# ON THE DIRECT NUMERICAL CALCULATION OF ELLIPTIC FUNCTIONS AND INTEGRALS

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*By*

LOUIS V. KING, M.A., D.Sc., F.R.S.

*Macdonald Professor of Physics,*
*McGill University, Montreal*

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# PREFACE

WHILE lecturing in 1913 on the calculation of self and mutual induction, the writer noticed that the integral in the case of co-axial circles was of such a form as to make the application of the arithmetico-geometrical means obvious. An extremely elegant formula (*Proc. Roy. Soc.* A, Vol. C. 1921, p. 63. Appendix, this book, No. 35) when put to numerical test justified the method as one well adapted to computation. The calculation of more complex cases of self and mutual induction led to several interesting formulae for the complete third elliptic integral. With a view to making a complete study of the group of recurrence formulae associated with the A.G.M. scales, further work was postponed until a vacation in 1921, when many of the results included in this book were obtained, and were thought by the writer to be new. On later consulting the *Collected Works* of Legendre, Gauss and Jacobi, the writer found that he had rediscovered many known formulae, but that quite a number, likely to be of service in computation, appeared to be hitherto unknown. In the summer of 1922 the writer decided, as a vacation task, to include in a single monograph the entire theory of elliptic functions associated with the A.G.M. scales, thus completing, and to some extent adding to, the work of Gauss left unpublished after his death, and placing before mathematicians in accessible form a mode of approach to Elliptic Function Theory directly related to the art of machine-computation. It is interesting to note in this connection that this subject was to have formed the content of the third volume of Halphen's *Fonctions Elliptiques*, unfortunately left incomplete at the time of his death.

While the present monograph contains many new formulae and methods of computation, no claim is made as to novelty in fundamental analytical treatment. References to more important books and memoirs consulted by the author are given in foot-notes. A complete bibliography is not given since the reader will find exhaustive references in the *Royal Society Index*, 1800–1900, Vol. I, Mathematics, and after 1900 in the mathematical volumes of the *International Catalogue of Scientific Literature*.

The author desires to express his thanks to his late mathematical colleague, Professor James Harkness of McGill University, for reading over the manuscript and proofs, and also to Mr Arthur Berry of King's College, Cambridge, for revising proofs, verifying formulae and for calling the writer's attention to several slips of calculation in the main text and appendix.

Finally, the author wishes to thank his colleague, Dr A. S. Eve, F.R.S., for his aid in obtaining from McGill University that financial assistance which has made possible the publication of this little volume. For the courtesy of the officers of the Cambridge University Press in undertaking to print this book in its usual impeccable style, the author is also deeply grateful. For assistance in the final revision of the proofs, the author is indebted to one of his students, Mr W. L. Robertson.

L. V. K.

McGill University,
*April* 18, 1924.

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