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Introduction and Main Results

This monograph provides a careful and accessible exposition of a functional analytic approach to initial boundary value problems for semilinear parabolic differential equations of second order. It focuses on the relationship between three interrelated subjects in analysis: elliptic boundary value problems and parabolic initial boundary value problems, with emphasis on the general study of analytic semigroups. This semigroup approach can be traced back to the pioneering work of Fujita–Kato [18] on the Navier–Stokes equation in fluid mechanics.

The approach here is distinguished by the extensive use of the techniques characteristic of recent developments in the theory of partial differential equations. The main technique used is the L^p theory of pseudo-differential operators which may be considered as a modern theory of classical potentials. The theory of pseudo-differential operators continues to be one of the most influential works in modern history of analysis, and is a very refined mathematical tool whose full power is yet to be exploited. Several recent developments in the theory of pseudo-differential operators have made possible further progress in the study of elliptic boundary value problems and hence the study of parabolic initial boundary value problems. The presentation of these new results is the main purpose of this book.

We study a class of *degenerate* boundary value problems for second-order elliptic differential operators in the framework of L^p Sobolev spaces which include as particular cases the Dirichlet and Neumann problems, and proves that these boundary value problems provide an interesting example of analytic semigroups in the L^p topology. As an application, we can apply these results to the initial boundary value problems for semilinear parabolic differential equations of second order in the framework of L^p spaces. We confined ourselves to the simple but important boundary condition. This makes it possible to develop our basic machinery with a minimum of bother and the principal ideas can be presented concretely and explicitly.

Let Ω be a bounded domain of Euclidean space $\mathbf{R}^n$, with C^∞ boundary $\Gamma = \partial\Omega$; its closure $\overline{\Omega} = \Omega \cup \Gamma$ is an n-dimensional, compact C^∞ manifold with boundary. We let

$$Au = \sum_{j,k=1}^{n} a^{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b^j(x) \frac{\partial u}{\partial x_j} + c(x)u$$

be a second-order, *uniformly elliptic differential operator* with real coefficients on $\overline{\Omega} = \Omega \cup \Gamma$ such that:

(1) $a^{jk} \in C^\infty(\overline{\Omega})$ and $a^{jk}(x) = a^{kj}(x)$ for all $x \in \overline{\Omega}$ and $1 \leq j,k \leq n$, and there exists a constant $a_0 > 0$ such that

$$\sum_{j,k=1}^{n} a^{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2 \quad \text{for all } (x,\xi) \in \overline{\Omega} \times \mathbf{R}^n.$$

(2) $b^j \in C^\infty(\overline{\Omega})$ for all $1 \leq j \leq n$.
(3) $c \in C^\infty(\overline{\Omega})$ and $c(x) \leq 0$ in Ω.

Throughout this book, for simplicity, we assume that

The function $c(x)$ does not vanish *identically* in Ω.

1.1 Formulation of the elliptic boundary value problem $(*)$

We consider the following elliptic boundary value problem: Given functions $f(x)$ and $\varphi(x')$ defined in Ω and on Γ, respectively, find a function $u(x)$ in Ω such that

$$\begin{cases} Au = f & \text{in } \Omega, \\ Bu := a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = \varphi & \text{on } \Gamma. \end{cases} \tag{$*$}$$

Here:

(1) $a(x')$ and $b(x')$ are real-valued, C^∞ functions on Γ.
(2) $\partial/\partial\boldsymbol{\nu}$ is the *conormal* derivative associated with the differential operator A:

$$\frac{\partial}{\partial \boldsymbol{\nu}} = \sum_{j,k=1}^{n} a^{jk}(x')n_k \frac{\partial}{\partial x_j},$$

$\mathbf{n} = (n_1, n_2, \ldots, n_n)$ being the unit outward normal to the boundary Γ (see Figure 1.1).

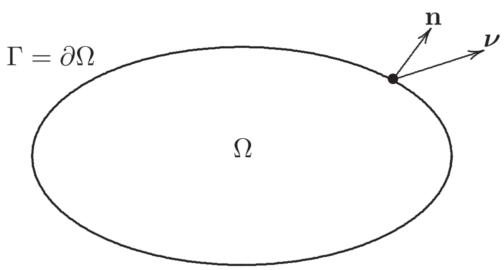

Fig. 1.1. The unit outward normal $\mathbf{n}$ and the conormal $\boldsymbol{\nu}$ to Γ

In this book, we assume that the following two conditions (H.1) and (H.2) are satisfied:

(H.1) $a(x') \geq 0$ and $b(x') \geq 0$ on Γ.
(H.2) $b(x') > 0$ on $\Gamma_0 = \{x' \in \Gamma : a(x') = 0\}$.

We give a simple example of such functions $a(x')$ and $b(x')$ on the boundary Γ:

Example 1.1 (K. Umezu). Let $\Omega = \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 < 1\}$ with the boundary $\Gamma = \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 = 1\}$. For a local coordinate system $x_1 = \cos\theta$, $x_2 = \sin\theta$ with $\theta \in (-\pi, \pi]$ on Γ, we define two functions $a(x') = a(x_1, x_2)$ and $b(x') = b(x_1, x_2)$ as follows:

$$a(x_1, x_2) = \begin{cases} e^{\frac{2}{\pi} + \frac{1}{\theta + \frac{\pi}{2}}} \left(1 - e^{\frac{2}{\pi} - \frac{1}{\theta + \pi}}\right) & \text{for } \theta \in \left(-\pi, -\frac{\pi}{2}\right), \\ 0 & \text{for } \theta \in \left[-\frac{\pi}{2}, 0\right], \\ e^{\frac{2}{\pi} - \frac{1}{\theta}} \left(1 - e^{\frac{2}{\pi} + \frac{1}{\theta - \frac{\pi}{2}}}\right) & \text{for } \theta \in \left(0, \frac{\pi}{2}\right), \\ 1 & \text{for } \theta \in \left[\frac{\pi}{2}, \pi\right], \end{cases}$$

and

$$b(x_1, x_2) = 1 - a(x_1, x_2).$$

Then, this is the case as desired.

It should be emphasized that the boundary value problem $(*)$ is a *degenerate* elliptic boundary value problem from an analytical point of view. This is due to the fact that the so-called *Lopatinski–Shapiro ellipticity condition* is violated at each point x' of the set $\Gamma_0 = \{x' \in \Gamma :$

$a(x') = 0\}$ (see Example 6.1 in Section 6.6). More precisely, the boundary value problem $(*)$ is elliptic (or non-degenerate) if and only if either $a(x') > 0$ on Γ (the Robin case) or $a(x') \equiv 0$ and $b(x') > 0$ on Γ (the Dirichlet case).

1.2 Existence and uniqueness theorem for problem $(*)$

The first purpose of this book is to prove an existence and uniqueness theorem for the elliptic boundary value problem $(*)$ with *degenerate* boundary condition in the framework of L^p Sobolev spaces. The essential point is how to define a function space which is a suitable tool to investigate the degenerate boundary condition B.

If $1 \leq p < \infty$, we let

$$\begin{aligned} L^p(\Omega) = &\text{the space of (equivalence classes of) Lebesgue measurable} \\ &\text{functions } u \text{ on } \Omega \text{ such that } |u|^p \text{ is integrable on } \Omega. \end{aligned}$$

The space $L^p(\Omega)$ is a Banach space with the norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}.$$

If s is a positive integer, we define the Sobolev space

$$\begin{aligned} W^{s,p}(\Omega) = &\text{the space of (equivalence classes of) functions} \\ &u \in L^p(\Omega) \text{ whose derivatives } D^\alpha u, \ |\alpha| \leq s, \text{ in the} \\ &\text{sense of distributions are in } L^p(\Omega). \end{aligned}$$

The space $W^{s,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{s,p} = \left(\sum_{|\alpha| \leq s} \int_\Omega |D^\alpha u(x)|^p \, dx \right)^{1/p}.$$

Furthermore, we let

$$\begin{aligned} B^{s-1/p,p}(\Gamma) = &\text{the space of the boundary values } \varphi \text{ of functions} \\ &u \in W^{s,p}(\Omega). \end{aligned}$$

In the space $B^{s-1/p,p}(\Gamma)$, we introduce a norm

$$|\varphi|_{s-1/p,p} = \inf \left\{ \|u\|_{s,p} : \ u \in W^{s,p}(\Omega), \ u = \varphi \text{ on } \Gamma \right\}.$$

The space $B^{s-1/p,p}(\Gamma)$ is a Banach space with respect to this norm $|\cdot|_{s-1/p,p}$. More precisely, it is a Besov space on Γ (cf. [7], [70]).

We introduce a subspace of $B^{s-1-1/p,p}(\Gamma)$ which is associated with the *degenerate boundary condition*

$$Bu = a(x')\frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u \quad \text{on } \Gamma.$$

We let

$$B_*^{s-1-1/p,p}(\Gamma) = \left\{\varphi = a(x')\varphi_1 + b(x')\varphi_2 : \varphi_1 \in B^{s-1-1/p,p}(\Gamma),\ \varphi_2 \in B^{s-1/p,p}(\Gamma)\right\},$$

and define a norm

$$|\varphi|^*_{s-1-1/p,p} = \inf\left\{|\varphi_1|_{s-1-1/p,p} + |\varphi_2|_{s-1/p,p} : \varphi = a(x')\varphi_1 + b(x')\varphi_2\right\}.$$

Then it is easy to verify (see Lemma 6.8) that the space $B_*^{s-1-1/p,p}(\Gamma)$ is a Banach space with respect to the norm $|\cdot|^*_{s-1-1/p,p}$. We remark that the space $B_*^{s-1-1/p,p}(\Gamma)$ is an "interpolation space" between the spaces $B^{s-1/p,p}(\Gamma)$ and $B^{s-1-1/p,p}(\Gamma)$. More precisely, we have the assertions

$$\begin{cases} B_*^{s-1-1/p,p}(\Gamma) = B^{s-1/p,p}(\Gamma) & \text{if } a(x') \equiv 0 \text{ on } \Gamma, \\ B_*^{s-1-1/p,p}(\Gamma) = B^{s-1-1/p,p}(\Gamma) & \text{if } a(x') > 0 \text{ on } \Gamma, \end{cases}$$

and, for general $a(x')$, the continuous injections

$$B^{s-1/p,p}(\Gamma) \subset B_*^{s-1-1/p,p}(\Gamma) \subset B^{s-1-1/p,p}(\Gamma).$$

Now we can state our existence and uniqueness theorem for the elliptic boundary value problem $(*)$:

Theorem 1.1 (the existence and uniqueness theorem). *Let* $1 < p < \infty$ *and* $s > 1 + 1/p$. *Assume that the conditions (H.1) and (H.2) are satisfied. Then the mapping*

$$(A, B) : W^{s,p}(\Omega) \longrightarrow W^{s-2,p}(\Omega) \times B_*^{s-1-1/p,p}(\Gamma)$$

is an algebraic and topological isomorphism. *In particular, for any* $f \in W^{s-2,p}(\Omega)$ *and any* $\varphi \in B_*^{s-1-1/p,p}(\Gamma)$ *there exists a unique solution* $u \in W^{s,p}(\Omega)$ *of problem* $(*)$.

1.3 Generation theorem for analytic semigroups of problem $(*)$

The second purpose of this book is to study the elliptic boundary value problem $(*)$ with degenerate boundary condition from the point of view of analytic semigroup theory in functional analysis. The generation theorem for analytic semigroups is well established in the non-degenerate case in the L^p topology (cf. [16], [67]). We shall generalize this generation theorem for analytic semigroups to the *degenerate* case.

First, we state a generation theorem for analytic semigroups in the L^p topology. We associate with problem $(*)$ an unbounded linear operator $\mathfrak{A}_p$ from the Banach space $L^p(\Omega)$ into itself as follows:

(a) The domain $D(\mathfrak{A}_p)$ of definition of $\mathfrak{A}_p$ is the set

$$D(\mathfrak{A}_p) = \left\{ u \in W^{2,p}(\Omega) : Bu = a(x')\frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 \text{ on } \Gamma \right\}.$$

(b) $\mathfrak{A}_p u = Au$ for every $u \in D(\mathfrak{A}_p)$.

The next generation theorem for analytic semigroups is an L^p version of [57, Theorem 1]:

Theorem 1.2 (the generation theorem for analytic semigroups). *Let $1 < p < \infty$. If the conditions (H.1) and (H.2) are satisfied, then we have the following two assertions (i) and (ii):*

(i) For every $0 < \varepsilon < \pi/2$, there exists a constant $r_p(\varepsilon) > 0$ such that the resolvent set of $\mathfrak{A}_p$ contains the set

$$\Sigma(\varepsilon) = \left\{ \lambda = r^2 e^{i\theta} : r \geq r_p(\varepsilon),\ |\theta| \leq \pi - \varepsilon \right\},$$

and that the resolvent $(\mathfrak{A}_p - \lambda I)^{-1}$ satisfies the estimate

$$\left\| (\mathfrak{A}_p - \lambda I)^{-1} \right\| \leq \frac{c_p(\varepsilon)}{|\lambda|}, \quad \lambda \in \Sigma(\varepsilon),$$

where $c_p(\varepsilon) > 0$ is a constant depending on ε.

(ii) The operator $\mathfrak{A}_p$ generates a semigroup $U(z)$ on the space $L^p(\Omega)$ which is analytic *in the sector*

$$\Delta_\varepsilon = \{ z = t + is : z \neq 0, |\arg z| < \pi/2 - \varepsilon \}$$

for any $0 < \varepsilon < \pi/2$ (see Figure 1.2 below).

It is worth pointing out here that the generation theorem for analytic semigroups is established in the topology of *uniform convergence* by Taira [64, Theorem 1.3]. As an application of this theorem, we can construct a strong Markov process corresponding to such a physical phenomenon that a Markovian particle moves both by jumps and continuously in the state space until it "dies" at the time when it reaches the set where the particle is definitely absorbed (see [63] and [65]).

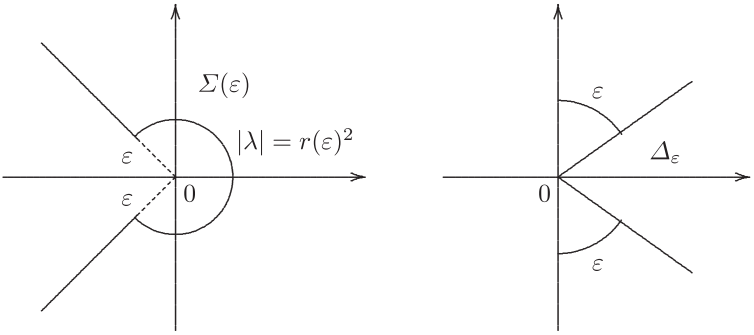

Fig. 1.2. The set $\Sigma(\varepsilon)$ and the sector Δ_ε

1.4 The semilinear initial boundary value problem for problem $(*)$

In this section, as an application of Theorem 1.2 we consider the following *semilinear* initial boundary value problem for a second-order parabolic differential equation associate with problem $(*)$: Given functions $f(x,t,u,\xi)$ and $u_0(x)$ defined in $\Omega \times [0,T) \times \mathbf{R} \times \mathbf{R}^n$ and in Ω, respectively, find a function $u(x,t)$ in $\Omega \times [0,T)$ such that

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - A\right) u(x,t) = f\,(x,t,u,\operatorname{grad} u) & \text{in } \Omega \times (0,T), \\ Bu(x',t) := a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}}(x',t) + b(x')u(x',t) = 0 & \text{on } \Gamma \times [0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{$**$}$$

By using the operator $\mathfrak{A}_p$, we can formulate the initial boundary value problem $(**)$ in terms of the *abstract Cauchy problem* in the Banach space $L^p(\Omega)$ as follows:

$$\begin{cases} \dfrac{du}{dt} = \mathfrak{A}_p\, u(t) + F(t,u(t)), \ 0 < t < T, \\ u|_{t=0} = u_0. \end{cases} \tag{$**'$}$$

Here $u(t) = u(\cdot,t)$ and $F(t,u(t)) = f(\cdot,t,u(t),\operatorname{grad} u(t))$ are functions defined on the interval $[0,T)$, taking values in the space $L^p(\Omega)$.

First, we consider the case where $p > n$, and prove the following *local* existence and uniqueness theorem for the initial boundary value problem $(**')$:

Theorem 1.3 (the local existence and uniqueness theorem). *Let $n < p < \infty$ and assume that the conditions (H.1) and (H.2) are satisfied.*

Moreover, we assume that $f(x,t,u,\xi)$ *is a locally Lipschitz continuous function of all its variables with the possible exception of the* x *variable. Then, for every function* u_0 *of* $D(\mathfrak{A}_p)$ *the Cauchy problem* $(**')$ *has a unique local solution* $u \in C([0,T']; L^p(\Omega)) \cap C^1((0,T'); L^p(\Omega))$ *where* $T' = T'(p,u_0) > 0$.

Here $C([0,T']; L^p(\Omega))$ denotes the space of continuous functions on the interval $[0,T']$ taking values in $L^p(\Omega)$ and $C^1((0,T'); L^p(\Omega))$ denotes the space of continuously differentiable functions on the interval $(0,T')$ taking values in $L^p(\Omega)$, respectively.

In the case where $p < n$, the domain $D(\mathfrak{A}_p)$ is large compared with the case $n < p < \infty$. Hence we must impose some *growth conditions* on the function $f(x,t,u,\xi)$. More precisely, we prove the following local existence and uniqueness theorem for the initial boundary value problem $(**')$:

Theorem 1.4 (the local existence and uniqueness theorem). *Let* $n/2 < p < n$ *and assume that the conditions (H.1) and (H.2) are satisfied. Moreover, we assume that* $f(x,t,u,\xi)$ *is a locally Lipschitz continuous function of all its variables with the possible exception of the* x *variable and further that there exist a continuous non-negative function* $\rho(t,r)$ *on* $[0,\infty) \times [0,\infty)$ *and a constant* $1 \leq \gamma < n/(n-p)$ *such that the following four conditions (a) through (d) hold true for all* $(x,t,u,\xi) \in \Omega \times [0,T) \times \mathbf{R} \times \mathbf{R}^n$*:*

(a) $|f(x,t,u,\xi)| \leq \rho(t,|u|)(1+|\xi|^\gamma)$.
(b) $|f(x,t,u,\xi) - f(x,s,u,\xi)| \leq \rho(t,|u|)(1+|\xi|^\gamma)|t-s|$.
(c) $|f(x,t,u,\xi) - f(x,t,u,\eta)| \leq \rho(t,|u|)\left(1+|\xi|^{\gamma-1}+|\eta|^{\gamma-1}\right)|\xi-\eta|$.
(d) $|f(x,t,u,\xi) - f(x,t,v,\xi)| \leq \rho(t,|u|+|v|)(1+|\xi|^\gamma)|u-v|$.

Then, for every function u_0 *of* $D(\mathfrak{A}_p)$ *the Cauchy problem* $(**')$ *has a unique local solution* $u \in C([0,T'];\, L^p(\Omega)) \cap C^1((0,T');\, L^p(\Omega))$ *where* $T' = T'(p,u_0) > 0$.

Theorems 1.3 and 1.4 are a generalization of Pazy [45, Section 8.4, Theorems 4.4 and 4.5] to the *degenerate* case.

1.5 The global existence and uniqueness theorems

In this section, as an application of Theorem 3.19 we prove a global version of Theorems 1.3 and 1.4 if an *a priori* bound for the nonlinear term $f(x,t,u,\xi)$ can be found.

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More precisely, we consider the case where $n < p < \infty$, and prove the following *global* existence and uniqueness theorem for the initial boundary value problem $(**')$:

Theorem 1.5 (the global existence and uniqueness theorem). *Let $n < p < \infty$ and assume that the conditions (H.1) and (H.2) are satisfied. Moreover, we assume that $f(x,t,u,\xi)$ is a locally Lipschitz continuous function of all its variables with the possible exception of the x variable and further that there exists a continuous, non-decreasing and non-negative function $\rho(t)$ on the interval $[0,\infty)$ such that the following condition (a) is satisfied:*

(a) $|f(x,t,u,\xi)| \leq \rho(t)(1+|\xi|)$ *for all* $(x,t,u,\xi) \in \Omega\times[0,\infty)\times\mathbf{R}\times\mathbf{R}^n$.

Then, for every function u_0 of $D(\mathfrak{A}_p)$ the Cauchy problem $(**')$ has a unique global *solution* $u \in C\left([0,\infty); L^p(\Omega)\right) \cap C^1\left((0,\infty); L^p(\Omega)\right)$.

In the case where $p < n$, we prove a global version of Theorem 1.4 if an *a priori* bound for the nonlinear term $f(x,t,u,\xi)$ can be found. Since the domain $D(\mathfrak{A}_p)$ is large compared with the case where $n < p < \infty$, we must impose some *growth conditions* on $f(x,t,u,\xi)$.

More precisely, we consider the case where $n/2 < p < n$, and prove the following *global* existence and uniqueness theorem for the initial boundary value problem $(**')$:

Theorem 1.6 (the global existence and uniqueness theorem). *Let $n/2 < p < n$ and assume that the conditions (H.1) and (H.2) are satisfied. Moreover, we assume that $f(x,t,u,\xi)$ is a locally Lipschitz continuous function of all its variables with the possible exception of the x variable and further that there exists a continuous, non-decreasing and non-negative function $\rho(t)$ on the interval $[0,\infty)$ such that the following four conditions (a) through (d) hold true for all $(x,t,u,\xi) \in \Omega\times[0,\infty)\times\mathbf{R}\times\mathbf{R}^n$:*

(a) $|f(x,t,u,\xi)| \leq \rho(t)(1+|\xi|)$.
(b) $|f(x,t,u,\xi) - f(x,s,u,\xi)| \leq \rho(t)\left(1+|\xi|\right)|t-s|$.
(c) $|f(x,t,u,\xi) - f(x,t,u,\eta)| \leq \rho(t)|\xi-\eta|$.
(d) $|f(x,t,u,\xi) - f(x,t,v,\xi)| \leq \rho(t)\left(1+|\xi|\right)|u-v|$.

Then, for every function u_0 of $D(\mathfrak{A}_p)$ the Cauchy problem $(**')$ has a unique global *solution* $u \in C\left([0,\infty);\, L^p(\Omega)\right) \cap C^1\left((0,\infty);\, L^p(\Omega)\right)$.

1.6 Summary of the contents

Now we give a chapter-by-chapter summary of what is covered in the rest of this book. A concise and elementary description of Chapters 2 through 5 is especially addressed to the advanced undergraduates and beginning-graduate students with interest in functional analysis and partial differential equations.

The purpose of Chapter 2 is to present fundamental results from functional analysis. The material in this chapter is standard and can be found in textbooks on functional analysis such as Folland [15], Friedman [17] and Yosida [75]. This preparatory chapter, included for the sake of completeness, should serve to settle questions of notation and such.

Sections 2.1 through 2.4 are devoted to a summary of the basic definitions and results about topological spaces, quasinormed and normed linear spaces and continuous (bounded) linear operators between Banach spaces which will be used throughout the book. The presentation here is intended as a reference rather than a systematic exposition.

In Section 2.5 we formulate three pillars of functional analysis – Banach's open mapping theorem (Theorem 2.25), Banach's closed graph theorem (Theorem 2.26) and Banach's closed range theorem (Theorem 2.29) for closed operators in Banach spaces.

In Section 2.6 we give two criteria for a closed subspace to be complemented in a Banach space (Theorem 2.31).

Section 2.7 is devoted to the Riesz–Schauder theory for compact operators. More precisely, for a compact operator T in Banach spaces, the eigenvalue problem can be treated fairly completely in the sense that the classical theory of Fredholm integral equations may be extended to the linear functional equation

$$Tx - \lambda x = y$$

with a complex parameter λ (Theorem 2.36).

In Section 2.8 we state important properties of Fredholm operators (Theorems 2.37 through 2.41). Moreover, we prove a very useful criterion for Fredholm operators due to Peetre (Theorem 2.42).

Section 2.9 is devoted to a review of standard topics in the Hilbert space setting such as the Riesz representation theorem (Theorem 2.47), the Gram–Schmidt orthogonalization (Theorem 2.49) and adjoint operators (transpose operators in the Hilbert space setting).

In the last Section 2.10 we present the Hilbert–Schmidt theory (The-