Cambridge University Press
978-1-316-61958-2 - Graded Rings and Graded Grothendieck Groups
Roozbeh Hazrat
Excerpt
More information

# Introduction

A bird's eye view of the theory of graded modules over a graded ring might give the impression that it is nothing but ordinary module theory with all its statements decorated with the adjective "graded". Once the grading is considered to be trivial, the graded theory reduces to the usual module theory. From this perspective, the theory of graded modules can be considered as an extension of module theory. However, this simplistic overview might conceal the point that graded modules come equipped with a *shift*, thanks to the possibility of partitioning the structures and then rearranging the partitions. This adds an extra layer of structure (and complexity) to the theory. This monograph focuses on the theory of the graded Grothendieck group $K_0^{\mathrm{gr}}$, that provides a sparkling illustration of this idea. Whereas the usual $K_0$ is an abelian group, the shift provides $K_0^{\mathrm{gr}}$ with a natural structure of a $\mathbb{Z}[\Gamma]$-module, where $\Gamma$ is the group used for the grading and $\mathbb{Z}[\Gamma]$ its group ring. As we will see throughout this book, this extra structure carries substantial information about the graded ring.

Let $\Gamma$ and $\Delta$ be abelian groups and $f : \Gamma \to \Delta$ a group homomorphism. Then for any $\Gamma$-graded ring $A$, one can consider a natural $\Delta$-grading on $A$ (see §1.1.2); in the same way, any $\Gamma$-graded $A$-module can be viewed as a $\Delta$-graded $A$-module. These operations induce functors

$$U_f : \mathrm{Gr}^{\Gamma}\text{-}A \longrightarrow \mathrm{Gr}^{\Delta}\text{-}A,$$
$$(-)_{\Omega} : \mathrm{Gr}^{\Gamma}\text{-}A \longrightarrow \mathrm{Gr}^{\Omega}\text{-}A_{\Omega},$$

(see §1.2.8), where $\mathrm{Gr}^{\Gamma}\text{-}A$ is the category of $\Gamma$-graded right $A$-modules, $\mathrm{Gr}^{\Delta}\text{-}A$ that of $\Delta$-graded right $A$-modules, and $\mathrm{Gr}^{\Omega}\text{-}A$ the category of $\Omega$-graded right $A_{\Omega}$-module with $\Omega = \ker(f)$.

One aim of the theory of graded rings is to investigate the ways in which these categories relate to one another, and which properties of one category can be lifted to another. In particular, in the two extreme cases when the group

$\Delta = 0$ or $f : \Gamma \to \Delta$ is the identity, we obtain the forgetful functors

$$U : \mathrm{Gr}^\Gamma\text{-}A \longrightarrow \mathrm{Mod}\text{-}A,$$
$$(-)_0 : \mathrm{Gr}^\Gamma\text{-}A \longrightarrow \mathrm{Mod}\text{-}A_0.$$

The category $\mathrm{Pgr}^\Gamma\text{-}A$ of graded finitely generated projective $A$-modules is an exact category. Thus Quillen's $K$-theory machinery [81] defines graded $K$-groups

$$K_i^{\mathrm{gr}}(A) := K_i(\mathrm{Pgr}^\Gamma\text{-}A),$$

for $i \in \mathbb{N}$. On the other hand, the shift operation on modules induces a functor on $\mathrm{Gr}^\Gamma\text{-}A$ that is an auto-equivalence (§1.2.2), so that these $K$-groups also carry a $\Gamma$-module structure. One can treat the groups $K_i(A)$ and $K_i(A_0)$ in a similar way. Quillen's $K$-theory machinery allows us to establish relations between these $K$-groups. In particular:

**Relating $K_*^{\mathrm{gr}}(A)$ to $K_*(A)$ for a positively graded rings §6.1.** For a $\mathbb{Z}$-graded ring with the positive support, there is a $\mathbb{Z}[x, x^{-1}]$-module isomorphism,

$$K_i^{\mathrm{gr}}(A) \cong K_i(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}[x, x^{-1}].$$

**Relating $K_*^{\mathrm{gr}}(A)$ to $K_*(A_0)$ for graded Noetherian regular rings §6.3.** Consider the full subcategory $\mathrm{Gr}_0\text{-}A$ of $\mathrm{Gr}\text{-}A$, of all graded modules $M$ as objects such that $M_0 = 0$. This is a Serre subcategory of $\mathrm{Gr}\text{-}A$. One can show that $\mathrm{Gr}\text{-}A/\,\mathrm{Gr}_0\text{-}A \cong \mathrm{Mod}\text{-}A_0$. If $A$ is a (right) regular Noetherian ring, the quotient category identity above holds for the corresponding graded finitely generated modules, *i.e.*, $\mathrm{gr}\text{-}A/\,\mathrm{gr}_0\text{-}A \cong \mathrm{mod}\text{-}A_0$ and the localisation theorem gives a long exact sequence of abelian groups,

$$\cdots \longrightarrow K_{n+1}(A_0) \xrightarrow{\delta} K_n(\mathrm{gr}_0\text{-}A) \longrightarrow K_n^{\mathrm{gr}}(A) \longrightarrow K_n(A_0) \longrightarrow \cdots.$$

**Relating $K_*^{\mathrm{gr}}(A)$ to $K_*(A)$ for graded Noetherian regular rings §6.4.** For a $\mathbb{Z}$-graded ring $A$ which is right regular Noetherian, there is a long exact sequence of abelian groups

$$\cdots \longrightarrow K_{n+1}(A) \longrightarrow K_n^{\mathrm{gr}}(A) \xrightarrow{\bar{i}} K_n^{\mathrm{gr}}(A) \xrightarrow{U} K_n(A) \longrightarrow \cdots.$$

The main emphasis of this book is on the group $K_0^{\mathrm{gr}}$ as a powerful invariant in the classification problems. This group is equipped with the extra structure of the action of the grade group induced by the shift. In many important examples, in fact this shift is all the difference between the graded Grothendieck group and the usual Grothendieck group, *i.e.,*

$$K_0^{\mathrm{gr}}(A)/\langle [P] - [P(1)] \rangle \cong K_0(A),$$

where $P$ is a graded projective $A$-module and $P(1)$ is the shifted module (Chapter 6, see Corollary 6.4.2).

The motivation to write this book came from recent activities that adopt the graded Grothendieck group as an invariant to classify the Leavitt path algebras [47, 48, 79]. Surprisingly, not much is recorded about the graded version of the Grothendieck group in the literature, despite the fact that $K_0$ has been used on many occasions as a crucial invariant, and there is a substantial amount of information about the graded version of other invariants such as (co)homology groups, Brauer groups, etc. The other surge of interest in this group stems from the recent activities on the (graded) representation theory of Hecke algebras. In particular for a quiver Hecke algebra, its graded Grothendieck group is closely related to its corresponding quantised enveloping algebra. For this line of research see the survey [54].

This book tries to fill this gap, by systematically developing the theory of graded Grothendieck groups. In order to do this, we have to carry over and work out the details of known results in the nongraded case to the graded setting, and gather together important results on the graded theory scattered in research papers.

The group $K_0$ has been successfully used in operator theory to classify certain classes of $C^*$-algebras. Building on work of Bratteli, Elliott in [36] used the pointed ordered $K_0$-groups (called dimension groups) as a complete invariant for AF $C^*$-algebras. Another cornerstone of using $K$-groups for the classifications of a wider range of $C^*$-algebras was the work of Kirchberg and Phillips [80], who showed that $K_0$ and $K_1$-groups together are a complete invariant for a certain type of $C^*$-algebras. The Grothendieck group considered as a module induced by a group action was used by Handelman and Rossmann [45] to give a complete invariant for the class of direct limits of finite dimensional, representable dynamical systems. Krieger [56] introduced (past) dimension groups as a complete invariant for the shift equivalence of topological Markov chains (shift of finite types) in symbolic dynamics. Surprisingly, we will see that Krieger's groups are naturally expressed by graded Grothendieck groups (§3.11).

We develop the theory for rings graded by abelian groups rather than arbitrary groups for two reasons, although most of the results could be carried over to nonabelian grade groups. One reason is that using the abelian grading makes the presentation and proofs much more transparent. In addition, in most applications of graded $K$-theory, the ring has an abelian grading (often a $\mathbb{Z}$-grading).

In Chapter 1 we study the basic theory of graded rings. Chapter 2 concentrates on graded Morita theory. In Chapter 3 we compute $K_0^{\mathrm{gr}}$ for certain graded rings, such as graded local rings and (Leavitt) path algebras. We study the pre-ordering available on $K_0^{\mathrm{gr}}$ and determine the action of $\Gamma$ on this group. Chapter 4 studies graded Picard groups and in Chapter 5 we prove that for the so-called graded ultramatricial algebras, the graded Grothendieck group is a complete invariant. Finally, in Chapter 6, we explore the relations between (higher) $K_n^{\mathrm{gr}}$ and $K_n$, for the class of $\mathbb{Z}$-graded rings. We describe a generalisation of the Quillen and van den Bergh theorems. The latter theorem uses the techniques employed in the proof of the fundamental theorem of $K$-theory, where the graded $K$-theory appears. For this reason we present a proof of the fundamental theorem in this chapter.

**Conventions** Throughout this book, unless it is explicitly stated, all rings have identities, homomorphisms preserve the identity and all modules are unitary. Moreover, all modules are considered right modules. For a ring $A$, the category of right $A$-modules is denoted by Mod-$A$. A full subcategory of Mod-$A$ consisted of all finitely generated $A$-modules is denoted by mod-$A$. By Pr-$A$ we denote the category of finitely generated projective $A$-modules.

For a set $\Gamma$, we write $\bigoplus_\Gamma \mathbb{Z}$ or $\mathbb{Z}^\Gamma$ to mean $\bigoplus_{\gamma\in\Gamma} \mathbb{Z}_\gamma$, where $\mathbb{Z}_\gamma = \mathbb{Z}$ for each $\gamma \in \Gamma$. We denote the cyclic group $\mathbb{Z}/n\mathbb{Z}$ with $n$ elements by $\mathbb{Z}_n$.

**Acknowledgements** I would like to acknowledge Australian Research Council grants DP150101598 and DP160101481. Part of this work was done at the University of Bielefeld, where the author was a Humboldt Fellow.

I learned about the graded techniques in algebra from Adrian Wadsworth. Judith Millar worked with me to study the graded $K$-theory of Azumaya algebras. Gene Abrams was a source of encouragement that the graded techniques would be fruitful in the study of Leavitt path algebras. Andrew Mathas told me how graded Grothendieck groups are relevant in representation theory and pointed me to the relevant literature. The discussions with Zuhong Zhang, who kindly invited me to the Beijing Institute of Technology in July 2013 and 2014, helped to improve the presentation. To all of them, I am grateful.

# 1

# Graded rings and graded modules

Graded rings appear in many circumstances, both in elementary and advanced areas. Here are two examples.

1 In elementary school when we distribute 10 apples giving 2 apples to each person, we have

$$10\ \textsc{Apples} : 2\ \textsc{Apples} = 5\ \textsc{People}.$$

The psychological problem caused to many kids as to exactly how the word "People" appears in the equation can be overcome by correcting it to

$$10\ \textsc{Apples} : 2\ \textsc{Apples} / \textsc{People} = 5\ \textsc{People}.$$

This shows that already at the level of elementary school arithmetic, children work in a much more sophisticated structure, *i.e.,* the graded ring

$$\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

of Laurent polynomial rings! (see the interesting book of Borovik [23, §4.7] on this).

2 If $A$ is a commutative ring generated by a finite number of elements of degree 1, then by the celebrated work of Serre [85], the category of quasicoherent sheaves on the scheme is equivalent to QGr-$A \cong$ Gr-$A/$ Fdim-$A$, where Gr-$A$ is the category of graded modules over $A$ and Fdim-$A$ is the Serre subcategory of (direct limits of) finite dimensional submodules. In particular when $A = K[x_0, x_1, \dots, x_n]$, where $K$ is a field, then QCoh-$\mathbb{P}^n$ is equivalent to QGr-$A[x_0, x_1, \dots, x_n]$ (see [85, 9, 79] for more precise statements and relations with noncommutative algebraic geometry).

This book treats graded rings and the category of graded modules over a

Cambridge University Press
978-1-316-61958-2 - Graded Rings and Graded Grothendieck Groups
Roozbeh Hazrat
Excerpt
More information



graded ring. This category is an abelian category (in fact a Grothendieck category). Many of the classical invariants constructed for the category of modules can be constructed, *mutatis mutandis*, starting from the category of graded modules. The general viewpoint of this book is that, once a ring has a natural graded structure, graded invariants capture more information than the non-graded counterparts.

In this chapter we give a concise introduction to the theory of graded rings. We introduce grading on matrices, study graded division rings and introduce gradings on graph algebras that will be the source of many interesting examples.

## 1.1 Graded rings

### 1.1.1 Basic definitions and examples

A ring $A$ is called a $\Gamma$-*graded ring*, or simply a *graded ring*, if $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$, where $\Gamma$ is an (abelian) group, each $A_\gamma$ is an additive subgroup of $A$ and $A_\gamma A_\delta \subseteq A_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$.

If $A$ is an algebra over a field $K$, then $A$ is called a *graded algebra* if $A$ is a graded ring and for any $\gamma \in \Gamma$, $A_\gamma$ is a $K$-vector subspace.

The set $A^h = \bigcup_{\gamma\in\Gamma} A_\gamma$ is called the set of *homogeneous elements* of $A$. The additive group $A_\gamma$ is called the $\gamma$-*component* of $A$ and the nonzero elements of $A_\gamma$ are called *homogeneous of degree* $\gamma$. We write $\deg(a) = \gamma$ if $a \in A_\gamma \backslash \{0\}$. We call the set

$$\boxed{\Gamma_A = \{\, \gamma \in \Gamma \mid A_\gamma \neq 0 \,\}}$$

the *support* of $A$. We say the $\Gamma$-graded ring $A$ has a *trivial grading*, or $A$ is *concentrated in degree zero*, if the support of $A$ is the trivial group, *i.e.*, $A_0 = A$ and $A_\gamma = 0$ for $\gamma \in \Gamma\backslash\{0\}$.

For $\Gamma$-graded rings $A$ and $B$, a $\Gamma$-*graded ring homomorphism* $f : A \to B$ is a ring homomorphism such that $f(A_\gamma) \subseteq B_\gamma$ for all $\gamma \in \Gamma$. A graded homomorphism $f$ is called a *graded isomorphism* if $f$ is bijective and, when such a graded isomorphism exists, we write $A \cong_{\mathrm{gr}} B$. Notice that if $f$ is a graded ring homomorphism which is bijective, then its inverse $f^{-1}$ is also a graded ring homomorphism.

If $A$ is a graded ring and $R$ is a commutative graded ring, then $A$ is called a *graded $R$-algebra* if $A$ is an $R$-algebra and the associated algebra homomorphism $\phi : R \to A$ is a graded homomorphism. When $R$ is a field concentrated in degree zero, we retrieve the definition of a graded algebra above.

**Proposition 1.1.1** *Let $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ be a $\Gamma$-graded ring. Then*

(1) *$1_A$ is homogeneous of degree $0$;*
(2) *$A_0$ is a subring of $A$;*
(3) *each $A_\gamma$ is an $A_0$-bimodule;*
(4) *for an invertible element $a \in A_\gamma$, its inverse $a^{-1}$ is homogeneous of degree $-\gamma$, i.e., $a^{-1} \in A_{-\gamma}$.*

*Proof* (1) Suppose $1_A = \sum_{\gamma\in\Gamma} a_\gamma$ for $a_\gamma \in A_\gamma$. Let $b \in A_\delta$, $\delta \in \Gamma$, be an arbitrary nonzero homogeneous element. Then $b = b1_A = \sum_{\gamma\in\Gamma} ba_\gamma$, where $ba_\gamma \in A_{\delta+\gamma}$ for all $\gamma \in \Gamma$. Since the decomposition is unique, $ba_\gamma = 0$ for all $\gamma \in \Gamma$ with $\gamma \neq 0$. But as $b$ was an arbitrary homogeneous element, it follows that $ba_\gamma = 0$ for all $b \in A$ (not necessarily homogeneous), and in particular $1_A a_\gamma = a_\gamma = 0$ if $\gamma \neq 0$. Thus $1_A = a_0 \in A_0$.

(2) This follows since $A_0$ is an additive subgroup of $A$ with $A_0A_0 \subseteq A_0$ and $1 \in A_0$.

(3) This is immediate.

(4) Let $b = \sum_{\delta\in\Gamma} b_\delta$, with $\deg(b_\delta) = \delta$, be the inverse of $a \in A_\gamma$, so that $1 = ab = \sum_{\delta\in\Gamma} ab_\delta$, where $ab_\delta \in A_{\gamma+\delta}$. By (1), since 1 is homogeneous of degree 0 and the decomposition is unique, it follows that $ab_\delta = 0$ for all $\delta \neq -\gamma$. Since $a$ is invertible, $b_{-\gamma} \neq 0$, so $b = b_{-\gamma} \in A_{-\gamma}$ as required. □

The ring $A_0$ is called the *0-component ring* of $A$ and plays a crucial role in the theory of graded rings. The proof of Proposition 1.1.1(4), in fact, shows that if $a \in A_\gamma$ has a left (or right) inverse then that inverse is in $A_{-\gamma}$. In Theorem 1.6.9, we characterise $\mathbb{Z}$-graded rings such that $A_1$ has a left (or right) invertible element.

**Example 1.1.2** Group rings

For a group $\Gamma$, the group ring $\mathbb{Z}[\Gamma]$ has a natural $\Gamma$-grading

$$\mathbb{Z}[\Gamma] = \bigoplus_{\gamma\in\Gamma} \mathbb{Z}[\Gamma]_\gamma, \text{ where } \mathbb{Z}[\Gamma]_\gamma = \mathbb{Z}\gamma.$$

In §1.1.4, we construct crossed products which are graded rings and are generalisations of group rings and skew groups rings. A group ring has a natural involution which makes it an involutary graded ring (see §1.9).

In several applications (such as $K$-theory of rings, Chapter 6) we deal with $\mathbb{Z}$-graded rings with support in $\mathbb{N}$, the so called *positively graded rings*.

**Example 1.1.3** Tensor algebras as positively graded rings

Let $A$ be a commutative ring and $M$ be an $A$-module. Denote by $T_n(M)$,

$n \geq 1$, the tensor product of $n$ copies of $M$ over $A$. Set $T_0(M) = A$. Then the natural $A$-module isomorphism $T_n(M) \otimes_A T_m(M) \to T_{n+m}(M)$, induces a ring structure on

$$T(M) := \bigoplus_{n \in \mathbb{N}} T_n(M).$$

The $A$-algebra $T(M)$ is called the *tensor algebra* of $M$. Setting

$$T(M)_n := T_n(M)$$

makes $T(M)$ a $\mathbb{Z}$-graded ring with support $\mathbb{N}$. From the definition, we have $T(M)_0 = A$.

If $M$ is a free $A$-module, then $T(M)$ is a free algebra over $A$, generated by a basis of $M$. Thus free rings are $\mathbb{Z}$-graded rings with the generators being homogeneous elements of degree 1. We will systematically study the grading of free rings in §1.6.1.

**Example 1.1.4** Formal matrix rings as graded rings

Let $R$ and $S$ be rings, $M$ a $R$–$S$-bimodule and $N$ a $S$–$R$-bimodule. Consider the set

$$T := \left\{ \begin{pmatrix} r & m \\ n & s \end{pmatrix} \,\middle|\, r \in R, s \in S, m \in M, n \in N \right\}.$$

Suppose that there are bimodule homomorphisms $\phi : M \otimes_S N \to R$ and $\psi : N \otimes_R M \to S$ such that $(mn)m' = m(nm')$, where we denote $\phi(m,n) = mn$ and $\psi(n,m) = nm$. One can then check that $T$ with matrix addition and multiplication forms a ring with an identity. The ring $T$ is called the *formal matrix ring* and denoted also by

$$T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}.$$

For example, the Morita ring of a module is a formal matrix ring (see §2.3 and (2.6)).

Considering

$$T_0 = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix},$$

it is easy to check that $T$ becomes a $\mathbb{Z}_2$-graded ring. In the cases that the images of $\phi$ and $\psi$ are zero, these rings have been extensively studied (see [57] and references therein).

When $N = 0$, the ring $T$ is called a *formal triangular matrix ring*. In this case there is no need to consider the homomorphisms $\phi$ and $\psi$. Setting further $T_i = 0$ for $i \neq 0, 1$ makes $T$ also a $\mathbb{Z}$-graded ring.

One specific example of such a grading on (subrings of) formal triangular matrix rings is used in representation theory. Recall that for a field $K$, a finite dimensional $K$-algebra $R$ is called *Frobenius algebra* if $R \cong R^*$ as right $R$-modules, where $R^* := \operatorname{Hom}_K(R, K)$. Note that $R^*$ has a natural $R$-bimodule structure.

Starting from a finite dimensional $K$-algebra $R$, one constructs the *trivial extension* of $R$ which is a Frobenius algebra and has a natural $\mathbb{Z}$-graded structure as follows. Consider $A := R \bigoplus R^*$, with addition defined component-wise and multiplication defined as

$$(r_1, q_1)(r_2, q_2) = (r_1r_2, r_1q_1 + q_2r_2),$$

where $r_1, r_2 \in R$ and $q_1, q_2 \in R^*$. Clearly $A$ is a Frobenius algebra with identity $(1, 0)$. Moreover, setting

$$\begin{aligned} A_0 &= R \oplus 0,\\ A_1 &= 0 \oplus R^*,\\ A_i &= 0, \text{ otherwise,} \end{aligned}$$

makes $A$ into a $\mathbb{Z}$-graded ring with support $\{0, 1\}$. In fact this ring is a subring of the formal triangular matrix ring

$$T_0 = \begin{pmatrix} R & R^* \\ 0 & R \end{pmatrix},$$

consisting of elements $\begin{pmatrix} a & q \\ 0 & a \end{pmatrix}$.

These rings appear in representation theory (see [46, §2.2]). The graded version of this contraction is carried out in Example 1.2.9.

**Example 1.1.5** <u>THE GRADED RING $A$ AS $A_0$-MODULE</u>

Let $A$ be a $\Gamma$-graded ring. Then $A$ can be considered as an $A_0$-bimodule. In many cases $A$ is a projective $A_0$-module, for example in the case of group rings (Example 1.1.2) or when $A$ is a strongly graded ring (see §1.1.3 and Theorem 1.5.12). Here is an example that this is not the case in general. Consider the formal matrix ring $T$

$$T = \begin{pmatrix} R & M \\ 0 & 0 \end{pmatrix},$$

where $M$ is a left $R$-module which is not a projective $R$-module. Then by Example 1.1.4, $T$ is a $\mathbb{Z}$-graded ring with $T_0 = R$ and $T_1 = M$. Now $T$ as a $T_0$-module is $R \oplus M$ as an $R$-module. Since $M$ is not projective, $R \oplus M$ is not a projective $R$-module. We also get that $T_1$ is not a projective $T_0$-module.

### 1.1.2 Partitioning graded rings

Let $A$ be a $\Gamma$-graded ring and $f:\Gamma\to\Delta$ be a group homomorphism. Then one can assign a natural $\Delta$-graded structure to $A$ as follows: $A=\bigoplus_{\delta\in\Delta}A_\delta$, where

$$A_\delta=\begin{cases}\bigoplus_{\gamma\in f^{-1}(\delta)}A_\gamma & \text{if } f^{-1}(\delta)\neq\varnothing;\\ 0 & \text{otherwise.}\end{cases}$$

In particular, for a subgroup $\Omega$ of $\Gamma$ we have the following constructions.

**Subgroup grading** The ring $A_\Omega:=\bigoplus_{\omega\in\Omega}A_\omega$ forms a $\Omega$-graded ring. In particular, $A_0$ corresponds to the trivial subgroup of $\Gamma$.

**Quotient grading** Considering

$$A=\bigoplus_{\Omega+\alpha\in\Gamma/\Omega}A_{\Omega+\alpha},$$

where

$$A_{\Omega+\alpha}:=\bigoplus_{\omega\in\Omega}A_{\omega+\alpha},$$

makes $A$ a $\Gamma/\Omega$-graded ring. (Note that if $\Gamma$ is not abelian, then for this construction, $\Omega$ needs to be a normal subgroup.) Notice that with this grading, $A_0=A_\Omega$. If $\Gamma_A\subseteq\Omega$, then $A$, considered as a $\Gamma/\Omega$-graded ring, is concentrated in degree zero.

This construction induces a *forgetful* functor (or with other interpretations, a *block*, or a *coarsening* functor) from the category of $\Gamma$-graded rings $\mathcal{R}^\Gamma$ to the category of $\Gamma/\Omega$-graded rings $\mathcal{R}^{\Gamma/\Omega}$, *i.e.,*

$$U:\mathcal{R}^\Gamma\to\mathcal{R}^{\Gamma/\Omega}.$$

If $\Omega=\Gamma$, this gives the obvious forgetful functor from the category of $\Gamma$-graded rings to the category of rings. We give a specific example of this construction in Example 1.1.8 and others in Examples 1.1.20 and 1.6.1.

**Example 1.1.6** TENSOR PRODUCT OF GRADED RINGS

Let $A$ be a $\Gamma$-graded and $B$ a $\Omega$-graded ring. Then $A\otimes_{\mathbb{Z}}B$ has a natural $\Gamma\times\Omega$-graded ring structure as follows. Since $A_\gamma$ and $B_\omega$, $\gamma\in\Gamma,\omega\in\Omega$, are $\mathbb{Z}$-modules then $A\otimes_{\mathbb{Z}}B$ can be decomposed as a direct sum

$$A\otimes_{\mathbb{Z}}B=\bigoplus_{(\gamma,\omega)\in\Gamma\times\Omega}A_\gamma\otimes B_\omega$$

(to be precise, $A_\gamma\otimes B_\omega$ is the image of $A_\gamma\otimes_{\mathbb{Z}}B_\omega$ in $A\otimes_{\mathbb{Z}}B$).