CHAPTER I

FUNDAMENTAL IDEAS

§ 1. The fundamental equations for free aether.

In Maxwell's electromagnetic theory the state of the aether in the vicinity of a point (x, y, z) at time t is specified by means of two vectors E and H which satisfy the circuital relations*

$$\text{rot}\, H = \frac{1}{c}\frac{\partial E}{\partial t}, \quad \text{rot}\, E = -\frac{1}{c}\frac{\partial H}{\partial t} \quad \text{..............(1)},$$

and the solenoidal or sourceless conditions

$$\text{div}\, E = 0, \qquad \text{div}\, H = 0.$$

If right-handed rectangular axes are used the symbol† rot H denotes the vector whose components are of type

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z},$$

the three components of H being H_x, H_y, H_z respectively. The symbol div H denotes the divergence of H, i.e. the quantity

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}.$$

The vector E is called the *electric displacement* or *electric force* and H the *magnetic force*. The quantity c represents the

* The equations are written in the symmetrical form in which they were presented by O. Heaviside, *Electrical Papers*, Vol. 1, § 30, and H. Hertz, *Electric Waves*, p. 138. Sir Joseph Larmor points out that a set of equations equivalent to these was first used by MacCullagh in 1838 as a scheme consistently covering the whole ground of Physical Optics, *Collected Works* of James MacCullagh (1880), p. 145.

† We use here the units and notation employed in Lorentz's *The Theory of Electrons*, Ch. I, except that large letters are used to denote vectors and E is written in place of D. Many writers use the symbol *curl* instead of rot.

velocity of propagation of homogeneous plane waves and is commonly called the velocity of light; we shall assume it to be a constant, although in the most recent speculations it is treated as variable*.

Some of the modern writers on the theory of relativity maintain that the introduction of the idea of an aether is unnecessary and misleading. Their criticisms are directed chiefly against the popular conception of the aether as a kind of fluid or elastic solid which can be regarded as practically stationary while material and electrified particles move through it. This idea has been very helpful as it presents us with a vivid picture of the processes which may be supposed to take place, it also has the advantage that with its aid we can attach a meaning to the term absolute motion, but herein lies its weakness. Larmor, Lorentz and Einstein have shown, in fact, that the differential equations of the electron theory admit of a group of transformations which can be interpreted to mean that there is no such thing as absolute motion.

If this be admitted, the popular idea of the aether must be regarded as incorrect, and so if we wish to retain the idea of a continuous medium to explain action at a distance we must frankly acknowledge that the simplest description we can give of the properties of our medium is that embodied in the differential equations (1).

If we abandon the idea of a continuous medium in the usual sense only two ways of explaining action at a distance readily suggest themselves. We may either think of the aether as a collection of tubes or filaments attached to the particles of matter as in the form of Faraday's theory which has been developed by Sir Joseph Thomson and N. R. Campbell; or we may suppose that some particle or entity which belonged to an active body at time t belongs to the body acted upon at a later time $t + \tau$. From one point of view these two theories are the same, for if particles are continually emitted from an active

* A. Einstein, *Ann. d. Phys.* Vol. 35 (1911), p. 898; Vol. 38 (1912), pp. 355 and 443. M. Abraham, *Phys. Zeitschr.* (1912), pp. 1—5, 310—314, 793—797; *Ann. d. Phys.* (1912), pp. 444 and 1056; Fifth International Congress of Mathematicians, *Proceedings*, Vol. 2, p. 256.

body they will form a kind of thread attached to it. The first form of the theory is, however, more general than the second.

At present we are unable to form a satisfactory picture of the processes that give rise to, or are represented by, the vectors E and H. We believe, however, that some points may be made clear by studying the properties of solutions of our differential equations.

It will be seen from the investigations of Chapter VIII that the mathematical analysis connected with these equations is suitable for the discussion of three distinct theories of the universe, which may be described briefly as follows:—

Aether	*Matter*
Continuous medium.	Aggregates of discrete particles.
Discontinuous medium consisting of a collection of tubes or filaments.	An aggregate of discrete particles attached to the tubes.
Continuous medium.	An aggregate of discrete particles to which tubes are attached.

The last theory may be supposed to include that form of the emission theory of light in which small entities are projected from the particles of matter under certain circumstances and produce waves in the surrounding medium. This theory might be justly ascribed to Newton*.

For other theories of the aether the reader is referred to Prof. E. T. Whittaker's recent work† *A History of the Theories of the Aether*.

In the first part of this book the analysis is adapted almost entirely to the first theory, the high development of which we owe to the pioneer work of Maxwell, FitzGerald, Hertz, Rayleigh, Heaviside, J. J. Thomson, Lorentz and Larmor. The other theories have not yet received much attention but it is hoped

* A form of the theory in which the entities are electric doublets has been developed by W. H. Bragg and applied to the X and γ rays. *British Association Reports* (1911), p. 340.

† Dublin Univ. Press; Longmans, Green and Co. (1910).

that the analysis of Chapter VIII will lead to further developments so that a comparison can be made between the different theories. It is quite likely that one theory will be enriched by the developments of another.

§ 2. Electromagnetic fields.

For many purposes it is convenient to work with a complex vector* $M = H \pm iE$, where $i = \sqrt{-1}$ and the ambiguous sign $\pm$ is independent of the ambiguity which occurs in the determination of $\sqrt{-1}$. The differential equations (1) may then be replaced by the simpler equations

$$\operatorname{rot} M = \mp \frac{i}{c}\frac{\partial M}{\partial t}, \quad \operatorname{div} M = 0 \qquad \text{(2)}.$$

When a solution of these equations has been found a pair of vectors E and H satisfying equations (1) may be obtained by equating coefficients of the ambiguous sign. In working with an ambiguous sign it must be remembered that when two ambiguous signs are multiplied together the ambiguity is removed. The chief advantage in using the two independent ambiguities $\pm$ and $\sqrt{-1}$ is that we can assume that the vectors E and H are the real parts of expressions of the form $Ae^{i\omega t}$ and we are at liberty to equate the coefficients of either i or $\pm$ in any of our equations.

Definition. A solution of the differential equations (2) or (1), which provides us with single-valued vector functions E and H for each space-time point (x, y, z, t) belonging to a certain domain D, is said to define an electromagnetic field in the domain D.

Since the differential equations are linear the sum of any number of solutions is also a solution. The physical meaning of this is that when two electromagnetic fields are superposed, they are together equivalent to an electromagnetic field.

Two superposed electromagnetic fields can of course be related to one another in some way. When electromagnetic

* The use of a complex vector $H - iE$ is recommended by L. Silberstein, *Ann. d. Phys.* Vols. 22 and 24 (1907); *Phil. Mag.* (6), Vol. 23 (1912), p. 790. He does not, however, use the ambiguous sign.

waves fall upon an obstacle, a secondary disturbance is produced which depends in character upon the nature of both the primary waves and the obstacle.

We shall find that in some cases it is possible to find two fields in which the vectors (E, H), (E', H') are connected by the two relations embodied in the equation

$$(MM') \equiv M_x M_x' + M_y M_y' + M_z M_z' = 0 \quad \text{.........(3)}$$

for all values of (x, y, z, t) belonging to some domain.

When this is the case the fields are said to be *conjugate* within this domain.

If we use the notation

$$(M^2) = M_x^2 + M_y^2 + M_z^2,$$

we may write

$$(M^2) = (H^2) - (E^2) \pm 2i\,(EH) = I_1 \pm 2i\,I_2,$$

where I_1 and I_2 are two quantities which we shall call the invariants*. It is easy to see that when two conjugate fields are superposed the invariant I_1 for the total field is the sum of the invariants I_1 for the two component fields. Similarly for the invariant I_2.

When the invariants are zero over a given domain the field may be called self-conjugate for this region†.

§ 3. The flow of energy.

An entity whose volume density‡ ρ is a function of (x, y, z, t) will vary in a manner which can be described as a simple flow with component velocities (u, v, w) if the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad \text{...........(4)}$$

is satisfied. This equation implies in fact that there is no

* They are invariants for the group of linear transformations which leave the electromagnetic equations unaltered in form. Cf. H. Minkowski, *Gött. Nachr.* (1908); E. Cunningham, *Proc. London Math. Soc.* (2), Vol. 8 (1910), p. 89; H. Poincaré, *Rend. Palermo* (1906); M. Planck, *Ann. d. Phys.* Vol. 26 (1908). Other invariants are given by these authors.

† Silberstein calls it a pure electromagnetic wave.

‡ The limitations to which the idea of density is subject and the question of the continuity of the function ρ are discussed by J. G. Leathem, "Volume integrals and their use in physics," *Cambridge Mathematical Tracts* (1905).

creation or annihilation of the entity in the neighbourhood of (x, y, z, t).

Now it is easy to see that the equation of continuity is satisfied in virtue of equations (1) if we put

$$\rho = \tfrac{1}{2}(E^2) + \tfrac{1}{2}(H^2), \quad \rho u = c(E_y H_z - E_z H_y),$$

and two similar equations. We shall regard ρ in this case as the volume density of the *energy* contained in the electromagnetic field. The vector Σ whose components are of the type $c(E_y H_z - E_z H_y)$ can then be supposed to indicate the rate at which energy flows through the field. Since

$$\rho^2 (c^2 - u^2 - v^2 - w^2) = \tfrac{1}{4} c^2 (E^2 - H^2)^2 + c^2 (EH)^2,$$

it appears that energy travels through the field with a velocity which is less than the velocity of light. The velocity c is attained only in the case of a self-conjugate field.

The vector Σ was introduced by Prof. Poynting* and is usually called Poynting's vector. The idea of describing the transfer of energy in this way also occurred to Prof. Lamb before the publication of Poynting's work.

Example. Prove that the equation of continuity may be satisfied by putting

$$\rho u = \frac{1}{c}\frac{\partial\theta}{\partial t} E_x - \frac{\partial\theta}{\partial y} H_z + \frac{\partial\theta}{\partial z} H_y, \qquad \rho v = \frac{1}{c}\frac{\partial\theta}{\partial t} E_y - \frac{\partial\theta}{\partial z} H_x + \frac{\partial\theta}{\partial x} H_z,$$

$$\rho w = \frac{1}{c}\frac{\partial\theta}{\partial t} E_z - \frac{\partial\theta}{\partial x} H_y + \frac{\partial\theta}{\partial y} H_x, \qquad \rho c = -\frac{\partial\theta}{\partial x} E_x - \frac{\partial\theta}{\partial y} E_y - \frac{\partial\theta}{\partial z} E_z,$$

where θ is an arbitrary function. Obtain a similar solution by replacing E by H and H by $-E$.

§ 4. First solution of the fundamental equations.

Let us use the symbol Ωu to denote the Dalembertian† of u, viz.

$$\Omega u \equiv \Delta u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$

* *Phil. Trans.* A, Vol. 175 (1884), p. 343. See also H. A. Lorentz, *The Theory of Electrons*, p. 22.

† This is the name suggested by Lorentz, *loc. cit.* p. 17. Many writers use Cauchy's symbol □ to denote the Dalembertian, but I think Ω is preferable because its form suggests a wave. Murphy's symbol Δ is also used here in place of the usual symbol ∇^2. E. B. Wilson and G. N. Lewis use the symbol $\diamondsuit^2 u$ to denote the Dalembertian of u. Cf. *Proc. Amer. Acad. of Arts and Sciences*, Vol. 48 (1912), p. 389.

and the symbol grad U to denote the vector whose components are

$$\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}$$

respectively. Let us also use $\Omega\Lambda$, where Λ is a vector with components $\Lambda_x, \Lambda_y, \Lambda_z$, to denote the vector whose components are $\Omega\Lambda_x, \Omega\Lambda_y, \Omega\Lambda_z$. The equation $\Omega u = 0$ will be called the *wave-equation* and a solution of this equation a *wave-function*. A vector function Λ will be said to satisfy the wave-equation when each of its components is a wave-function, i.e. if $\Omega\Lambda = 0$. We may now satisfy equations (1) and (2) by writing

$$M = \pm i \operatorname{rot} L \equiv \frac{1}{c}\frac{\partial L}{\partial t} + \operatorname{grad} \Lambda \quad \text{..............(5)},$$

where the *scalar potential* $\Lambda = \Psi \mp i\Phi$ and the *vector potential* $L = B \mp iA$ satisfy the equations

$$\Omega\Lambda = 0, \quad \Omega L = 0, \quad \operatorname{div} L + \frac{1}{c}\frac{\partial \Lambda}{\partial t} = 0 \quad \text{........(6)}.$$

The last three equations may be solved in a general way by writing

$$\left.\begin{aligned} L &= \frac{1}{c}\frac{\partial G}{\partial t} \pm i \operatorname{rot} G + \operatorname{grad} K \\ \Lambda &= -\operatorname{div} G - \frac{1}{c}\frac{\partial K}{\partial t} \end{aligned}\right\} \quad \text{..............(7)},$$

where the vector $G \equiv \Gamma \mp i\Pi$ and the scalar K satisfy the *wave-equation*

$$\Omega u = 0 \quad \text{..........................(8)}.$$

The solution of equations (1) which is embodied in (5), (6) and (7) is a simple extension of Hertz's solution* and is suggested by Whittaker's solution† in terms of two scalar potentials. It is clear that the function K drops out when we differentiate to find M and so the electric and magnetic forces depend only on the vector G. The form of this vector indicates that the electromagnetic field can be regarded as the sum of two partial fields; one of these is derived from the vector Π and

* *Ann. d. Phys.* Vol. 36 (1888), p. 1. The general solution is given by Righi, *Bologna Mem.* (5), t. 9 (1901), p. 1; *Il Nuovo Cimento* (5), t. 2 (1901), p. 2. He finds suitable expressions for the vectors Π and Γ in a number of cases.

† *Proc. London Math. Soc.* Ser. 2, Vol. 1 (1903).

will be called a field of *electric type*, the other is derived from the function Γ and will be called a field of *magnetic type*.

This resolution of an electromagnetic field into two partial fields is analogous to the one used by H. M. Macdonald* in the study of the effect of an obstacle on a train of electric waves. The component fields are then of such a type that in one case the magnetic force normal to the obstacle vanishes over the surface of the latter, in the other case it is the electric force normal to the obstacle that vanishes. The same idea has been used recently by Mie† and Debye‡ in the treatment of the case of a spherical obstacle.

In Hertz's solution we have $\Gamma = 0$, $K = 0$ and Π has components $(0, 0, S)$.

The components of E and H are consequently given by the formulae

$$\left.\begin{aligned} E_x &= \frac{\partial^2 S}{\partial x \partial z}, & H_x &= \frac{1}{c}\frac{\partial^2 S}{\partial y \partial t} \\ E_y &= \frac{\partial^2 S}{\partial y \partial z}, & H_y &= -\frac{1}{c}\frac{\partial^2 S}{\partial x \partial t} \\ E_z &= \frac{\partial^2 S}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 S}{\partial t^2}, & H_z &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots(9).$$

Hertz uses Euler's wave-function§

$$S = \frac{1}{r}\sin \kappa (r - ct), \quad r^2 = x^2 + y^2 + z^2,$$

and obtains in this way a theory of his oscillator‖. The electric and magnetic forces become infinite at the origin which is therefore a singularity of the electromagnetic field. A singularity of this type is called a *vibrating electric doublet* and is regarded as the simplest model of a source of light or electromagnetic waves.

* *Electric Waves*, Ch. VI.

† *Ann. d. Phys.* Vol. 25 (1908), p. 382. ‡ *Ibid.* Vol. 30 (1909), p. 57.

§ Periodic solutions representing a disturbance sent out from n-fold poles had been used previously by H. A. Rowland and applied to the elucidation of optical phenomena. *Amer. Journal of Mathematics*, Vol. 6, p. 359; *Phil. Mag.* Vol. 17 (1884), p. 423. Cf. also Stokes, *Cambr. Phil. Trans.* (1849).

‖ To deal with the case in which the vibrations are damped we assume $S = \frac{1}{r} e^{+\nu(r-ct)} \sin \kappa (r - ct)$. Cf. K. Pearson and A. Lee, *Phil. Trans.* A, Vol. 193 (1900), p. 159.

The solutions of equations (1) which are obtained by superposing elementary solutions of this type are of great importance in physical optics.

When r is very great the most important terms in the expressions (9) are

$$E_x = -\frac{\kappa^2 xzs}{r^3}, \qquad H_x = \frac{\kappa^2 ys}{r^2},$$

$$E_y = -\frac{\kappa^2 yzs}{r^3}, \qquad H_y = -\frac{\kappa^2 xs}{r^2},$$

$$E_z = \frac{\kappa^2 (x^2 + y^2) s}{r^3}, \qquad H_z = 0,$$

where $s = \sin \kappa (r - ct)$. All the other terms are of order $1/r^2$ or $1/r^3$. These expressions give

$$(EH) = 0, \qquad (E^2) - (H^2) = 0.$$

Hence at a very great distance from the origin the field is practically a self-conjugate field and so the energy travels with a velocity very nearly equal to the velocity of light. The expressions indicate that Poynting's vector is ultimately along the radius from the origin; now the electric and magnetic forces are at right angles to Poynting's vector and so the vibrations of the light-vector, whether we take it to be the electric or magnetic force, are at right angles to the radius. The waves sent out from the source have, then, the character of monochromatic light at a great distance from the origin*. The amplitudes of the vibrations at points on the same radius are proportional to the quantities $1/r$ when r is large, and so if the intensity of the light be measured by the square of the amplitude the inverse square law is fulfilled.

Since the electric force is ultimately at right angles to the radius there is no total charge associated with the singularity, for the charge is equal to the surface-integral of the normal electric force over a large sphere concentric with the origin and this integral is evidently zero. We are consequently justified in regarding the singularity as a doublet and in fact

* For a fuller discussion see Larmor, *Phil. Mag.* (5), Vol. 44 (1897), p. 503; *Aether and Matter*, Chap. XIV, where it is shown that energy is radiated from a moving charge only when the velocity of the charge alters in either magnitude or direction.

as a simple electric doublet of varying moment as is indicated by the way in which the electric and magnetic forces become infinite*. The axis of the doublet is along the axis of z.

The electric lines of force due to a vibrating electric doublet have been drawn by Hertz† for various stages of the motion. The general character of the lines of force is indicated in Fig. 1.

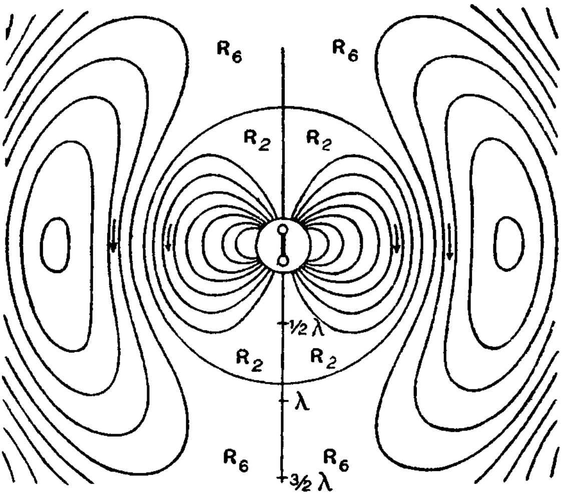

Fig. 1‡.

It will be noticed that the lines are all at right angles to a plane perpendicular to the axis of the doublet. M. Abraham§ has used a Hertzian doublet to obtain a model of the electromagnetic field produced by the oscillations in a vertical antenna, the plane just mentioned being supposed to represent the earth which is regarded as a perfect conductor. Zenneck‖ has, however, pointed out that when the imperfect conductivity of the earth is taken into account the circumstances of the

* See § 42.

† *Ann. d. Phys.* Vol. 36 (1888), p. 1. The case of damped vibrations is considered by K. Pearson and A. Lee, *loc. cit.*

‡ I am indebted to the Macmillan Company and A. Gray, Esq., for permission to reproduce this diagram.

§ *Phys. Zeitschr.* Vol. 2 (1901), p. 329; *Theorie der Elektrizität*, Vol. 2, § 34; *Encyklop. d. Math. Wiss.* Band 5, § 18.

‖ *Ann. d. Phys.* Vol. 23 (1907), p. 846; *Phys. Zeitschr.* Vol. 9 (1908), p. 50; *Ibid.* p. 553.