ON THE DIRECT NUMERICAL CALCULATION OF ELLIPTIC FUNCTIONS AND INTEGRALS

SECTION I

INTRODUCTION

Although the numerical evaluation of elliptic functions and integrals is dealt with in many of the standard treatises, it cannot be said that the use of tables is entirely satisfactory in dealing with computations required in several branches of physics and astronomy. Many of the integrals appearing in these problems can only be expressed as somewhat complicated expressions involving the complete elliptic integrals of the first and second kinds, and the corresponding functions of complementary modulus. These formulae give rise in many cases to differences of nearly equal quantities: the numerical calculation of such expressions to a sufficient number of significant figures is thus often a matter of some difficulty, requiring tedious and sometimes uncertain interpolations.

While the numerical calculation of the Theta-functions is extremely rapid by the use of the highly convergent series involving the *nome* q, these are not usually the functions which make their appearance directly in physical or astronomical problems. The necessary transformations required to express the results in terms of the q-series require in many cases somewhat complicated analysis, while numerical computation by this method necessitates the exact calculation of a large number of auxiliary quantities*.

The forthcoming tables in course of construction under the auspices of the British Association† are so arranged as to allow a large number of functions to be evaluated, and will very greatly aid numerical work in many practical problems involving the use of elliptic integrals and functions. While sufficient for most purposes, the use of these tables will still involve considerable interpolation when it is required to take out values corresponding to arguments and moduli expressed by six or

* On this point see Rosa, E. H., and Grover, F. N., *Bulletin* 169, Bureau of Standards, Washington (1913), pp. 67 and 73; Nagaoka, H., *Tok. Coll. Sc. J.* 27 (1909), No. 6, and *Jour. Tok. Math. Phys. Soc.* 4 (1908), p. 284; 6 (1911), p. 10; Nagaoka and Sakurai, *Scientific Papers of the Institute of Physical and Chemical Research*, Vol. II. Dec. 1922, pp. 1–67; Olshausen, G. R., *Phys. Rev.* 51 (Dec. 1910), pp. 617–636.

† See report (by Sir A. G. Greenhill), *Brit. Ass. Report*, 1912 (Dundee), pp. 39–55.

seven significant figures, as is now necessary in several modern applications of elliptic functions.

For several reasons it is advantageous to have available a direct method of computation, independent of auxiliary tables, based on the use of the modern high-speed calculating machine. The writer has therefore thought it desirable to set out in detail the method of calculation based on Landen's quadric transformation and the use of the scale of arithmetico-geometrical means, recapitulating briefly the original developments of Lagrange, Legendre, Gauss, Jacobi and others, extending them in several directions and keeping in mind the ultimate use of the calculating machine for numerical work.

SECTION II

HISTORICAL NOTE ON LANDEN'S TRANSFORMATION AND THE VARIOUS SCALES OF MODULI AND AMPLITUDES

The importance of realizing rapid and accurate methods of calculating the elliptic integrals now denoted by

$$u = F(\phi, k) = \int_0 d\phi/\Delta(\phi, k) \text{ and } E(\phi, k) = \int_0^\phi \Delta(\phi, k)\, d\phi \quad \ldots(1),$$

where $$\Delta(\phi, k) = \sqrt{(1 - k^2 \sin^2\phi)},$$

was first remarked by Euler* (1766), although it was not until several years later that Landen† (1775) discovered in geometrical form the transformation which forms the basis of the existing methods of the numerical calculation of the elliptic integrals.

A method of successive transformations for the ultimate reduction of the algebraic forms of these integrals to elementary integrals was published by Lagrange‡ in 1784–85. This memoir contains an exposition of the scales of arithmetico-geometrical means, together with two types of algebraic transformations corresponding to the increasing and diminishing amplitudes of Landen's trigonometrical forms: it is shown that the given integral must lie between two limits which may be made to approach each other as closely as may be desired: the limits thus ob-

* Euler, *Novi Comm. Acad. Sc. Petrop.* vol. x. 1766.

† Landen, *Phil. Trans. Roy. Soc.* vol. LXV. p. 283, 1775; *Mathematical Memoirs*, London, 1780.

‡ Lagrange, "Sur une nouvelle méthode de calcul intégral pour les différentielles affectées d'un radical carré sous lequel la variable ne passe pas le quatrième degré," *Mém. de l'Acad. roy. des Sc. de Turin*, t. II. 1784–5; *Œuvres* (Gauthier-Villars, Paris, 1868), t. II. pp. 253–312.

tained for the integrals of the first and second kinds are practically identical with the limits and series obtained about the same time by Legendre.

Landen's transformation was applied by Legendre* to the numerical calculation of the elliptic integrals, and furnished the method of computation by means of which the latter's well-known tables were constructed. The scales of moduli and amplitudes employed by Legendre are briefly described below.

If we form successive moduli and amplitudes according to the recurrence formulae

$$k_{n+1} = (1 - k_n')/(1 + k_n') \quad \text{and} \quad \tan(\phi_{n+1} - \phi_n) = k_n' \tan \phi_n \quad \ldots(2),$$

Landen's transformation in trigonometrical form leads to the result

$$F(\phi_{n+1}, k_{n+1}) = (1 + k_n')\, F(\phi_n, k_n) \quad \ldots\ldots\ldots\ldots\ldots(3).$$

As n increases, the successive moduli k_n converge to zero, and the complementary moduli k_n' to unity, while the amplitudes increase so that $\phi_n/2^n$ tends rapidly to a definite limit: ultimately we may write $F(\phi_n, k_n) = \phi_n$ and thus obtain the result

$$F(\phi_0, k_0) = (2K/\pi)\, \phi_n/2^n, \text{ where } K = \tfrac{1}{2}\pi(1 + k_1)(1 + k_2) \ldots (1 + k_n) \quad \ldots\ldots\ldots(4).$$

The result of calculating successive moduli and amplitudes in reverse order is equivalent to calculating in forward order these quantities according to the recurrence formulae

$$k_{n+1} = 2\sqrt{k_n}/(1 + k_n) \quad \text{and} \quad \sin(2\psi_{n+1} - \psi_n) = k_n \sin \psi_n \quad \ldots(5).$$

This transformation then gives

$$F(\psi_{n+1}, k_{n+1}) = \tfrac{1}{2}(1 + k_n)\, F(\psi_n, k_n) \quad \ldots\ldots\ldots\ldots\ldots(6).$$

As n increases the successive moduli k_{n+1} converge to unity and the amplitudes to a limit ψ_n: we thus obtain, when n is sufficiently large,

$$F(\psi_n, k_n) = \int_0^{\psi_n} \sec \psi_n \, d\psi_n = \log \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\psi_n) \quad \ldots\ldots(7),$$

giving finally

$$F(\psi_0, k_0) = \sqrt{(k_1 k_2 k_3 \ldots k_n)}/\sqrt{k_0} \,.\, \log \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\psi_n) \quad \ldots(8).$$

Legendre also developed for purposes of numerical calculation extremely convergent series for $E(\phi, k)$ which he employed in the tabulation of this function.

* Legendre, "Mémoire sur les intégrations par arcs d'ellipse" and "Second mémoire, etc.," *Mém. de l'Acad. des Sciences de Paris*, ann. 1786 (Paris, 1788), pp. 616–643 and 644–683; *Traité des Fonctions Elliptiques*, Paris, 1825, t. I. p. 79 *et seq.*

The method discovered independently by Gauss and called by him the "algorithm of the arithmetico-geometrical mean" originated in connection with the evaluation of complete elliptic integrals arising from a problem in attractions required in planetary theory*: in a note he mentions that the results were obtained by him as part of a more comprehensive theory, independently of the results of Lagrange and Legendre with which they are closely connected. Gauss employed a trigonometrical transformation along the following lines.

Commencing with two numbers (a_0, b_0), successive numbers (a_n, b_n) are calculated from the recurrence formulae,

$$a_{n+1} = \tfrac{1}{2}(a_n + b_n), \quad b_{n+1} = \sqrt{(a_n b_n)} \quad \ldots\ldots\ldots\ldots\ldots(9).$$

Writing $\Delta_n = \sqrt{(a_n{}^2 \cos^2 \phi_n + b_n{}^2 \sin^2 \phi_n)}$, Gauss employs the recurrence formula

$$a_n \tan \phi_{n+1} = \Delta_{n+1} \tan \phi_n,$$

or $$\tan \phi_{n+1} = \tan \phi_n \sqrt{(a_{n+1}/a_n)} \,.\, \sqrt{(b_n + \Delta_n)}/\sqrt{(a_n + \Delta_n)} \quad \ldots(10).$$

From the relation

$$\tan \phi_n = (\Delta_1 \Delta_2 \ldots \Delta_n \ldots)/(a_0 a_1 \ldots a_n \ldots) \,.\, \tan \phi_0$$

we have ultimately, since the ϕ's tend to a limit ϕ_n,

$$\int_0^{\phi_0} d\phi_0/\Delta_0 = \phi_n/a_n,$$

and hence

$$a_n \int_0^{\phi_0} d\phi_0/\Delta_0 = \tan^{-1}[(\Delta_1 \Delta_2 \ldots \Delta_n \ldots)/(a_0 a_1 \ldots a_n \ldots) \,.\, \tan \phi_0] \quad \ldots(11).$$

Extremely convergent series for the elliptic integral of the second kind were also obtained by Gauss. The application of this method to the numerical calculation of the elliptic integrals was considered by Jacobi some years later†.

In the course of researches carried out between 1797 and 1808, but not published until after his death considerably later ‡, Gauss developed the theory of elliptic functions to a remarkable extent and derived a large number of formulae which, at a later date, were obtained independently by Abel and Jacobi and which are now associated with the

* Gauss, "Determinatio Attractionis quam in Punctum quodvis Positionis datae exerceret Planeta, etc.," *Comm. Gott. Soc. Reg. Scient.* IV. 1818; *Ges. Werke* (Göttingen, 1866), Bd. III. pp. 331–355. The note mentioned above is dated Feb. 9, 1818; *Ges. Werke*, Bd. III. pp. 357–360.

† Jacobi, *Fundamenta Nova*, 1829, sections 38 and 52; *Ges. Werke* (Berlin, 1881), Bd. I. pp. 154 and 203. Also "Numerische Berechnung der Elliptischen Functionen," *Crelle*, Bd. XXVI. pp. 93–114; *Ges. Werke* (1861), Bd. I. pp. 345–368.

‡ Gauss, *Ges. Werke* (1866), Bd. III. pp. 361–403 and 433–469.

theory of the Theta-functions. In one of these posthumous papers, Gauss* employs the recurrence formula $\tan 2\phi_n = \sqrt{(b_{n+1}/a_{n+1})} \tan \phi_{n+1}$, which is much more convenient for purposes of numerical calculation than (10). The last mentioned method is considerably simpler than that of Legendre's scales of moduli, especially when a calculating machine is available, although in theory these two methods of computation are practically identical.

Although a few formulae based on the scales of arithmetico-geometrical means were briefly considered by Jacobi †, no attempt seems to have been made by him to apply the results to actual computation.

As the method of evaluating the elliptic functions by the use of the A.G.M. scales is only briefly dealt with in existing text-books ‡, and does not seem to have been developed to the extent it deserves in various memoirs which have appeared on the subject in recent years, a tolerably full account of the subject is given in the present book, together with a brief outline of the main analytical steps. Several new formulae have been obtained by the writer which facilitate the application of this method of numerical computation and considerably extend its scope.

From the point of view of pure analysis, the modern Weierstrassian notation presents many advantages in regard to elegance and symmetry, a superiority which no longer holds, however, when it is required to proceed to numerical evaluation. As the main point of the present paper is concerned with improvements in methods of computation, the older Jacobian notation has been adhered to as far as possible, especially as it expresses most readily the elliptic functions and integrals which arise from integrations involving circular and hyperbolic functions. The results are set out in such a manner as to provide a fairly complete compendium of formulae intended to be of service to the computer in reducing to numerical results such expressions involving elliptic functions and integrals as may arise in physical and astronomical problems.

* Gauss, *loc. cit.* p. 388.

† Jacobi, *Ges. Werke* (1881), p. 357.

‡ Cayley, A., *Elliptic Functions*, 2nd Ed. (G. Bell and Sons, London, 1895), pp. 326–338; Enneper-Müller, *Elliptische Functionen* (2nd ed. 1890), pp. 361–364; Tannery-Molk, *Fonctions Elliptiques*, t. IV. 1902, pp. 269–275. According to a foot-note in Halphen's *Fonctions Elliptiques*, t. II. 1888, p. 310, the subject of the arithmetico-geometrical mean was to have received special attention in the author's third volume, unfortunately incomplete at the time of his death.

SECTION III

ON THE SCALE OF ARITHMETICO-GEOMETRICAL MEANS

In forming the scale of arithmetico-geometrical means, we start with the positive numbers (a_0, b_0), of which a_0 is the greater, and form successively

$$\left.\begin{array}{lll} a_1 = \frac{1}{2}(a_0 + b_0), & b_1 = \sqrt{(a_0 b_0)}, & c_1 = \frac{1}{2}(a_0 - b_0) \\ a_2 = \frac{1}{2}(a_1 + b_1), & b_2 = \sqrt{(a_1 b_1)}, & c_2 = \frac{1}{2}(a_1 - b_1) \\ \dots\dots & \dots\dots & \dots\dots \\ a_n = \frac{1}{2}(a_{n-1} + b_{n-1}) & b_n = \sqrt{(a_{n-1} b_{n-1})} & c_n = \frac{1}{2}(a_{n-1} - b_{n-1}) \\ \dots\dots, & \dots\dots, & \dots\dots \end{array}\right\} \quad \dots(12).$$

We obviously have the relations

$$\left.\begin{array}{ll} a_n = a_{n+1} + c_{n+1}, & c_n^2 = a_n^2 - b_n^2 \\ b_n = a_{n+1} - c_{n+1}, & a_n c_n = \frac{1}{4} c_{n-1}^2 \end{array}\right\} \quad \dots\dots\dots\dots(13).$$

The a's and b's tend to the same limit denoted by

$$M(a_0, b_0) = \lim_{n \to \infty} a_n \dots\dots\dots\dots\dots\dots\dots\dots(14)$$

with extraordinary rapidity, even when a_0 and b_0 are initially numbers of very different magnitudes*.

We notice in passing that $M(a_0, b_0)$ satisfies a homogeneity relation of the form

$$\epsilon M(a_0, b_0) = M(\epsilon a_0, \epsilon b_0) \dots\dots\dots\dots\dots\dots(15),$$

ϵ being any number. The array of numbers (12) will be referred to as the "scale of arithmetico-geometrical means (a_0, b_0)," or more briefly as the A.G.M. scale (a_0, b_0). Also, the limit $M(a_0, b_0)$ will be denoted by a_n, as long as by so doing no ambiguity is involved.

If we calculate in the same way an array of numbers commencing with $\{a_0' = a_0,\ b_0' = c_0 = \sqrt{(a_0^2 - b_0^2)}\}$, we derive what will be called the "complementary A.G.M. scale (a_0', b_0')," the symbols being in this case denoted by accented letters, and the limit $M(a_0', b_0')$ by a_n'.

* The usual arithmetical process of extracting square roots is easily adapted to most calculating machines of modern type. An account of these is given in *Modern Instruments and Methods of Calculation*, edited by E. M. Horseburgh (London, G. Bell and Sons, 1914); also by d'Ocagne, *Le Calcul Simplifié* (Gauthier-Villars, Paris, 1905). In carrying out the process the work is considerably shortened by keeping in mind the rule that if the first p digits out of the number n required in the square root have been obtained by the usual process, the next $p-1$ digits can be obtained by division only, with a possible error of 1 in the last digit (Chrystal's *Algebra*, Part I. 5th ed. 1904, p. 210). To illustrate the rapid convergence of the A.G.M. scales, Gauss works out an example starting with $a_0 = 1$ and $b_0 = 0{\cdot}2$: a_5 and b_5 are represented by the same number to fifteen significant figures.

It is obvious that the operations of the A.G.M. scale may be carried out backwards: if the entries are denoted by negative suffixes, they are connected with those of the complementary scale by the relations

$$a_{-n} = 2^n a_n', \qquad b_{-n} = 2^n c_n', \qquad c_{-n} = 2^n b_n' \quad \text{.........(16).}$$

These relations evidently hold good when accented and unaccented symbols are interchanged. As a result of these relations it will be seen that there is no advantage to be gained from considering any other than the A.G.M. scale of positive suffixes (a_n, b_n) and the corresponding complementary scale (a_n', b_n').

SECTION IV

LANDEN'S SCALE OF INCREASING AMPLITUDES

(i) *Calculation of the Elliptic Integrals of the First and Second kinds, modulus k.*

If we write $\Delta_n = \sqrt{(a_n^2 \cos^2\phi_n + b_n^2 \sin^2\phi_n)}$, the recurrence formula

$$\tan(\phi_{n+1} - \phi_n) = (b_n/a_n)\tan\phi_n \quad \text{...............(17)}$$

may be written in either of the forms

$$\sin(2\phi_n - \phi_{n+1}) = (c_{n+1}/a_{n+1})\sin\phi_{n+1},$$
$$\cos(2\phi_n - \phi_{n+1}) = \Delta_{n+1}/a_{n+1} \quad \text{......(18).}$$

From these we derive

$$\Delta_{n+1} + c_{n+1}\cos\phi_{n+1} = \Delta_n \quad \text{and} \quad \Delta_{n+1} - c_{n+1}\cos\phi_{n+1} = a_n b_n/\Delta_n \quad \text{...(19),}$$

which give on differentiation,

$$d\phi_0/\Delta_0 = \tfrac{1}{2} d\phi_1/\Delta_1 = \tfrac{1}{4} d\phi_2/\Delta_2 = \ldots = (\tfrac{1}{2})^n d\phi_n/\Delta_n = \ldots \quad \text{......(20).}$$

From the relation

$$\Delta^2_{n+1} + c_{n+1}\Delta_{n+1}\cos\phi_{n+1} = \tfrac{1}{2}\Delta_n^2 + \tfrac{1}{2} a_n b_n$$

we obtain, making use of (20),

$$\Delta_n d\phi_n - \Delta_{n+1} d\phi_{n+1} = c_{n+1}\cos\phi_{n+1} d\phi_{n+1} - a_n b_n d\phi_n/\Delta_n \quad \text{...(21).}$$

Subtracting from each side the identity

$$a_n^2 d\phi_n/\Delta_n - a^2_{n+1} d\phi_{n+1}/\Delta_{n+1} = \tfrac{1}{2} c_n^2 d\phi_n/\Delta_n - a_n b_n d\phi_n/\Delta_n \text{...(22)}$$

we obtain

$$(\Delta_n - a_n^2/\Delta_n) d\phi_n - (\Delta_{n+1} - a^2_{n+1}/\Delta_{n+1}) d\phi_{n+1}$$
$$= c_{n+1}\cos\phi_{n+1} d\phi_{n+1} - \tfrac{1}{2} c_n^2 d\phi_n/\Delta_n \quad \text{...(23).}$$

Writing down a series of such equations, commencing with $n = 0$, and noting that as n increases, $(\Delta_n - a_n^2/\Delta_n)$ tends to zero in the limit, we have, on integrating,

$$\int_0^{\phi_0} \Delta_0 d\phi_0 - a_0^2 \int_0^{\phi_0} d\phi_0/\Delta_0 = c_1 \sin\phi_1 + c_2 \sin\phi_2 + \ldots + c_n \sin\phi_n + \ldots$$
$$- \tfrac{1}{2}(c_0^2 + 2c_1^2 + 4c_2^2 + \ldots + 2^n c_n^2 + \ldots)\int_0^{\phi_0} d\phi_0/\Delta_0 \quad \ldots\ldots(24).$$

If we now construct the A.G.M. scale ($a_0 = 1$, $b_0 = k'$), and calculate successive amplitudes commencing with $\phi_0 = \phi$, we have $\Delta_0 = \Delta(\phi, k)$. As n increases, the a's and b's tend to the same limit a_n, while the angle $\phi_n/2^n$ tends to a finite limit. We thus obtain from (20) and (24),

$$u = F(\phi, k) = (1/a_n)(\phi_n/2^n) \quad \ldots\ldots\ldots\ldots\ldots\ldots(25),$$

$$E(\phi, k) - F(\phi, k) = (c_1 \sin\phi_1 + c_2 \sin\phi_2 + \ldots + c_n \sin\phi_n + \ldots)$$
$$- \tfrac{1}{2}(c_0^2 + 2c_1^2 + 4c_2^2 + \ldots + 2^n c_n^2 + \ldots) F(\phi, k) \quad \ldots\ldots(26).$$

If we commence with $\phi_0 = \frac{1}{2}\pi$, we have

$$\phi_1 = \pi,\ \phi_2 = 2\pi,\ \phi_3 = 4\pi \ldots \phi_n/2^n = \tfrac{1}{2}\pi,$$

so that $K = \frac{1}{2}\pi/a_n$, and

$$(K - E)/K = \tfrac{1}{2}(c_0^2 + 2c_1^2 + 4c_2^2 + \ldots + 2^n c_n^2 + \ldots) \quad \ldots\ldots(27).$$

The series (24) and (27) converge with extreme rapidity, resembling in this respect the q-series with which they are closely connected*.

Jacobi's integral $Z(\phi, k)$ is defined by the relation

$$Z(\phi, k) = E(\phi, k) - (E/K) F(\phi, k) \quad \ldots\ldots\ldots\ldots(28)$$

or, if the angle ϕ is connected with the argument u by the relation

$$\sin\phi = \operatorname{sn}(u, k),$$

$$Z(u, k) = \int_0^u \operatorname{dn}^2(u, k)\, du - (E/K)\, u \quad \ldots\ldots\ldots\ldots(29).$$

From (26) and (27) we derive the expansion

$$Z(\phi, k) = c_1 \sin\phi_1 + c_2 \sin\phi_2 + \ldots + c_n \sin\phi_n + \ldots \quad \ldots(30).$$

In terms of Jacobi's Theta-function $\Theta(u, k)$,

$$\int_0^u Z(u, k)\, du = \int_0^\phi Z(\phi, k)\, d\phi/\Delta(\phi, k) = \log[\Theta(u, k)/\Theta(0, k)] \quad \ldots(31)$$

where $$\Theta(0, k) = \sqrt{(2k'K/\pi)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(32).$$

* These relations are discussed by Legendre (*Traité des Fonctions Elliptiques*, t. III. Deuxième Supplément, 1828, p. 111 *et seq.*). On the connection between the A.G.M. scale and the modern transformation theory of the Theta-functions, see Tannery and Molk, *Fonctions Elliptiques*, t. IV. 1902, note 2, pp. 269–273. A development of the theory of the Theta-functions based on Legendre's scales of moduli is outlined by Richelot, *Correspondenz mit Herrn Professor Schröter* (Königsberg, 1868). See Examples 19–24 of the Appendix.

From (20) we have $d\phi/\Delta(\phi, k) = (\frac{1}{2})^n d\phi_n/\Delta_n$, so that the series (30) enables us to write

$$\int_0^{\phi} Z(\phi, k)\, d\phi/\Delta(\phi, k) = \sum_1^{\infty} (c_n/2^n) \int_0^{\phi_n} \sin\phi_n d\phi_n/\Delta_n.$$

On integrating and making use of (19) and (18), we obtain

$$\int_0^{\phi_n} \sin\phi_n d\phi_n/\Delta_n = (1/c_n)\log(a_{n-1}/\Delta_{n-1}) = (1/c_n)\log\sec(2\phi_{n-2} - \phi_{n-1}),$$

leading to the expansions*

$$\int_0^u Z(u, k)\, du = \sum_1^{\infty} (\tfrac{1}{2})^n \log(a_{n-1}/\Delta_{n-1}) \qquad \text{.........(33)},$$

or

$$\log[\Theta(u, k)/\Theta(0, k)] = \tfrac{1}{2}\log(a_0/\Delta_0) + \tfrac{1}{4}\log\sec(2\phi_0 - \phi_1)$$
$$+ \tfrac{1}{8}\log\sec(2\phi_1 - \phi_2) + \ldots + (\tfrac{1}{2})^{n+1}\log\sec(2\phi_{n-1} - \phi_n) + \ldots \qquad \text{...(34)}.$$

In the series (34) it is more convenient to write

$$a_0/\Delta_0 = \cos(\phi_1 - \phi_0)/\cos\phi_0$$

for purposes of numerical calculation.

(ii) *Calculation of the Jacobian Functions* sn (u, k), cn (u, k), dn (u, k) *etc. in terms of the argument* u.

When the argument u is given and the modulus k is known, it is only necessary to compute the A.G.M. scale ($a_0 = 1$, $b_0 = k'$) to such a value of n that c_n is less than the small quantity determining the order of accuracy of the calculations. Since $u = F(\phi, k)$, equation (25) enables us to calculate in circular measure the angle ϕ_n from the relation

$$\phi_n = 2^n a_n u \qquad \text{.........................(35)}.$$

From the recurrence formula (18),

$$\sin(2\phi_{n-1} - \phi_n) = (c_n/a_n)\sin\phi_n,$$

we are enabled to calculate successively the angles $\phi_{n-1} \ldots \phi_2, \phi_1$ and finally ϕ_0, in terms of which

$$\text{sn}(u, k) = \sin\phi_0, \quad \text{cn}(u, k) = \cos\phi_0, \quad \text{dn}(u, k) = \Delta_0 = \cos\phi_0/\cos(\phi_1 - \phi_0) \qquad \text{......(36)}.$$

$Z(u, k)$ may then be computed from the series (30), and $\Theta(u, k)$ from formulae (32) and (34).

* Formulae (30), (33) and (34) are practically the same as those given by Legendre (*Traité des Fonctions Elliptiques*, t. I. section 90), and are contained implicitly in formulae given in Gauss' posthumous papers. Formula (34) is equivalent to that given by Jacobi (*Crelle*, Bd. XXVI. pp. 93–114; *Gesammelte Werke* (1881), Bd. II. p. 357).

SECTION V

THE HYPERBOLIC SCALE OF INCREASING AMPLITUDES

(i) *Calculation of the Elliptic Integrals of the First and Second kinds, modulus* k'.

Writing $i\phi_n$ for ϕ_n throughout the formulae (17) to (24), we obtain, instead of (17),

$$\tanh(\phi_{n+1}-\phi_n)=(b_n/a_n)\tanh\phi_n \quad \ldots\ldots\ldots\ldots\ldots(37),$$

which may also be written

$$\sinh(2\phi_n-\phi_{n+1})=(c_{n+1}/a_{n+1})\sinh\phi_{n+1},\ \cosh(2\phi_n-\phi_{n+1})=\Delta_{n+1}/a_{n+1} \quad \ldots\ldots(38).$$

If we denote $\Delta_n=\surd(a_n^2\cosh^2\phi_n-b_n^2\sinh^2\phi_n)$, it is easily proved, as before, that

$$d\phi_0/\Delta_0=\tfrac{1}{2}d\phi_1/\Delta_1=\tfrac{1}{4}d\phi_2/\Delta_2=\ldots=(\tfrac{1}{2})^n d\phi_n/\Delta_n=\ldots \quad \ldots(39)$$

from which it follows that as n increases, we have in the limit,

$$\int_0^{\phi_0} d\phi_0/\Delta_0=(1/a_n)(\phi_n/2^n).$$

If we now write

$$\sinh\phi_0=\tan\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40)$$

it is easily proved that

$$\Delta_0=\Delta(\phi,k')/\cos\phi,\quad d\phi_0/\Delta_0=d\phi/\Delta(\phi,k') \quad \ldots\ldots\ldots(41),$$

so that

$$u=F(\phi,k')=(1/a_n)(\phi_n/2^n) \quad \ldots\ldots\ldots\ldots\ldots(42).$$

Formula (24) becomes,

$$\int_0^{\phi_0}\Delta_0 d\phi_0-a_0^2\int_0^{\phi_0}d\phi_0/\Delta_0=(c_1\sinh\phi_1+c_2\sinh\phi_2+\ldots+c_n\sinh\phi_n+\ldots)$$
$$-\tfrac{1}{2}(c_0^2+2c_1^2+4c_2^2+\ldots+2^n c_n^2+\ldots)\int_0^{\phi_0}d\phi_0/\Delta_0 \quad \ldots(43).$$

It is easily shown that

$$\int_0^{\phi_0}\Delta_0 d\phi_0=\int_0^{\phi}\Delta(\phi,k')\,d\phi/\cos^2\phi$$
$$=\int_0^u \mathrm{dn}^2(u,k')\,du/\mathrm{cn}^2(u,k')=\int_0^u \mathrm{dn}^2(iu,k)\,du,$$

or in terms of the Z-function of imaginary argument,

$$\int_0^{\phi_0}\Delta_0 d\phi_0=(1/i)Z(iu,k)+(E/K)\,u.$$

Substituting in (43), and making use of (27), we derive the series, corresponding to that of (30),

$$(1/i)Z(iu,k)=c_1\sinh\phi_1+c_2\sinh\phi_2+\ldots+c_n\sinh\phi_n+\ldots \quad \ldots(44).$$