I. INDICES, LOGARITHMS AND SURDS

1.1. The index laws.

If n is a positive integer, a^n means $aaa\ldots$ (n factors).

If m, n are positive integers, it is easy to see that

$$a^m \times a^n = a^{m+n};$$
$$a^m \div a^n = a^{m-n} \text{ (provided that } m > n);$$
$$(a^m)^n = a^{mn};$$
$$\sqrt[n]{(a^m)} = a^{m/n} \text{ (provided that } m \text{ is a multiple of } n).$$

These are the 'index laws'.

Expressions such as a^0, a^{-n}, $a^{m/n}$ (where m is not a multiple of n) are given such meanings as fit in with the index laws:

$$a^0 = 1 \text{ (for } a^0 = a^{3-3} = a^3 \div a^3 = 1);$$
$$a^{-n} = 1/a^n \text{ (for } a^{-n} = a^{0-n} = a^0 \div a^n = 1/a^n);$$
$$a^{m/n} = \sqrt[n]{(a^m)} \text{ [for } (a^{m/n})^n = a^m].$$

1.2. Logarithmic form of the index laws.

If $x = 10^a$, a is called the 'logarithm of x to the base 10'.

Logarithms are thus indices, and the index laws may be rewritten as follows:

$$\log xy = \log x + \log y;$$
$$\log (x/y) = \log x - \log y;$$
$$\log x^n = n \log x;$$
$$\log \sqrt[n]{x} = \frac{1}{n} \log x.$$

Proofs: We shall assume that the base of the logarithms is 10, but the proofs are general.

Let $\quad \log x = a, \quad \log y = b.$

Then $\quad x = 10^a, \quad y = 10^b$

(for when we say, for example, that $\log 2 = 0{\cdot}3010$, we mean that $2 = 10^{0\cdot3010}$).

Therefore $\quad xy = 10^a \times 10^b = 10^{a+b}.$

Therefore $\quad \log xy = a + b = \log x + \log y.$

And similarly for the others.

1.3. Change of base.

It was assumed, in the above proof, that the base of the logarithms was 10. A similar proof would, of course, hold good if another base were used. The only other base of practical importance is e ($= 2{\cdot}718\ldots$, the sum of the infinite series $1+1/1!+1/2!+1/3!+\ldots$), chosen on account of the properties that

$$\frac{d}{dx}e^x = e^x, \quad \text{whereas} \quad \frac{d}{dx}10^x \simeq 2{\cdot}302\times 10^x,$$

and
$$\frac{d}{dx}\log_e x = 1/x, \quad \text{whereas} \quad \frac{d}{dx}\log_{10} x \simeq 0{\cdot}4343\times 1/x.$$

A change of base may be easily effected by the following rule:

$$\log_b x = \log x/\log b,$$

the logarithms on the right-hand side being to any base.

Proof: Let $\quad \log_b x = y.$

Then $\quad x = b^y.$

Taking logarithms to 10 or any other base,

$$\log x = y \log b.$$

Therefore $\quad y = \log x/\log b.$ Q.E.D.

If the base chosen for the right-hand side is x, we have the special case

$$\log_b x = 1/\log_x b.$$

Thus, for example, $\log_e 10$ is the reciprocal of $\log_{10} e$.

1.31. It should be noted that $\log_e x$ and e^x are 'inverse functions' (i.e. if x and y are interchanged in the equation $y = \log_e x$, we obtain $x = \log_e y$, which is equivalent to $y = e^x$). This explains the relationship between the graphs of the two functions.

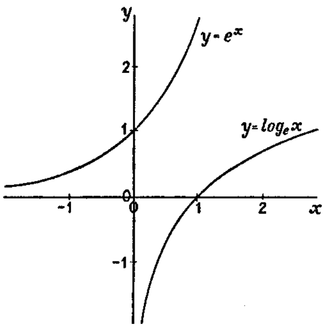

Fig. 1.11

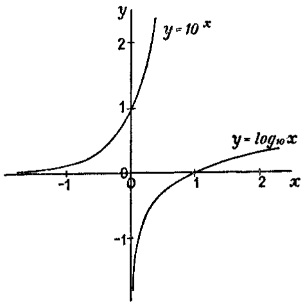

Fig. 1.12

1.4. Surds.

Such numbers as $\sqrt{a}$, $1/\sqrt[3]{a}$, $(\sqrt[3]{a})^2$ are usually dealt with more conveniently by means of indices, in the forms $a^{\frac{1}{2}}$, $a^{-\frac{1}{3}}$, $a^{\frac{2}{3}}$ respectively, but it is also worth noting that a denominator in surd form can be 'rationalised' if we multiply numerator and denominator by a suitable factor.

For example,

$$1/\sqrt{2} = \sqrt{2}/2 \text{ (multiplying numerator and denominator by } \sqrt{2});$$

or again, $$1/(\sqrt{x}+\sqrt{a}) = (\sqrt{x}-\sqrt{a})/(x-a)$$

(multiplying numerator and denominator by $\sqrt{x}-\sqrt{a}$).

1.5. Equations involving surds.

In such an equation as

$$\sqrt{(2x+4)}+\sqrt{(x-5)} = 5,$$

it is usual to regard the $\sqrt{}$ sign as meaning the positive square root. The solution of such an equation involves at some point the squaring of both sides, and as this introduces extra solutions, it is essential to test any results so obtained by substituting in the original equation.

Thus the above equation may be solved as follows:

$$\sqrt{(2x+4)} = 5-\sqrt{(x-5)}.$$

Squaring, $$2x+4 = 25-10\sqrt{(x-5)}+x-5.$$

Therefore $$10\sqrt{(x-5)} = 16-x.$$

Squaring again, $$100x-500 = 256-32x+x^2.$$

Therefore $$x^2-132x+756 = 0.$$

Therefore $$(x-6)(x-126) = 0,$$

and $$x = 6 \quad \text{or} \quad 126.$$

If $x = 6$, $\sqrt{(2x+4)}+\sqrt{(x-5)} = \sqrt{16}+\sqrt{1} = 5.$

If $x = 126$, $\sqrt{(2x+4)}+\sqrt{(x-5)} = \sqrt{256}+\sqrt{121} = 27.$

The solution $x = 126$ must therefore be rejected. It is evidently a solution, not of the given equation, but of

$$\sqrt{(2x+4)}-\sqrt{(x-5)} = 5.$$

Cambridge University Press
978-1-316-61273-6 — Algebra for Science and Engineering Students
E. H. Lockwood
Excerpt
More Information

EXAMPLES I

1. Express in index notation:

(1) $\sqrt[3]{a^2}$, (2) $1/\sqrt{(x-a)}$, (3) $\sqrt{(a^7/a^3)}$,

(4) $\sqrt{a^6}/\sqrt[3]{a^{27}}$, (5) $(a^2)^5 \div (a^3 \times a^7)$, (6) $1/(a^{-\frac{2}{3}})^{\frac{3}{4}}$.

2. Simplify:

(1) $x^m \div \sqrt{x^n}$, (2) $(x^2-2ax+a^2)^m \div (x-a)^n$, (3) $\sqrt[3]{x^{2n}} \times \sqrt{x^{3n}}$,

(4) $(x^{m/n})^{n/m}$, (5) $\sqrt[n]{x^{m+n}} \div \sqrt[n]{x^{2m}}$, (6) $1/x^{-n}$.

3. Find x from each of the following equations, working always in indices:

(1) $x^2 = a^{\frac{3}{4}}$, (2) $x^{\frac{2}{3}} = a^{\frac{3}{4}}$, (3) $x^{-2} = a^{\frac{3}{4}}$,

(4) $a/x = \sqrt{x}$, (5) $(a/x)^{\frac{1}{2}} = (x/a)^{\frac{1}{3}}$, (6) $a^3x^{-n} = ax$.

4. Multiply $x^{\frac{2}{3}}+2+3x^{-\frac{2}{3}}$ by $x^{\frac{2}{3}}-2+3x^{-\frac{2}{3}}$, and find the value of the product when $x = 8$.

5. If $v = a^3$ and $a^2b = 1$, obtain an expression (not containing a) for b, in the form v^n, giving the value of n.

6. Find the value of x if $2 \times 4^{x-1} = 8^{-x}$.

7. If $b = 100a^{-\frac{2}{3}}$, find the value of a when $b = 1$, and the value of b when $a = 8$. Find also, by means of logarithms, the value of b when $a = 0{\cdot}8$.

8. If $a = b^{\frac{2}{3}}$ and $b = (ac)^{-\frac{1}{2}}$, express c as a power of a, not containing b.

9. Express each of the following as a single logarithm:

(1) $\log(a+x) - \log(a-x)$, (2) $5\log(a+x)$,

(3) $-\frac{1}{5}\log(a+x)$, (4) $m\log x^2 + n\log x^3$,

(5) $m\log(a+x) - n\log(a-x)$, (6) $-k\log y$.

10. Find x from each of the following equations:

(1) $\log x + \log a = \log b$, (2) $n\log x = \log y$,

(3) $\log x = \log a + n\log b$, (4) $m\log x = n\log a$.

11. Evaluate (1) $\log_2 5$, (2) $\log_7 10$.

12. Prove (1) that $\log_a x - \log_a y = \log_a x/y$, (2) that $\log_a x^n = n\log_a x$.

13. Prove that $\log_e x = \log_e 10 \log_{10} x$.

14. Prove in the manner of §1.3 that $\log_a x = \log_a b \, . \log_b x$.

15. Given that $\log_{10} e = 0{\cdot}4343$, evaluate:

(1) $\log_e 10$, (2) $\log_e 8$, (3) $\log_e 0{\cdot}8$.

16. Find, to 3 places of decimals, the value of x which satisfies the equation $\log_x 10 = -4$.

17. Rationalise the denominators of:

(1) $1/\sqrt{a}$, (2) $\sqrt{(x+a)}/\sqrt{(x-a)}$, (3) $1/(\sqrt{x}-\sqrt{a})$.

18. Given that $\sqrt{2} \simeq 1{\cdot}414$, evaluate:

(1) $1/\sqrt{2}$, (2) $1/(\sqrt{2}+1)$, (3) $(\sqrt{2}+1)/(\sqrt{2}-1)$.

19. Solve the following equations:

(1) $\sqrt{(x+2)} = \sqrt{(2x-3)}-1$,

(2) $\sqrt{(x+7)}+\sqrt{(3x+3)} = 6$,

(3) $\sqrt{(x+1)}+\sqrt{(2x)} = \sqrt{(6x+1)}$.

20. If $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, express x in terms of y.

21. Given that the differential coefficient of x^n is nx^{n-1} for all values of n, write down the differential coefficients of

(1) $1/x^2$, (2) $\sqrt[3]{x}$, (3) $1/\sqrt{x}$.

22. If air is expanded at constant temperature, $pv =$ const., where p, v are the pressure and volume; if it is expanded adiabatically, i.e. without gain or loss of heat, $pv^{1{\cdot}4} =$ const. If a given quantity of air is at pressure p_0 and is expanded from volume v_0 to v, find the ratio of the final pressure when the expansion is done at constant temperature to the final pressure when it is done adiabatically.

23. If the growth of a part of a living organism is represented by the formula $y = a.b^{c^t}$, where b and c are fractions, and t is the time, measured from the first observation, consider the value of y when $t = 0$ and when t is very large, and hence interpret the quantity a and the fraction b.

24. It is found by plotting $\log y$ against $\log x$ that the variables x and y are related by the equation

$$\log y = 1{\cdot}301 + 3 \log x.$$

Express y in terms of x.

25. Some air is expanded and the temperature and volume are found to follow the law $\log t + 0{\cdot}4 \log v = c$, where c is a constant. Obtain an equivalent relation not involving logarithms and using the fact that if p is the pressure, pv/t is constant, prove that the expansion is adiabatic (i.e. that $pv^{1{\cdot}4}$ is constant).

26. Coulomb's Law for the force F between two electric charges e, e' at a distance r apart in a vacuum is $\mathsf{F} = ee'/r^2$. The 'dimensions' of F and r are respectively $[\mathsf{MLT}^{-2}]$ and $[\mathsf{L}]$. Show that the unit of electric charge must be regarded as having dimensions $[\mathsf{M}^{\frac{1}{2}}\mathsf{L}^{\frac{3}{2}}\mathsf{T}^{-1}]$.

The force between two magnetic poles m, m' is likewise given by $\mathsf{F} = mm'/r^2$, and magnetic intensity H by $\mathsf{H} = m/r^2$. Find the dimensions of H.

An electric current is measured by q/t (where q is charge and t is time) and also by $r\mathsf{H}\tan\theta/2n\pi$, where r is a length, H a magnetic intensity and n, π, $\tan\theta$ are of no dimensions. Show that on these assumptions q is of dimensions $[\mathsf{M}^{\frac{1}{2}}\mathsf{L}^{\frac{1}{2}}]$. What are the dimensions of the ratio of these two units of charge?

27. If the pressure p of the atmosphere at height z is given by $p = p_0 e^{-z/\mathsf{H}}$, where p_0 and H are constants, prove that the difference in height of two stations (heights z_1, z_2) at which the pressures are p_1, p_2 respectively is given by

$$z_2 - z_1 = 2{\cdot}3026\mathsf{H}\log_{10}(p_1/p_2).$$

28. If I_1, I_2 measure the 'brightness' of two stars whose 'magnitudes' are m_1, m_2, then $\mathsf{I}_1/\mathsf{I}_2 = k^{m_2 - m_1}$, where $\log k = 0{\cdot}4$. If one star is 100 times as 'bright' as another, by how many units do their 'magnitudes' differ?

29. The gradient of the chord joining two points $(a, \sqrt{a})$ and $(b, \sqrt{b})$ on the curve $y = \sqrt{x}$ is $(\sqrt{b} - \sqrt{a})/(b-a)$. If $b \to a$, the chord becomes the tangent at $(a, \sqrt{a})$. What is the gradient of this tangent?

30. If the sizes x and y of two parts of an organism are represented by the formulae $x = \mathsf{A}.\mathsf{B}^{kt}$ and $y = a.b^{kt}$, where t represents the time and the other letters are constants, prove that $y = \mathsf{C}x^n$, where $n = \log b/\log \mathsf{B}$ and $\mathsf{C} = a/\mathsf{A}^n$.

II. VARIATION

2.1. It is assumed that the reader has some knowledge of variation, obtained from work on graphs. The present chapter is a summary of some of the more important types of variation, arranged particularly with a view to showing how a formula may be obtained to fit a given set of values of two variables.

2.11. The variable plotted horizontally (x in the examples given) is called the 'independent variable'. We eventually express the other one (called the 'dependent variable') in terms of it (i.e. y in terms of x).

2.12. The term 'gradient', applied to a straight line graph, means

$$\frac{\text{increase in } y}{\text{corresponding increase in } x},$$

the increases being measured according to the scales chosen for y and x respectively. The gradient is taken as $+$ if y increases as x increases, and $-$ if y decreases as x increases.

2.13. The table of values, the graph and the equation all give the same information, but in different ways and with different degrees of completeness.

2.2. Direct proportion.

$$y = ax.$$

x	2	3	4	6	-2
y	6	9	12	18	-6
y/x	3	3	3	3	3

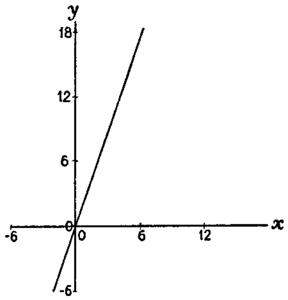

Fig. 2.1

Characteristics: As x increases, y increases in the same ratio.

As x increases by equal steps, so does y.

When $x = 0$, $y = 0$.

Arithmetical test: The quotient y/x or x/y is constant.

Graphical test: The graph is a straight line through the origin.

To find the equation from the graph:

If the gradient is m, $y/x = m$, or $y = mx$.

In the example given, $y = 3x$.

2.3. The linear law. $y = ax + b$.

x	-10	0	10	20
y	0	3	6	9

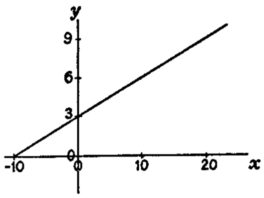

Fig. 2.2

Characteristics: As x increases by equal steps, so does y.

Arithmetical test: Differences between values of y are proportional to those between the corresponding values of x.

Graphical test: The graph is a straight line.

To find the equation from the graph:

Method 1. If the graph is seen to cut the y-axis at the point $y = c$, and the gradient is m: then $y-c$ is the height of any point (x, y) on the line above the point $(0, c)$, and $(y-c)/x = m$. Therefore $y = mx + c$.

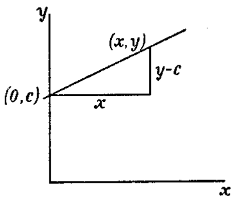

Fig. 2.21

In the example given, gradient $= \frac{3}{10}$ and the starting-point is (0, 3). Therefore $y = \frac{3}{10}x + 3$.

Method 2. This method is useful when the origin, (0, 0), is off the paper.

If the graph is seen to pass through a point (x_1, y_1) and to have gradient m: then $y - y_1$ and $x - x_1$ are the distances of any point (x, y) above and to the right of (x_1, y_1). Hence for any point on the line, $(y-y_1)/(x-x_1) = m$, or $y - y_1 = m(x - x_1)$.

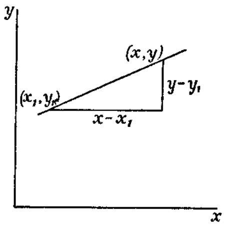

Fig. 2.22

In the example given, the gradient is $\frac{3}{10}$, and if we take (10, 6) as the starting-point, $(y-6)/(x-10) = \frac{3}{10}$. Therefore $y = \frac{3}{10}x + 3$.

Method 3. An alternative method, which some find easier, is to select two points on the line, e.g. (0, 3) and (20, 9), and substitute in the equation $y = ax + b$.

Thus $3 = a.0 + b$ and $9 = a.20 + b$, whence $b = 3$ and $a = \frac{3}{10}$.

The equation $y = ax + b$ may then be rewritten as $y = \frac{3}{10}x + 3$.

2.4. Inverse proportion. $y = a/x$.

x	2	3	4	6	-2
y	6	4	3	2	-6
xy	12	12	12	12	12

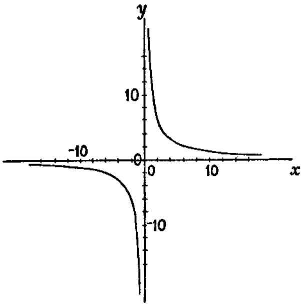

Fig. 2.3

This curve is called a 'hyperbola'.

Characteristics: As x increases, y decreases in inverse ratio.

As $x \to \infty$, $y \to 0$.

As $x \to 0$, $y \to \infty$.

If x changes sign, so does y. (The graph has central symmetry about the origin.)

Arithmetical test: Product xy = constant.

Graphical test: If y is plotted against $1/x$, the graph is a straight line through the origin.

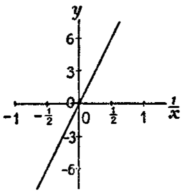

Fig. 2.31

To find the equation from the graph:

If the gradient of the graph in Fig. 2.31 is m, $y \div 1/x = m$,

or $xy = m$,

or $y = m/x$.

In the example given, the gradient is 12. Therefore $y = 12/x$.