FINITE DIFFERENCES

Unless otherwise stated, the interval of differencing is to be taken as unity.

1. Find $$\Delta^2 (2^{4x} - x^2 + x - 1).$$

2. Express in its simplest form $\dfrac{\Delta^3}{E^2} u_x - \dfrac{\Delta^3 u_x}{E^2 u_x}$, where

   (i) $u_x = a + bx^3$; (ii) $u_x = e^{hx}$ (interval h).

3. Obtain, for an interval α,

   (i) $\Delta \sin x \cos x$; (ii) $\dfrac{1}{\Delta \cot x} - \dfrac{1}{\Delta \tan x}$.

4. Find $$\Delta^3 \frac{4x+17}{(2x+1)(2x+3)(2x+5)(2x+7)}.$$

5. Obtain simple expressions for the first differences of

   (i) $(4-x)\{(x-2)^2 - \frac{1}{4}\}^{-1}$;

   (ii) $[\Delta E^{-1} \log_e (ax+b)]^2$ (interval a).

6. Find the value of

   $$\Delta x^{(m)} - 2\Delta^2 x^{(m)} + 3\Delta^3 x^{(m)} - 4\Delta^4 x^{(m)} + \ldots \text{ to } m \text{ terms.}$$

7. Calculate $\Delta E^{-2} f(x)$, when

   (i) $f(x) = xe^x$; (ii) $f(x) = (3+x)[(x+2)^2 - \frac{1}{9}]^{-1}$.

8. If $u_x = e^x$ and $v_x = x^3$, find the value of

   (i) $\dfrac{Eu_x}{\Delta v_x} - \dfrac{\Delta^2}{E^2} u_x + \dfrac{\Delta^2}{E^2} v_x$; (ii) $\Delta (u_x v_x) - u_x \Delta v_x - v_{x+1} \Delta u_x$.

9. Show that

$$u_0 + nu_1 x + \frac{n(n-1)}{2!} u_2 x^2 + \ldots$$
$$= (1+x)^n u_0 + n(1+x)^{n-1} x\Delta u_0 + \frac{n(n-1)}{2!}(1+x)^{n-2} x^2 \Delta^2 u_0 + \ldots.$$

10. (i) Given $u_0 = 5$, $u_1 = 10$, $u_2 = 20$, $u_3 = -8$, obtain u_4 by constructing a difference table.

   (ii) $u_x = \cdot 1126$, $u_{x+1} = \cdot 1365$, $u_{x+3} = \cdot 2143$. Find u_{x+2}.

11. Without constructing a difference table find $\Delta^6 u_3$, given:

x	9	8	7	6	5	4	3
u_x	57	54	41	26	13	4	0

12. Corresponding values of x and $f(x)$ are as in the following table:

x	10	11	12	13	14	15	16
$f(x)$	0	10	α	51	100	200	β

If fifth differences are constant and equal to 100, find α and β.

13. $u_0=0$, $u_1=79$, $u_2=146$, $u_3=204$, $u_5=310$, $u_8=534$. Find u_4, u_6 and u_7.

14. Given that as x takes the values 0, 1, 2, 3, 4, 5, u_x takes the values 9, 1, 25, 81, 169, 289 respectively, obtain the form of u_x in the following cases:

(*a*) u_x is a rational integral function of x of the second degree;

(*b*) u_x is a rational integral function of x of degree less than 6;

(*c*) u_x is a rational integral function of x;

(*d*) no information is available regarding the nature of u_x.

15. $q_{55}=\cdot 02111$, $q_{57}=\cdot 02444$, $q_{58}=\cdot 02629$, $q_{59}=\cdot 02827$. Find q_{56}.

16. Supply the annuity values for ages 40 and 43:

Age x	38	39	41	42	44
a_x	13·798	13·408	12·605	12·194	11·355

17. The following values of x and $f(x)$ are given:

x	0	1	2	3	4	5
$f(x)$	−66	0	0	0	354	1824

Assuming that $f(x)$ is a rational integral function of x, find the form of $f(x)$, and hence obtain $f(\frac{1}{4})$ to the nearest whole number.

18. $u_{-2}=\cdot 1126$, $u_{-1}=\cdot 1365$, $u_1=\cdot 1849$, $u_3=\cdot 2381$. Find u_0 and u_2.

19. Find A_{51}, given that $A_{43}=\cdot 49136$, $A_{44}=\cdot 50515$, $A_{45}=\cdot 51851$, $A_{46}=\cdot 53148$, $A_{47}=\cdot 54410$.

20. Complete the series for $f(x)$ from $x=0$ to $x=12$ from the data:

x	0	4	8	12
$f(x)$	2·714	2·884	3·037	3·175

21. Express $\dfrac{x^3+x+5}{x^4+10x^3+35x^2+50x+24}$ as the sum of a number of inverse factorials, and hence find its second difference in the form of a single fraction.

22. Six consecutive terms of a series are given:

0, 1, 3, 8, 20, 47.

An estimate is required of the seventh term, and *A* gives 103 for his answer. *B* states that a more probable figure is 105. Examine these estimates and give reasons for your preference, if any, for one over the other.

23. The following table gives the number of children under five years of age per 1000 women between the ages of 16 and 44 resident in a certain country at the undermentioned dates:

Year	1800	1805	1815	1830	1860
Number of children	976	976	952	877	714

Estimate the number of children per 1000 women for the year 1810.

24. Find u_x as a rational integral function of x from the following data:

x	2	6	4	8	7
u_x	5	205	57	497	330

25. What simple form of function gives $u_0=-4$, $u_3=23$, $u_4=60$ and $u_{10}=996$?

26. Find x when $u_x=160$, given that u_x has the values 153, 157, 164 and 177 when x has the values 10, 11, 13 and 16 respectively.

27. Four values of a function u_a, u_x, u_b, u_c are given, corresponding to the values a, x, b, c of the variable. Find the third divided difference of u_a and deduce Lagrange's formula for u_x in terms of u_a, u_b and u_c.

28. Prove the interpolation formula:

$$u_x=u_0+x\,.\,\tfrac{1}{2}(\Delta u_0+\Delta u_1)+\frac{x^2}{2}\Delta^2 u_{-1}$$
$$+\frac{x(x^2-1)}{6}\,.\,\tfrac{1}{2}(\Delta^3 u_{-1}+\Delta^3 u_{-2})+\frac{x^2(x^2-1)}{24}\Delta^4 u_{-2}+\ldots.$$

Set out a table of differences and find q_{51} from the values

$q_{50}=\cdot015714$, $q_{48}=\cdot015152$, $q_{52}=\cdot016303$,
$q_{54}=\cdot016915$, $q_{46}=\cdot014625$.

Cambridge University Press
978-1-316-61278-1 - Examples in Finite Differences, Calculus and Probability: Supplement to an Elementary Treatise on Actuarial Mathematics
Harry Freeman
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29. Find $f(1{\cdot}4375)$, given the following data:

x	1·40	1·41	1·42	1·43	1·44	1·45	1·46
$f(x)$	1053	808	608	452	339	268	238

30. Given n corresponding values of a variable and of a rational integral function of the variable, in what circumstances would you use the following formulae to interpolate between two given values of the function to find a value corresponding to an intermediate value of the variable:

(i) Newton's advancing difference formula;

(ii) Lagrange's formula;

(iii) Newton's divided difference formula;

(iv) A central difference formula?

31. Use an appropriate interpolation formula to find u_1, given $u_x = 0, 108, 200, 316, 508, 840$ for the values of $x = -4, -2, 0, 2, 4, 6$ respectively.

32. Show that

$$u_x = \tfrac{1}{2}(u_0 + u_1) + (x - \tfrac{1}{2})\,\Delta u_0 + \frac{x(x-1)}{2!}\cdot\frac{\Delta^2 u_{-1} + \Delta^2 u_0}{2} + \ldots.$$

Extend the formula to include terms involving the third and fourth differences of the function.

Use the formula to find $f(50)$ from the values:

x	40	48	56	64
$f(x)$	14·27	15·81	17·72	19·96

33. From the following data calculate a_{26}:

x	15	20	25	30	35	40
a_x	24·584	23·473	22·285	20·943	19·468	17·881

using (i) two of the values;

(ii) four of the values;

(iii) all the values given.

34. Construct a formula of interpolation expressing u_x in terms of u_0, Δu_{-1}, $\Delta^2 u_{-1}$, $\Delta^3 u_{-2}$, $\Delta^4 u_{-2}$, $\Delta^5 u_{-3}$ and $\Delta^6 u_{-3}$, and hence determine the entry corresponding to the argument $4\tfrac{1}{2}$, given:

Argument	1	2	3	4	5	6	7
Entry	1901	1264	29	−304	5125	27056	85349

35. Interpolate by means of Gauss's forward formula to find the present value of 1 due 27 years hence at 5 per cent. compound interest from the following data:

No. of years	15	20	25	30	35	40
Present value	10·3797	12·4622	14·0939	15·3725	16·3742	17·1591

36. Employ an appropriate formula to obtain successive approximations to $f(28{\cdot}3)$, given the table:

x	26	27	28	29	30
$f(x)$	·038462	·037037	·035714	·034483	·033333

37. Prove Everett's formula:

$$u_x = xu_1 + \frac{x(x^2-1)}{3!}\Delta^2 u_0 + \frac{x(x^2-1)(x^2-4)}{5!}\Delta^4 u_{-1} + \dots$$

$$+ \xi u_0 + \frac{\xi(\xi^2-1)}{3!}\Delta^2 u_{-1} + \frac{\xi(\xi^2-1)(\xi^2-4)}{5!}\Delta^4 u_{-2} + \dots.$$

What are the practical advantages of this formula?

Note. In the next five questions Everett's formula is to be used.

38. Find the present value of £1 per annum at the end of 20 years at 3·2 per cent., given the following extract from tables of compound interest:

Rate per cent.	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
Present value of 1 per annum (£)	16·3514	15·5892	14·8775	14·2124	13·5903	13·0079

39. The annual premium (P_x) for an assurance of 1 at 4 per cent. being available for the undermentioned ages, find P_x for ages 36 to 44 inclusive:

Age x	25	30	35	40	45	50	55
P_x	·01431	·01662	·01961	·02351	·02865	·03550	·04471

40. The following is an extract from the prospectus of a Life Assurance Company, giving the annual premiums for an assurance of £100:

Age at entry	Annual premium £	s.	d.
20	1	19	6
25	2	3	11
30	2	9	5
35	2	16	2
40	3	4	10
45	3	15	8
50	4	10	7

Obtain the premiums for ages 31 to 39 inclusive.

41. Calculate q_{71}, q_{72}, q_{73} and q_{74}, given:

Age x	60	65	70	75	80	85
q_x	·03108	·04468	·06558	·09747	·14525	·21504

42. From the following table of tan x obtain the values of the tangents of all angles from 55° to 65° inclusive:

Angle x (degrees)	45	50	55	60	65	70	75
tan x	1·0000	1·1918	1·4281	1·7320	2·1445	2·7475	3·7320

43. Values of the yearly pension secured to a wife after her husband's death by a yearly contribution of £1 are given for a husband aged 30 next birthday, according to two different tables:

Age of wife next birthday	20	30	40	50	60
Pension on Table I (£)	4·39	5·00	6·03	7·77	10·74
Pension on Table II (£)	3·90	4·47	5·44	7·13	10·15

Find, for age of wife 33 next birthday, the percentage increase in the yearly pension resulting from the adoption of Table I in place of Table II.

44. Use Stirling's formula to find a_{32}, given:

Age x	20	25	30	35	40	45
a_x	14·035	13·674	13·257	12·734	12·089	11·309

How would you alter your method, if at all, if you were required to find the values for all ages from 20 to 45 inclusive?

45. If a_x is the value of an annuity on a life aged x, payable yearly, an approximation to the value of the annuity payable continuously—i.e. at very frequent intervals—is

$$\bar{a}_x = \tfrac{1}{2} + a_x - \tfrac{1}{12}(\mu_x + \delta).$$

a_x is available for the undermentioned ages:

Age x	75	80	85	90	95
a_x	4·747	3·417	2·419	1·676	1·045

and the corrective factor for a different set of ages, thus:

Age x	77	82	87	92
$\frac{1}{12}(\mu_x+\delta)$	·013	·019	·027	·039

From these data calculate $\bar{a}_{83}$.

46. $f(8) = 1{\cdot}4422$, $f(9) = 1{\cdot}4581$, $f(10) = 1{\cdot}4736$, $f(11) = 1{\cdot}4888$ and $f(12) = 1{\cdot}5037$. Find $f(8{\cdot}5)$, using all the values given.

47. Use a central difference formula to obtain successive approximations to $(1+i)^{17}$ from the table:

x	0	5	10	15	20	25	30
$(1+i)^x$	1	1·27628	1·62889	2·07893	2·65330	3·38635	4·32194

48. From the following values of the expectation of life by English Life Table No. 8, estimate e_{28}:

x	20	25	30	35
e_x	44·21	40·00	35·81	31·71

If in addition the values for ages 10 (53·08) and 15 (48·57) are given, obtain a revised estimate for e_{28}.

49. Construct a table of divided differences from the data:

x	0	5	8	9	25
$f(x)$	−179	1	13	37	7221

Extend the table to include arguments $x=5$, repeated as many times as may be necessary, and thus find $f(x)$ in powers of $(x-5)$.

50. Prove Everett's second formula:

$$u_{p-\frac{1}{2}} = u_0 + \frac{p^2-\frac{1}{4}}{2!}\Delta u_0 + \frac{(p^2-\frac{1}{4})(p^2-\frac{9}{4})}{4!}\Delta^3 u_{-1} + \dots$$
$$- \frac{q^2-\frac{1}{4}}{2!}\Delta u_{-1} - \frac{(q^2-\frac{1}{4})(q^2-\frac{9}{4})}{4!}\Delta^3 u_{-2} - \dots.$$

When would you use this formula in practice?

51. Comment on the following extracts from a text-book on statistics:

(*a*) In order to apply the method of interpolation to statistics, it is assumed that the formulae may be employed with respect to figures which do not obey any definite law.

(*b*) When the argument proceeds by unequal intervals, the most convenient formula for interpolation is Lagrange's.

52. Establish the formula:

$$u_x = -\tfrac{1}{6}(x^2-\tfrac{1}{4})(x-\tfrac{3}{2})\,u_{-\frac{3}{2}} + \tfrac{1}{2}(x-\tfrac{1}{2})(x^2-\tfrac{9}{4})\,u_{-\frac{1}{2}}$$
$$-\tfrac{1}{2}(x+\tfrac{1}{2})(x^2-\tfrac{9}{4})\,u_{\frac{1}{2}} + \tfrac{1}{6}(x^2-\tfrac{1}{4})(x+\tfrac{3}{2})\,u_{\frac{3}{2}}.$$

Use the formula to find $f(1\tfrac{1}{4})$, given that $f(0)=896$, $f(1)=1000$, $f(2)=1216$, $f(3)=1508$.

53. Given $f(-\tfrac{5}{2})$, $f(-\tfrac{3}{2})$, $f(-\tfrac{1}{2})$, $f(\tfrac{1}{2})$, $f(\tfrac{3}{2})$ and $f(\tfrac{5}{2})$, obtain a six-term formula for $f(x)$ similar to the four-term formula in Qu. 52.

54. In a certain series $u_0=a$, $\Delta u_0=b$, $\Delta^2 u_x=c$ for all values of x from 0 to $m-1$, and $\Delta^2 u_x=d$ for all subsequent values; find u_{m+n}, where m and n are positive.

If the series in question is

$$6,\ 11,\ 20,\ 33,\ 50,\ u,\ 105,\ 143,\ 188,\ 240,\ \ldots,$$

find u.

55. y is a function of x. When x is 30, y is 1; when x is 31, y is 2; and when x is 32, y is 103. Find a value or values for x when y is 50, illustrating your answer by a diagram.

56. The following table gives the present value of £1 due 20 years hence, at varying rates of interest. Find, correct to three decimal places, the rate of interest for which the present value is £·4, by a method of inverse interpolation:

Rate per cent.	$4\frac{3}{8}$	$4\frac{1}{2}$	$4\frac{5}{8}$	$4\frac{3}{4}$	$4\frac{7}{8}$	5
Present value of £1	·42469	·41464	·40485	·39529	·38598	·37689

57. Find approximate values of the real roots of the equations

(*a*) $x^3+x=3$ (root between 1·2 and 1·3);

(*b*) $x^3+2x=20$ (root between 2·4 and 2·5).

58. If $u_x = p + x + x^2$, find, by inverse interpolation from u_0, u_1 and u_4, the value of x corresponding to $u_x = 3$.

What must p be in order that this value of x may satisfy the equation

$$3 = p + x + x^2?$$

59. Explain the method of elimination of third differences as used in inverse interpolation, and employ the method to find the real root of $x^3 + 12x = 12$, correct to three decimal places.

60. One root of the equation $10(x^3 - 7x + 6) = x^2$ is approximately 1. Find its value correct to four decimal places.

61. The value of an annuity-certain of £1 per annum for 30 years is given as £20, and it is desired to find the rate of interest involved. Using the following extract from a table of annuity values, obtain an approximate value for the rate of interest:

Rate per cent.	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$
Value of a 30-year annuity (£)	22·40	20·93	19·60	18·39

62. Find the real roots of

$$(a)\ x^3 = 6x + 11; \qquad (b)\ 13x^3 + 68x = 57x^2 + 36;$$

each correct to three decimal places.

63. u is the present value of a given series of payments, and it is known that u lies between u_0 and u_1, which are the present values of the given series at rates i and $i+h$ respectively. If $i+k$ be the rate of interest for which the present value is u, prove that, approximately,

$$k = \frac{h\,(1 + \frac{1}{2}\Delta^2 u_0/\Delta u_0)}{\{\Delta u_0/(u - u_0)\} + \{\frac{1}{2}\Delta^2 u_0/\Delta u_0\}}.$$

64. If $\alpha = \Delta u_0$ and $\beta = \frac{1}{2}\Delta^2(u_{-1} + u_0)$, show that the relation between k and h in Qu. 63 may be expressed as

$$k = h\,\frac{u - u_0}{\alpha - \frac{1}{2}\beta + \frac{1}{2}\beta\,\dfrac{u - u_0}{\alpha - \frac{1}{2}\beta}} \text{ approximately.}$$

65. Show, by means of examples, that the following are not universally true:

$$\text{(i)}\ \Sigma\Delta \equiv 1; \qquad \text{(ii)}\ u_x = (1 + \Delta)^x\, u_0.$$

66. Sum to n terms

(*a*) 3, 8, 11, 18, 35, 68, ...; (*b*) 0, 0, $1\frac{1}{2}$, 5, $10\frac{3}{4}$, $18\frac{7}{8}$, $29\frac{7}{16}$, ...

67. Find the sum of n terms of the series whose xth term is

$$3x(3x-3)(3x-6)-2x(2x-1)(2x-2).$$

68. Sum to n terms

(*a*) 2, 12, 22, 36, 62, 116, 230, ...;

(*b*) 1, 2, 4, 14, 40, 92, 184, 338, ...

69. Prove the formula for summation by parts in finite differences, and apply it to obtain the sum of n terms of the series

$$5+2.8+4.12+8.17+16.23+....$$

70. Sum to n terms

$$(a)\quad \frac{3}{1.2.4}+\frac{4}{2.3.5}+\frac{5}{3.4.6}+\frac{6}{4.5.7}+...;$$

$$(b)\quad \frac{15}{1.2.3}\left(\frac{6}{7}\right)+\frac{16}{2.3.4}\left(\frac{6}{7}\right)^2+\frac{17}{3.4.5}\left(\frac{6}{7}\right)^3+....$$

71. Find the general term of each of the series

(*a*) 67, 39, 29, 29, 35, 47, 69, ...;

(*b*) 67, 39, 29, 29, 35, 45, 58, ...

Sum series (*b*) to n terms.

72. The series 10, 30, 98, 346, 1278, 4838, ... is the sum of two series. The xth term of one of the series is of the form $p+qx+r^x$, while the other series is a geometrical progression. Find the general term of the combined series and its sum to n terms.

73. Sum to n terms

$$1.1+32.32+64.64+97.98+131.136+166.182+....$$

74. Find

(i) $\Delta^{-1}\{x!\,(x^2+x+1)\}$; (ii) $\Delta^{-n}3^x-\dfrac{1}{\Delta^n 2^x}$.

75. Evaluate $\sum\limits_1^n\{\Sigma x^2a^x\}$.

76. Prove that $2\sum\limits_1^n\{x2^x+3x\}=(n-1)\,2^{n+2}+3n^2+3n+4$.