

Contents

<i>Preface</i>	<i>page xi</i>
1 Introduction	1
2 Lie groups: basic definitions	4
2.1. Reminders from differential geometry	4
2.2. Lie groups, subgroups, and cosets	5
2.3. Lie subgroups and homomorphism theorem	10
2.4. Action of Lie groups on manifolds and representations	10
2.5. Orbits and homogeneous spaces	12
2.6. Left, right, and adjoint action	14
2.7. Classical groups	16
2.8. Exercises	21
3 Lie groups and Lie algebras	25
3.1. Exponential map	25
3.2. The commutator	28
3.3. Jacobi identity and the definition of a Lie algebra	30
3.4. Subalgebras, ideals, and center	32
3.5. Lie algebra of vector fields	33
3.6. Stabilizers and the center	36
3.7. Campbell–Hausdorff formula	38
3.8. Fundamental theorems of Lie theory	40
3.9. Complex and real forms	44
3.10. Example: $\mathfrak{so}(3, \mathbb{R})$, $\mathfrak{su}(2)$, and $\mathfrak{sl}(2, \mathbb{C})$	46
3.11. Exercises	48

4	Representations of Lie groups and Lie algebras	52
4.1.	Basic definitions	52
4.2.	Operations on representations	54
4.3.	Irreducible representations	57
4.4.	Intertwining operators and Schur's lemma	59
4.5.	Complete reducibility of unitary representations: representations of finite groups	61
4.6.	Haar measure on compact Lie groups	62
4.7.	Orthogonality of characters and Peter–Weyl theorem	65
4.8.	Representations of $\mathfrak{sl}(2, \mathbb{C})$	70
4.9.	Spherical Laplace operator and the hydrogen atom	75
4.10.	Exercises	80
5	Structure theory of Lie algebras	84
5.1.	Universal enveloping algebra	84
5.2.	Poincaré–Birkhoff–Witt theorem	87
5.3.	Ideals and commutant	90
5.4.	Solvable and nilpotent Lie algebras	91
5.5.	Lie's and Engel's theorems	94
5.6.	The radical. Semisimple and reductive algebras	96
5.7.	Invariant bilinear forms and semisimplicity of classical Lie algebras	99
5.8.	Killing form and Cartan's criterion	101
5.9.	Jordan decomposition	104
5.10.	Exercises	106
6	Complex semisimple Lie algebras	108
6.1.	Properties of semisimple Lie algebras	108
6.2.	Relation with compact groups	110
6.3.	Complete reducibility of representations	112
6.4.	Semisimple elements and toral subalgebras	116
6.5.	Cartan subalgebra	119
6.6.	Root decomposition and root systems	120
6.7.	Regular elements and conjugacy of Cartan subalgebras	126
6.8.	Exercises	130
7	Root systems	132
7.1.	Abstract root systems	132
7.2.	Automorphisms and the Weyl group	134
7.3.	Pairs of roots and rank two root systems	135

Contents

ix

7.4. Positive roots and simple roots	137
7.5. Weight and root lattices	140
7.6. Weyl chambers	142
7.7. Simple reflections	146
7.8. Dynkin diagrams and classification of root systems	149
7.9. Serre relations and classification of semisimple Lie algebras	154
7.10. Proof of the classification theorem in simply-laced case	157
7.11. Exercises	160
8 Representations of semisimple Lie algebras	163
8.1. Weight decomposition and characters	163
8.2. Highest weight representations and Verma modules	167
8.3. Classification of irreducible finite-dimensional representations	171
8.4. Bernstein–Gelfand–Gelfand resolution	174
8.5. Weyl character formula	177
8.6. Multiplicities	182
8.7. Representations of $\mathfrak{sl}(n, \mathbb{C})$	183
8.8. Harish–Chandra isomorphism	187
8.9. Proof of Theorem 8.25	192
8.10. Exercises	194
Overview of the literature	197
Basic textbooks	197
Monographs	198
Further reading	198
Appendix A Root systems and simple Lie algebras	202
A.1. $A_n = \mathfrak{sl}(n + 1, \mathbb{C}), n \geq 1$	202
A.2. $B_n = \mathfrak{so}(2n + 1, \mathbb{C}), n \geq 1$	204
A.3. $C_n = \mathfrak{sp}(n, \mathbb{C}), n \geq 1$	206
A.4. $D_n = \mathfrak{so}(2n, \mathbb{C}), n \geq 2$	207
Appendix B Sample syllabus	210
List of notation	213
<i>Bibliography</i>	216
<i>Index</i>	220