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Unless otherwise stated, the interval of differencing is to be taken as unity.

1. Find \( \Delta^2 (2^4x - x^2 + x - 1) \).

2. Express in its simplest form \( \frac{\Delta^2 u_x}{E^2} u_x - \frac{\Delta^2 u_y}{E^2} u_y \), where
   (i) \( u_x = a + bx^3 \);
   (ii) \( u_x = e^{ax} \) (interval \( h \)).

3. Obtain, for an interval \( a \),
   (i) \( \Delta \sin x \cos x \);
   (ii) \( \frac{1}{\Delta \cot x} - \frac{1}{\Delta \tan x} \).

4. Find \( \Delta^3 \frac{4x+17}{(2x+1)(2x+3)(2x+5)(2x+7)} \).

5. Obtain simple expressions for the first differences of
   (i) \( (4-x)^7 \{ (x-2)^6 - \frac{1}{2} \}^{-1} \);
   (ii) \( [\Delta E^{-1} \log_e (ax+b)]^8 \) (interval \( a \)).

6. Find the value of
   \( \Delta x^{(m)} - 2 \Delta^2 x^{(m)} + 3 \Delta^3 x^{(m)} - 4 \Delta^4 x^{(m)} + \ldots \) to \( m \) terms.

7. Calculate \( \Delta^2 e^x \), when
   (i) \( f(x) = xe^x \);
   (ii) \( f(x) = (3 + x) \left[ (x+2)^2 - \frac{1}{2} \right]^{-1} \).

8. If \( u_x = e^x \) and \( v_x = x^3 \), find the value of
   (i) \( \frac{E u_x}{E v_x} - \frac{\Delta^2}{E^2} u_x + \frac{\Delta^2}{E^2} v_x \);
   (ii) \( \Delta (u_x v_x) - u_x \Delta v_x - v_x + 1 \Delta u_x \).

9. Show that
   \[ u_0 + nu_1 x + \frac{n(n-1)}{2!} u_2 x^2 + \ldots \]
   \[ = (1 + x)^n u_0 + n (1 + x)^{n-1} x \Delta u_0 + \frac{n(n-1)}{2!} (1 + x)^{n-2} x^2 \Delta^2 u_0 + \ldots \]

10. Given \( u_0 = 5, u_1 = 10, u_2 = 20, u_3 = -8 \), obtain \( u_4 \) by constructing a difference table.
   (i) \( u_x = 1126, u_{x+1} = 1365, u_{x+3} = 2143 \). Find \( u_{x+2} \).
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11. Without constructing a difference table find $\Delta^4 u_3$, given:

\[
\begin{array}{c}
x & 9 & 8 & 7 & 6 & 5 & 4 & 3 \\
u_x & 57 & 54 & 41 & 26 & 13 & 4 & 0 \\
\end{array}
\]

12. Corresponding values of $x$ and $f(x)$ are as in the following table:

\[
\begin{array}{c}
x & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
f(x) & 0 & 10 & x & 51 & 100 & 200 & \beta \\
\end{array}
\]

If fifth differences are constant and equal to 100, find $x$ and $\beta$.

13. $u_0=0$, $u_1=79$, $u_2=146$, $u_3=204$, $u_4=310$, $u_5=534$. Find $u_4$, $u_5$ and $u_7$.

14. Given that as $x$ takes the values 0, 1, 2, 3, 4, 5, $u_x$ takes the values 9, 1, 25, 81, 169, 289 respectively, obtain the form of $u_x$ in the following cases:

(a) $u_x$ is a rational integral function of $x$ of the second degree;
(b) $u_x$ is a rational integral function of $x$ of degree less than 6;
(c) $u_x$ is a rational integral function of $x$;
(d) no information is available regarding the nature of $u_x$.

15. $q_{50}=0.02111$, $q_{51}=0.02444$, $q_{56}=0.02629$, $q_{59}=0.02827$. Find $q_{55}$.

16. Supply the annuity values for ages 40 and 43:

<table>
<thead>
<tr>
<th>Age</th>
<th>$a_x$</th>
<th>$13.798$</th>
<th>$13.408$</th>
<th>$12.605$</th>
<th>$12.194$</th>
<th>$11.355$</th>
</tr>
</thead>
</table>

17. The following values of $x$ and $f(x)$ are given:

\[
\begin{array}{c}
x & 0 & 1 & 2 & 3 & 4 & 5 \\
f(x) & -66 & 0 & 0 & 0 & 354 & 1824 \\
\end{array}
\]

Assuming that $f(x)$ is a rational integral function of $x$, find the form of $f(x)$, and hence obtain $f(\frac{1}{2})$ to the nearest whole number.

18. $u_2=1.126$, $u_3=1.365$, $u_4=1.849$, $u_5=2.381$. Find $u_0$ and $u_2$.

19. Find $A_{51}$, given that $A_{43}=0.49136$, $A_{44}=0.50515$, $A_{45}=0.51851$, $A_{46}=0.53148$, $A_{47}=0.54410$.

20. Complete the series for $f(x)$ from $x=0$ to $x=12$ from the data:

\[
\begin{array}{c}
x & 0 & 4 & 8 & 12 \\
f(x) & 2.714 & 2.884 & 3.037 & 3.175 \\
\end{array}
\]
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21. Express \( \frac{x^3 + x + 5}{x^4 + 10x^3 + 35x^2 + 50x + 24} \) as the sum of a number of inverse factorials, and hence find its second difference in the form of a single fraction.

22. Six consecutive terms of a series are given:
   \( 0, 1, 3, 8, 20, 47. \)
   An estimate is required of the seventh term, and \( A \) gives 103 for his answer. \( B \) states that a more probable figure is 105. Examine these estimates and give reasons for your preference, if any, for one over the other.

23. The following table gives the number of children under five years of age per 1000 women between the ages of 16 and 44 resident in a certain country at the undermentioned dates:

<table>
<thead>
<tr>
<th>Year</th>
<th>1800</th>
<th>1805</th>
<th>1815</th>
<th>1830</th>
<th>1860</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of children</td>
<td>976</td>
<td>976</td>
<td>952</td>
<td>877</td>
<td>714</td>
</tr>
</tbody>
</table>

Estimate the number of children per 1000 women for the year 1810.

24. Find \( u_x \) as a rational integral function of \( x \) from the following data:

\[
\begin{array}{cccccc}
   x & 2 & 6 & 4 & 8 & 7 \\
   u_x & 5 & 205 & 57 & 497 & 330 \\
\end{array}
\]

25. What simple form of function gives \( u_0 = -4, u_3 = 23, u_4 = 60 \) and \( u_{10} = 996 \)?

26. Find \( x \) when \( u_x = 160 \), given that \( u_x \) has the values 153, 157, 164 and 177 when \( x \) has the values 10, 11, 13 and 16 respectively.

27. Four values of a function \( u_a, u_x, u_b, u_c \) are given, corresponding to the values \( a, x, b, c \) of the variable. Find the third divided difference of \( u_x \) and deduce Lagrange’s formula for \( u_x \) in terms of \( u_a, u_b \) and \( u_c \).

28. Prove the interpolation formula:

\[
u_x = u_0 + x \cdot \frac{1}{2} (\Delta u_0 + \Delta u_1) + \frac{x^2}{2} \Delta^2 u_{-1} + \frac{x(x^2 - 1)}{6} \frac{1}{2} (\Delta^3 u_{-1} + \Delta^3 u_{-2}) + \frac{x^2(x^2 - 1)}{24} \Delta^4 u_{-2} + \ldots \]

Set out a table of differences and find \( q_{01} \) from the values

\[
q_{00} = 0.015714, \quad q_{10} = 0.015152, \quad q_{02} = 0.016303, \\
q_{04} = 0.016915, \quad q_{06} = 0.014625.
\]
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29. Find \( f(1.4375) \), given the following data:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.40</th>
<th>1.41</th>
<th>1.42</th>
<th>1.43</th>
<th>1.44</th>
<th>1.45</th>
<th>1.46</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1053</td>
<td>808</td>
<td>608</td>
<td>452</td>
<td>339</td>
<td>268</td>
<td>238</td>
</tr>
</tbody>
</table>

30. Given \( n \) corresponding values of a variable and of a rational integral function of the variable, in what circumstances would you use the following formulae to interpolate between two given values of the function to find a value corresponding to an intermediate value of the variable:

(i) Newton’s advancing difference formula;
(ii) Lagrange’s formula;
(iii) Newton’s divided difference formula;
(iv) A central difference formula?

31. Use an appropriate interpolation formula to find \( u_1 \), given \( u_e = 0, 108, 200, 316, 508, 840 \) for the values of \( x = -4, -2, 0, 2, 4, 6 \) respectively.

32. Show that

\[
u_e = \frac{1}{2} (u_0 + u_1) + (x - \frac{1}{2}) \Delta u_0 + \frac{x(x-1)}{2!} \cdot \frac{\Delta^2 u_{-1} + \Delta^2 u_{0}}{2} + \ldots\]

Extend the formula to include terms involving the third and fourth differences of the function.

Use the formula to find \( f(50) \) from the values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>40</th>
<th>48</th>
<th>56</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>14.27</td>
<td>15.81</td>
<td>17.72</td>
<td>19.96</td>
</tr>
</tbody>
</table>

33. From the following data calculate \( a_e \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_x )</td>
<td>24.584</td>
<td>23.473</td>
<td>22.285</td>
<td>20.943</td>
<td>19.468</td>
<td>17.881</td>
</tr>
</tbody>
</table>

using (i) two of the values;
(ii) four of the values;
(iii) all the values given.

34. Construct a formula of interpolation expressing \( u_e \) in terms of \( u_0, \Delta u_{-1}, \Delta^2 u_{-1}, \Delta^3 u_{-2}, \Delta^4 u_{-3} \) and \( \Delta^5 u_{-3} \), and hence determine the entry corresponding to the argument \( 4\frac{1}{2} \), given:

<table>
<thead>
<tr>
<th>Argument</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entry</td>
<td>1901</td>
<td>1264</td>
<td>29</td>
<td>-304</td>
<td>5125</td>
<td>27056</td>
<td>85349</td>
</tr>
</tbody>
</table>
35. Interpolate by means of Gauss’s forward formula to find the present value of £1 due 27 years hence at 5 per cent. compound interest from the following data:

<table>
<thead>
<tr>
<th>No. of years</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
</tr>
</thead>
</table>

36. Employ an appropriate formula to obtain successive approximations to \( f(28.3) \), given the table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>26</th>
<th>27</th>
<th>28</th>
<th>29</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.38462</td>
<td>0.37037</td>
<td>0.35714</td>
<td>0.34483</td>
<td>0.33333</td>
</tr>
</tbody>
</table>

37. Prove Everett’s formula:

\[
\begin{align*}
ux &= xu_1 + \frac{x(x^2 - 1)}{3!} \Delta^2 u_0 + \frac{x(x^2 - 1)(x^2 - 4)}{5!} \Delta^4 u_{-1} + \ldots \\
&+ \xi u_0 + \frac{\xi(x^2 - 1)}{3!} \Delta^2 u_{-1} + \frac{\xi(x^2 - 1)(x^2 - 4)}{5!} \Delta^4 u_{-2} + \ldots
\end{align*}
\]

What are the practical advantages of this formula?

*Note.* In the next five questions Everett’s formula is to be used.

38. Find the present value of £1 per annum at the end of 20 years at 3.2 per cent., given the following extract from tables of compound interest:

<table>
<thead>
<tr>
<th>Rate per cent.</th>
<th>2</th>
<th>2(\frac{1}{2})</th>
<th>3</th>
<th>3(\frac{1}{2})</th>
<th>4</th>
<th>4(\frac{1}{2})</th>
</tr>
</thead>
</table>

39. The annual premium \( P_x \) for an assurance of £1 at 4 per cent. being available for the undermentioned ages, find \( P_x \) for ages 36 to 44 inclusive:

<table>
<thead>
<tr>
<th>Age ( x )</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
<th>55</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_x )</td>
<td>0.1431</td>
<td>0.1662</td>
<td>0.1961</td>
<td>0.2351</td>
<td>0.2865</td>
<td>0.3550</td>
<td>0.4471</td>
</tr>
</tbody>
</table>
6  

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40. The following is an extract from the prospectus of a Life Assurance Company, giving the annual premiums for an assurance of £100:

<table>
<thead>
<tr>
<th>Age at entry</th>
<th>Annual premium £ s. d.</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1 19 6</td>
</tr>
<tr>
<td>25</td>
<td>2 3 11</td>
</tr>
<tr>
<td>30</td>
<td>2 9 5</td>
</tr>
<tr>
<td>35</td>
<td>2 16 2</td>
</tr>
<tr>
<td>40</td>
<td>3 4 10</td>
</tr>
<tr>
<td>45</td>
<td>3 15 8</td>
</tr>
<tr>
<td>50</td>
<td>4 10 7</td>
</tr>
</tbody>
</table>

Obtain the premiums for ages 31 to 39 inclusive.

41. Calculate $q_{71}$, $q_{72}$, $q_{73}$ and $q_{74}$, given:

<table>
<thead>
<tr>
<th>Age $x$</th>
<th>$60$</th>
<th>$65$</th>
<th>$70$</th>
<th>$75$</th>
<th>$80$</th>
<th>$85$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_x$</td>
<td>0.3108</td>
<td>0.4468</td>
<td>0.6558</td>
<td>0.9747</td>
<td>1.4525</td>
<td>2.1504</td>
</tr>
</tbody>
</table>

42. From the following table of $\tan x$ obtain the values of the tangents of all angles from 55° to 65° inclusive:

<table>
<thead>
<tr>
<th>Angle $x$ (degrees)</th>
<th>$45$</th>
<th>$50$</th>
<th>$55$</th>
<th>$60$</th>
<th>$65$</th>
<th>$70$</th>
<th>$75$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tan x$</td>
<td>1.0000</td>
<td>1.1918</td>
<td>1.4281</td>
<td>1.7320</td>
<td>2.1445</td>
<td>2.7475</td>
<td>3.7320</td>
</tr>
</tbody>
</table>

43. Values of the yearly pension secured to a wife after her husband’s death by a yearly contribution of £1 for a husband aged 30 next birthday, according to two different tables:

<table>
<thead>
<tr>
<th>Age of wife next birthday</th>
<th>$20$</th>
<th>$30$</th>
<th>$40$</th>
<th>$50$</th>
<th>$60$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pension on Table I (£)</td>
<td>4.39</td>
<td>5.00</td>
<td>6.03</td>
<td>7.77</td>
<td>10.74</td>
</tr>
<tr>
<td>Pension on Table II (£)</td>
<td>3.90</td>
<td>4.47</td>
<td>5.44</td>
<td>7.13</td>
<td>10.15</td>
</tr>
</tbody>
</table>

Find, for age of wife 33 next birthday, the percentage increase in the yearly pension resulting from the adoption of Table I in place of Table II.

44. Use Stirling’s formula to find $a_{73}$, given:

<table>
<thead>
<tr>
<th>Age $x$</th>
<th>$20$</th>
<th>$25$</th>
<th>$30$</th>
<th>$35$</th>
<th>$40$</th>
<th>$45$</th>
</tr>
</thead>
</table>

How would you alter your method, if at all, if you were required to find the values for all ages from 20 to 45 inclusive?
45. If \( a_x \) is the value of an annuity on a life aged \( x \), payable yearly, an approximation to the value of the annuity payable continuously—i.e. at very frequent intervals—is

\[ \ddot{a}_x = \frac{1}{2} + a_x - \frac{1}{2} (\mu_x + \delta). \]

\( a_x \) is available for the undermentioned ages:

\[
\begin{array}{cccccc}
\text{Age } x & 75 & 80 & 85 & 90 & 95 \\
a_x & 4.747 & 3.417 & 2.419 & 1.676 & 1.045 \\
\end{array}
\]

and the corrective factor for a different set of ages, thus:

\[
\begin{array}{cccc}
\text{Age } x & 77 & 82 & 87 & 92 \\
\frac{1}{2} (\mu_x + \delta) & .013 & .019 & .027 & .039 \\
\end{array}
\]

From these data calculate \( \ddot{a}_{93} \).

46. \( f(8) = 1.4422 \), \( f(9) = 1.4581 \), \( f(10) = 1.4736 \), \( f(11) = 1.4888 \) and \( f(12) = 1.5037 \). Find \( f(8.5) \), using all the values given.

47. Use a central difference formula to obtain successive approximations to \((1+i)^9\) from the table:

\[
(1+i)^x = 1.27628 \quad 1.62889 \quad 2.07893 \quad 2.53320 \quad 3.00354 \quad 3.52194
\]

48. From the following values of the expectation of life by English Life Table No. 8, estimate \( e_{85} \):

\[
\begin{array}{cccccc}
x & 20 & 25 & 30 & 35 & 40 \\
e_x & 44.21 & 40.00 & 35.81 & 31.71 & \\
\end{array}
\]

If in addition the values for ages 10 (53.08) and 15 (48.57) are given, obtain a revised estimate for \( e_{85} \).

49. Construct a table of divided differences from the data:

\[
\begin{array}{cccccc}
x & 0 & 5 & 8 & 9 & 25 \\
f(x) & -179 & 1 & 13 & 37 & 7221 \\
\end{array}
\]

Extend the table to include arguments \( x = 5 \), repeated as many times as may be necessary, and thus find \( f(x) \) in powers of \((x - 5)\).

50. Prove Everett’s second formula:

\[
u_{n-\frac{1}{2}} = u_0 + \frac{p^2 - \frac{1}{2}}{2!} \Delta u_0 + \frac{(p^2 - \frac{1}{2})(p^2 - \frac{3}{2})}{4!} \Delta^2 u_{-1} + \ldots
\]

\[
- \frac{q^2 - \frac{1}{2}}{2!} \Delta u_{-1} - \frac{(q^2 - \frac{1}{2})(q^2 - \frac{3}{2})}{4!} \Delta^2 u_{-2} - \ldots
\]

When would you use this formula in practice?
51. Comment on the following extracts from a text-book on statistics:

(a) In order to apply the method of interpolation to statistics, it is assumed that the formulae may be employed with respect to figures which do not obey any definite law.

(b) When the argument proceeds by unequal intervals, the most convenient formula for interpolation is Lagrange’s.

52. Establish the formula:

\[ u_x = -\frac{1}{6} (x^2 - \frac{1}{4}) (x - \frac{3}{2}) u_{-\frac{3}{4}} + \frac{1}{6} (x - \frac{1}{4}) (x^2 - \frac{3}{4}) u_{-\frac{3}{4}} \]

\[ -\frac{1}{6} (x + \frac{1}{4}) (x^2 - \frac{1}{4}) u_{\frac{1}{4}} + \frac{1}{6} (x^2 - \frac{3}{4}) (x + \frac{3}{4}) u_{\frac{3}{4}}. \]

Use the formula to find \( f \left(1\frac{1}{4}\right)\), given that \( f(0) = 896, f(1) = 1000, f(2) = 1216, f(3) = 1508.\)

53. Given \( f \left(-\frac{1}{3}\right), f \left(-\frac{3}{4}\right), f \left(-\frac{1}{2}\right), f \left(\frac{1}{2}\right), f \left(\frac{3}{4}\right)\) and \( f \left(\frac{5}{3}\right)\), obtain a six-term formula for \( f(x)\) similar to the four-term formula in Qu. 52.

54. In a certain series \( u_0 = a, \Delta u_0 = b, \Delta^2 u_x = c \) for all values of \( x \) from 0 to \( m - 1 \), and \( \Delta^2 u_x = d \) for all subsequent values; find \( u_{m+n}, \) where \( m \) and \( n \) are positive.

If the series in question is

\[ 6, 11, 20, 33, 50, u, 105, 143, 188, 240, \ldots, \]

find \( u \).

55. \( y \) is a function of \( x \). When \( x \) is 30, \( y \) is 1; when \( x \) is 31, \( y \) is 2; and when \( x \) is 32, \( y \) is 103. Find a value or values for \( x \) when \( y \) is 50, illustrating your answer by a diagram.

56. The following table gives the present value of \( £1 \) due 20 years hence, at varying rates of interest. Find, correct to three decimal places, the rate of interest for which the present value is \( £\cdot4 \), by a method of inverse interpolation:

<table>
<thead>
<tr>
<th>Rate per cent.</th>
<th>4(\frac{3}{8})</th>
<th>4(\frac{1}{2})</th>
<th>4(\frac{3}{4})</th>
<th>4(\frac{7}{8})</th>
<th>4(\frac{5}{2})</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present value of ( £1 )</td>
<td>42469</td>
<td>41464</td>
<td>40485</td>
<td>39529</td>
<td>38598</td>
<td>37689</td>
</tr>
</tbody>
</table>

57. Find approximate values of the real roots of the equations

(a) \( x^3 + x = 3 \) (root between 1·2 and 1·3);
(b) \( x^3 + 2x = 20 \) (root between 2·4 and 2·5).
58. If \( u_x = p + x + x^2 \), find, by inverse interpolation from \( u_0, u_1 \) and \( u_4 \), the value of \( x \) corresponding to \( u_x = 3 \).

What must \( p \) be in order that this value of \( x \) may satisfy the equation

\[ 3 = p + x + x^2. \]

59. Explain the method of elimination of third differences as used in inverse interpolation, and employ the method to find the real root of \( x^3 + 12x = 12 \), correct to three decimal places.

60. One root of the equation \( 10(x^3 - 7x + 6) = x^2 \) is approximately 1. Find its value correct to four decimal places.

61. The value of an annuity-certain of \( £1 \) per annum for 30 years is given as \( £20 \), and it is desired to find the rate of interest involved. Using the following extract from a table of annuity values, obtain an approximate value for the rate of interest:

<table>
<thead>
<tr>
<th>Rate per cent.</th>
<th>2</th>
<th>2½</th>
<th>3</th>
<th>3½</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of a 30-year annuity (( £ ))</td>
<td>22.40</td>
<td>20.93</td>
<td>19.60</td>
<td>18.39</td>
</tr>
</tbody>
</table>

62. Find the real roots of

\[
(a) \quad x^3 = 6x + 11; \quad (b) \quad 13x^3 + 68x = 57x^2 + 36;
\]
each correct to three decimal places.

63. \( u \) is the present value of a given series of payments, and it is known that \( u \) lies between \( u_0 \) and \( u_1 \), which are the present values of the given series at rates \( i \) and \( i+h \) respectively. If \( i+k \) be the rate of interest for which the present value is \( u \), prove that, approximately,

\[
k = \frac{h}{\Delta u_0 \left( u - u_0 \right) + \frac{1}{2} \Delta^2 u_0}.
\]

64. If \( x = \Delta u_0 \) and \( \beta = \frac{1}{2} \Delta^2 (u_{-1} + u_0) \), show that the relation between \( k \) and \( h \) in Qu. 63 may be expressed as

\[
k = h \frac{u - u_0}{\alpha - \frac{1}{2} \beta + \frac{1}{2} \beta} \quad \text{approximately}.
\]

65. Show, by means of examples, that the following are not universally true:

(i) \( \Sigma \Delta \equiv 1 \);

(ii) \( u_x = (1 + \Delta)^x u_0 \).
10. **FINITE DIFFERENCES**

66. Sum to $n$ terms

(a) 3, 8, 11, 18, 35, 68, ...; (b) 0, 0, $1\frac{1}{8}$, 5, $10\frac{1}{8}$, $18\frac{7}{8}$, $29\frac{7}{8}$, ....

67. Find the sum of $n$ terms of the series whose $x$th term is

$$3x(x-3)(x-6)-2x(2x-1)(2x-2).$$

68. Sum to $n$ terms

(a) 2, 12, 22, 36, 62, 116, 230, ...;
(b) 1, 2, 4, 14, 40, 92, 184, 338, ...

69. Prove the formula for summation by parts in finite differences, and apply it to obtain the sum of $n$ terms of the series

$$5+2.8+4.12+8.17+16.23+...$$

70. Sum to $n$ terms

(a) \[
\frac{3}{1\cdot2\cdot4} + \frac{4}{2\cdot3\cdot5} + \frac{5}{3\cdot4\cdot6} + \frac{6}{4\cdot5\cdot7} + \ldots;
\]

(b) \[
\frac{15}{1\cdot2\cdot3\cdot4\cdot7} + \frac{16}{2\cdot3\cdot4\cdot7} + \frac{17}{3\cdot4\cdot5\cdot7} + \ldots
\]

71. Find the general term of each of the series

(a) 67, 39, 29, 29, 35, 47, 69, ...
(b) 67, 39, 29, 29, 35, 45, 58, ...

Sum series (b) to $n$ terms.

72. The series $10, 30, 98, 346, 1278, 4838, ...$ is the sum of two series. The $x$th term of one of the series is of the form $p + qx + rx^2$, while the other series is a geometrical progression. Find the general term of the combined series and its sum to $n$ terms.

73. Sum to $n$ terms

$$11 + 32 + 32 + 64 + 64 + 97 + 98 + 131 + 136 + 166 + 182 + ...$$

74. Find

(i) $\Delta^{-1}\{x!(x^2+x+1)\}$; (ii) $\Delta^{-3}x^n = \frac{1}{\Delta^3x^n}$

75. Evaluate $\sum_{1}^{n} \{\Sigma x^2a^n\}$.

76. Prove that $2\Sigma_{1}^{n} \{x^2 + 3x\} = (n-1)2^{n+2} + 3n^2 + 3n + 4$. 

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