessary before Analysis can be founded on a satisfactory basis. The definition of an irrational number which will subsequently be given is due to B. Russell, and it is a slight modification of Dedekind's method.

The other method of pursuing the study of mathematics is the reverse of the historical order of progress. Instead of pursuing the constructive process towards increasing complexity, we proceed, by analysing, to greater abstractness and logical simplicity. Instead of considering what can be defined and deduced from our initial assumptions, we examine whether more general ideas and principles can be found in terms of which our original starting-point can be defined or deduced. This second method is what characterises the study which has come to be known as mathematical philosophy. Thus, if our foundations are to be based farther back than on the mere postulation of the existence of the natural numbers, it can only be done by considering some of the questions with which mathematical philosophy is concerned.

1·2. Fundamental notions.

We now state what concepts must be taken as fundamental in order to give a definition of number. The following remarks may perhaps clear the ideas of the beginner, and help him to appreciate the definition which will subsequently be given. A trio of men is an instance of the number 3, and the number 3 is an instance of number, but the trio itself is not an instance of number. The number 3 is not identical with any collection of terms having that number; it is something which all trios have in common, and which distinguishes them from other collections. It brings us a step nearer to the correct definition when we realise that number is to be regarded as something which characterises certain *collections*. The reference to a "collection" of objects introduces the first important concept, that of an aggregate.

An *aggregate* (or *collection*) of objects which is conceived of as containing more or fewer objects is a concept which will be taken as primitive, and no attempt is made to *define* it. A great deal can be known about an aggregate without our being able to enumerate its members. The elements composing an aggregate need not possess any parity as regards size or any other special quality. For example, an aggregate may be "all the living creatures in the city of London," "all the trees in a certain garden," or any other collection of entities of entirely diverse characteristics.

An aggregate, considered quite apart from the order of its members, is termed a CLASS or SET.

A great part of mathematical philosophy is concerned with RELATIONS, and although only a few important relations enter into the discussions in this book, a few remarks may be helpful to the beginner.

Amongst the most important kinds of relations is the class of "one-many," "many-one," and "one-one" relations*. If A and B denote two sets of entities, the relation between these two sets is "one-many," if more than one member of the set B bears the given relation to each member of the set A.

The following examples will make the ideas clearer. In countries where monogamy is practised the relation of *husband to wife* is one-one, in polygamous countries the relation is one-many, and in Tibet, where polyandry is practised, the relation is many-one. The relation of *father to son* is one-many, that of *son to father* is many-one, but that of *eldest son to father* is one-one.

The *domain* of a relation consists of all those terms which have the relation to something or other, and the *converse domain* consists of all those terms to which something or other has the relation.

The *field* of a relation consists of the domain and the converse domain together.

* The use of the word "one" in the description of these relations is justifiable, for a meaning can be assigned to the above relations which does not require any concept of the cardinal number "1." For an interesting remark on this point, see Littlewood's *Elements of the Theory of Real Functions* (1926), p. 2.

A further notion which is required is that of CORRESPONDENCE: this is the notion which underlies the process of tallying. The elements of one aggregate may be made to stand in some logical relation with those of another so that a definite element of one aggregate is regarded as correspondent to a definite element of another aggregate.

Two aggregates which are such that to each element of the first there corresponds one and only one element of the second, and to each element of the second there corresponds only one element of the first, are said to be in ONE-ONE CORRESPONDENCE.

1·21. Definition of number.

Two aggregates are said to be SIMILAR when there is a one-one correspondence which correlates their elements. Suppose now that all couples are in one bundle, all trios in another, and so on. In this way we obtain various bundles of collections. Each bundle is a set whose members are classes; thus each is a *set of classes*. To decide when two collections are to belong to the same bundle we use the notion of similarity defined above. Given any aggregate, we can define the bundle to which it must belong as being the set of all those aggregates which are similar to it.

We therefore give the following definition:

The NUMBER *of a class is the set of all those classes that are similar to it.*

According to this definition the set of *all* couples is the number 2; the set of *all* trios is the number 3, and so on.

Numbers in general have been defined as bundles into which similarity collects classes. A number is a set of classes such that any two of the classes are similar to each other, and none outside the set is similar to any inside the set.

On the same lines the number 0 can be defined. The number 0 is the number of terms in a class which has no members, and this class is called the *null class*. By the general definition of number, the number of terms in the null class is the set of all classes which are similar to the null class, and this is easily seen to be the set whose *only* member is the null class. The purely logical definition of the number 0 may therefore be given as follows:

The number 0 is the set whose only member is the null class.

1·3. Relations.

The important type of relation known as a one-one relation has already been mentioned. The concept of one-one correspondence is of fundamental importance, for upon it depends the definition of number given above. There are, however, many other kinds of relations, and one very important type, "serial relations," will be needed when we define "order."

The following examples of "one-one" relations may assist the reader to assimilate some of the essential ideas involved.

Take the first ten integers (excluding 0),

$$1, 2, 3, \ldots, 8, 9, 10 \quad \ldots\ldots\ldots\ldots(1).$$

This set can clearly be correlated with the set of integers

$$2, 3, 4, \ldots, 9, 10, 11 \quad \ldots\ldots\ldots\ldots(2);$$

and the relation which correlates these two classes can be described as *the relation of n to $n+1$*. The relation is clearly one-one; also the domain and converse domain overlap, for all the members of class (1), save the first, are repeated in class (2), and class (2) contains only one new member, 11.

If, instead of the relation of n to $n+1$, we take *the relation of a number to its square*, we obtain the set of integers

$$1, 4, 9, \ldots, 64, 81, 100 \quad \ldots\ldots\ldots\ldots(3),$$

which also bears a one-one relation to the set (1).

1·31. Serial relations.

An idea which obviously calls for attention is that of an aggregate whose members are arranged in a certain order. In defining number we considered aggregates quite apart from any question of order among their members. The numbers which we were able to define in this way are called CARDINAL NUMBERS.

An ordered aggregate, which appears to be a greater complexity than a class, is resolved in quite a simple way; but before we can deal with ordered aggregates, serial relations must be understood. It will be seen that an ordered aggregate leads naturally to the definition of an ordinal number.

In seeking a definition of order, the first thing to realise is that no set of terms has just *one* order to the exclusion of others; a set of terms has all the orders of which it is capable. It is true that the natural numbers (due to their employment for the purpose of counting) occur to us most readily in order of magnitude, but they are capable of an unlimited number of other arrangements. The definition of order is not therefore to be sought in the nature of the set of terms to be ordered, since the same set of terms has many different orders. The order lies, not in the *set* of terms, but in a relation among the members of the set, in respect of which some appear as earlier and some as later.

The essential characteristics of a relation which is to give rise to order may be discovered by considering that in respect of such a relation we must be able to say, of any two terms in the set which is to be ordered, that one "precedes" and the other "follows." We require the ordering relation to have the following three properties:

(1) ASYMMETRY. If x precedes y, y must not also precede x. We say that *any given relation is asymmetrical when, if it holds from x to y, then it does not hold from y to x.*

Relations which do not give rise to series often do not have the property of asymmetry. The relation "is the cousin of" is an example of this, for if x is the cousin of y, y is also the cousin of x.

(2) TRANSITIVENESS. If x precedes y and y precedes z, then x must also precede z. *A relation is transitive when, if it holds from x to y and from y to z, it also holds from x to z.*

The relation considered above is not transitive, for if x is the cousin of y, and y of z, x may not be the cousin of z, for x and z may be the same person. The relation "sameness of height" is transitive, but not asymmetrical. The relation "father" is asymmetrical but not transitive. The reader is advised to construct examples for himself.

(3) CONNECTIVITY. Given any two terms of the class which is to be ordered, there must be one which precedes and the other which follows.

A relation is connected, if given any two members x and y of its field, then either the relation holds from x to y or from y to x.

The relation "ancestor" has the first two properties, but not the third*. Its failure to possess the third property makes it an insufficient relation to arrange the human race in a series.

DEFINITION. *A relation is serial if it is asymmetrical, transitive and connected.*

A *series* is the same thing as a serial relation.

* We imply of course that the field of the relation considered is the human race.

1·32. Order.

The three properties required in order that any given relation may be serial have now been examined. It is not possible, without greatly increasing the number of new ideas and technical terms, to go very deeply into the question of *order*; but to be able to understand the fundamental relation "less than" the following definitions are required:

(1) A property is said to be *hereditary* in the natural number series, if whenever it belongs to a number n it also belongs to $n+1$, the successor of n.

(2) The *successor* of the number of terms in the class A is the number of terms in the class consisting of A together with x, where x is any term not belonging to A.

(3) A number m is said to be *less than* another number n when n possesses every hereditary property possessed by the successor of m.

It is not difficult to prove that the relation "less than" so defined is serial, and that it has the finite cardinal numbers for its field. By means of the relation "less than" the cardinal numbers acquire an *order*, and this order is the so-called "natural" order, or order of magnitude*.

The generation of series by means of relations more or less resembling that of n to $n+1$ is very common. The series of kings of England, for example, is generated by relations of each to his successor. This is probably the easiest way, where it is applicable, of conceiving the generation of a series.

1·33. Cardinal and ordinal numbers.

The distinction between a cardinal and an ordinal number is rendered difficult by the fact that each finite positive integer is made to serve two distinct purposes; it may be used to *count*, when it is acting in the ordinal sense, and it may be used to *number*, when it is acting as a cardinal number. Symbolically there is no distinction whatever between a cardinal and an ordinal number, but logically there is a fundamental difference between them. As we have already seen, the question of order is irrelevant to the definition of a cardinal number; and it is partly in order to secure this important distinction between cardinal and ordinal numbers that the discussion of "serial relations" and "order" has been given.

* The questions considered in this section are only briefly mentioned. For fuller reference the reader is referred to Russell's book, *loc. cit.* Chs. I–VI.

Briefly it may be said that cardinal numbers are obtained from the idea of *equivalence*, ordinal numbers from the idea of *likeness*. The precise significance to be attached to these two terms can be seen in the following definitions:

(1) *Two aggregates A and B are* EQUIVALENT (SIMILAR) *if there is a one-one correspondence between their members.*

(2) *Two series S and T are* LIKE (ORDINALLY SIMILAR) *if there is a one-one correspondence between their members, preserving the order**.

So long as our attention is confined to *finite* aggregates, rearrangement of the members of the aggregate cannot alter the ordinal number; in whatever way we count the members of a finite aggregate we always end up with the same number. Thus all the possible series which arrange the members of a finite class are *like* series.

Two finite series which are like, are said to have the same *ordinal number*, and by analogy with the definition of a cardinal number we can give the following definition:

The ORDINAL NUMBER *of a series is the set of all those ordered aggregates that are like (ordinally similar to) it.*

The important distinction between cardinal and ordinal numbers is best seen by considering *infinite aggregates*. A simple example will suffice.

Consider the two series

$$\text{(A)}\quad 1, 2, 3, \ldots, n, \ldots, 0,$$

$$\text{(B)}\quad 1, 2, 3, \ldots, n, \ldots.$$

The series (A) and (B) are not ordinally similar, for there is not a one-one correspondence which correlates each term of (A) with each term of (B) and which *preserves the order*, for there is no term in (B) with which the term 0 of (A) tallies. The ordinal number of the series (B) is usually denoted by ω, and the ordinal number of the series (A) is then denoted by $\omega+1$.

If these two series are considered as classes, no account being taken of order, there is a one-one correspondence between them.

$$\text{(A}'\text{)}\quad 0, 1, 2, 3, \ldots, n, \ldots,$$

$$\text{(B)}\quad 1, 2, 3, 4, \ldots, n+1, \ldots.$$

When (A) has been rewritten as (A′), the classes (A′) and (B) are correlated by the one-one relation of n to $n+1$. Thus both (A′) and (B) are classes which belong to the same set of classes, and the cardinal number which is characteristic of this particular set is denoted by $\aleph_0$ (Aleph zero).

* That is to say, a one-one correspondence in which, if x', y' correspond to any x, y, then x' precedes y' if x precedes y.