CHAPTER I

PRELIMINARY IDEAS

§ 1. Introduction: the Continuous Variable.

Before endeavouring to explain the processes of the Differential Calculus, we must first examine a few of the fundamental ideas which will continually be needed in this subject.

The reader is, no doubt, already familiar with the idea of a "variable" or "varying quantity." In dealing with elementary graphs, for instance, the idea must have occurred before. In the Calculus, however, we have to make one further restriction; namely, that the variable shall be a "continuous" variable. If x is the variable under consideration and if a and b are two "boundary values" of x (i.e. if only values of x lying between a and b are considered), then x must pass in succession through every conceivable value lying between a and b, irrational values included. In all that follows we shall assume that our variables are continuous.

There is a useful convention by which we denote variables by letters near the end of the alphabet, and also by the Greek equivalents of these letters, as, u, v, w, x, y, z, ξ, η, ζ, and so on; reserving the letters at the beginning of both alphabets for constants.

§ 2. Functions.

Many of the problems of Mathematics are concerned with the dependence of one magnitude on another; or on the inter-dependence of two magnitudes. When this dependence is such that, given one of the magnitudes, we can always determine the other, then the second of these magnitudes is called a *function* of the first. Many examples of functions will at once occur to the reader. If for example the radius of a circle is known, then its area can at once be found. Thus the area of a circle is a function of its radius. Or again we can have a function of two or more variables, e.g. the volume of a cylinder is a joint function of the height and base-radius.

This definition of a function can usually be replaced by two simpler statements. These are

(1) If y is a function of a continuous variable x, then y and x must be connected by some equation;

and (2) The relationship between a function y and its variable (or, as it is sometimes styled, its argument) can be represented by a graph.

The proof of these two statements lies somewhat beyond the scope of this work, though on general grounds they are seen to be true. We shall therefore assume, in general, that the functions with which we deal can be represented graphically and also by an equation, for all except special values of the variables.

§ 3. Implicit and Explicit Functions.

There are, generally speaking, two types of equation that may occur as the definition of functions. These are (a) explicit equations, (b) implicit equations.

An explicit equation is one in which y is given directly in terms of x, and y is then called an *explicit function of* x.

The following are examples of equations defining *explicit functions*:

(1) $y = x^2 + x + 1.$

(2) $y = \sin x.$

(3) $y = \pm\sqrt{1 - x^2}.$

(4) $y = -\dfrac{2x+1}{x+1}.$

An implicit equation is one in which neither variable is given explicitly in terms of the other; but an equation connecting the two variables is given instead.

The following are equations defining *implicit functions*:

(5) $x^2 + y^2 = 1.$

(6) $xy + 2x + y + 1 = 0.$

(7) $x^2 + 2xy - y^2 - 1 = 0.$

The reader may notice that (3) and (5), also (4) and (6), are identical. In fact, almost any implicit equation can theoretically be converted into an explicit one, but the labour of performing the conversion is often prohibitive and the result too cumbersome to use, as can be seen by solving (7) for y in terms of x.

§ 4. Many-valued Functions.

One further point may be noticed. The implicit equation (5) when converted to the explicit form (3) gives rise to an ambiguous sign. In such cases y is spoken of as being a "double-valued" function of x, there being two distinct values of y to every value of x. [The fact that these values are equal in magnitude is incidental and is not necessarily always the case.] If the graphical mode of representation is adopted it will be found that the graph of such a function is a curve consisting of two branches.

The "inversion" of a single-valued function often gives rise to a many-valued function. For example, take the function $\sin x$. Now $y = \sin x$ and $x = \sin^{-1} y$ are on the face of it the same equation. But one value of y corresponds to every value of x, whereas in the second any number of values of x appear to correspond to a value of y. In such cases we usually restrict the meaning of the inverse function. Thus we restrict $\sin^{-1} y$ to mean that angle *lying between* $\pm \frac{\pi}{2}$ which has y as its sine.

In general, if we have to prove any result about a function, we are quite safe in proving it for a single-valued function only.

§ 5. Definition of a Limit.

Consider now the functional equation

$$y = \frac{x^2 - 1}{x - 1}.$$

For every value of x, except one, a single value of y can be found without difficulty, being in fact the same as the value of $x + 1$. The function $\frac{x^2 - 1}{x - 1}$ has thus a well-defined value for every value of x, except one.

The value excepted is evidently unity; for, if $x = 1$, numerator and denominator both vanish and we are left with $\frac{0}{0}$, which, of course, is meaningless. We cannot even get out of the difficulty by dividing out the common factor and leaving $y = x + 1$. For by the ordinary principles of Algebra, division by the zero factor $x - 1$ is invalid. In fact the expressions $\frac{x^2 - 1}{x - 1}$ and $x + 1$ are only identical just so long as we exclude the possibility $x = 1$.

We thus see that for the value $x=1$ the function $\frac{x^2-1}{x-1}$ assumes an "indeterminate" form, and the function cannot be said to have a value at all when $x=1$, in the ordinary sense of the term "value."

Let us try another method of approach. Take a series of values of x converging down to unity, and evaluate y for each of these values. The results are tabulated as follows:

x	1·1	1·01	1·001	and so on
y	2·1	2·01	2·001	

Again, take a series of values below unity and converging up to it and we get the following table:

x	·9	·99	·999	and so on
y	1·9	1·99	1·999	

We see that the closer x approaches the value 1, the closer y approaches 2, whether we come down to the value $x=1$ from above, or up to it from below. In fact, however close we may wish y to be to 2, we can find a value of x close to unity which will give us this value. For suppose we wish to have y within ·000001 of 2; it is necessary merely to take x between 2·000001 and 1·999999 and the required standard is attained.

Summing up, we see that the following facts are true:

(1) As x approaches unity, either from above or below, the values of y become closer and closer to 2.

(2) However close to 2 we may wish y to be, a value of x can be found near unity to give the desired result.

We express this state of affairs shortly by saying that the "Limiting value" or "Limit" of y as x "approaches" or "tends to" 1 is 2. This is written in the form

$$\underset{x\to 1}{\mathrm{Lt}}\frac{x^2-1}{x-1}=2.$$

§ 6. Limits and Values.

The reader should make his mind quite clear on this subject before proceeding further. "Limits" should not be confused with values, and for this reason the term "Limit" is to be preferred to the alternative "Limiting value." The function never actually assumes the value 2, but can be made to approach it as closely as we like. The reader will probably have encountered a similar state of affairs in dealing with an infinite G.P. (though in this case n, the number of terms, is not a continuous variable).

It may perhaps be found easy to consider the question as if it were a game played by two opponents. First of all, your opponent states his standard of accuracy. He says: "I will accept the limit as 2 if you can get y within ·001 of 2." You counter him by saying "This can be done by taking x between ·999 and 1·001." He then changes his ground and says "I will increase the standard of accuracy to ·00001." You can then counter him by saying "Let x lie between ·99999 and 1·00001." And so on.

If you can always win, however stringent the standard of accuracy imposed, then the limit is 2, as you state it to be. But if at any stage your opponent defeats you, the limit is not 2.

This idea cannot easily be put into precise mathematical language without making the definition rather involved. But, for interest, we give here a complete definition of a limit.

Given a small number ϵ and two definite numbers a and A and that $f(x) \sim A < \epsilon$, if it is possible to find another small number η such that $x \sim a < \eta$, however small ϵ may be; and conversely if for all values of x lying within η of a the difference $f(x) \sim A < \epsilon$, then A is said to be the limit of the function as x approaches a.

But the reader will probably find the idea easier to remember than the definition.

Note that it is necessary for $f(x)$ to approach the *same* number A whether x is just greater or just less than a; otherwise the function undergoes a sudden change in value and is said to be discontinuous. But we will revert to this subject a little later.

A similar mode of argument is seen to hold for a function which "increases without limit" or "tends to infinity" as x tends to some particular value. It is, of course, a contradiction in terms to speak of "infinity" as a "value" of the function, for this implies that "infinity" is a definite number.

Here again your opponent states his standard, which in this case is a very large number, let us say, 10000. You then have to prove the existence of a value of x close to the particular value mentioned such that $f(x) > 10000$, and so on. If this can be done, however large a number your opponent chooses, then the function is said to "become infinite" or "tend to infinity" at the value of x concerned. This state of affairs is written

$$\operatorname*{Lt}_{x \to a} f(x) \to \infty$$

or simply $$f(x) \to \infty \text{ as } x \to a.$$

In many cases however the function changes sign as x passes through the value a, and this, and other reasons we shall consider in § 11, cause an "infinity" to be regarded as a "discontinuity" of a function.

§ 7. Rules for use of Limits.

We give here, without proof, the main rules for working with limits. It can be seen that the proof of these rules is somewhat beyond the scope of this work, but it can be safely assumed that these results are true for any functions which the student is likely to encounter in the course of ordinary work.

I. *The limit of the algebraic sum of any finite number of functions is equal to the like algebraic sum of their separate limits.*

II. *The limit of the product of any finite number of functions is equal to the product of their separate limits.*

III. *The limit of the quotient of two functions is equal to the quotient of their limits, provided that the limit of the divisor is not zero.*

A fourth rule, whilst not exactly fundamental, is usually assumed to hold.

IV. *The limit of the logarithm of any function is equal to the logarithm of the limit of the function.*

§ 8. An important Limit.

We give here as a worked example a very important Trigonometrical Limit, viz.

$$\underset{\theta \to 0}{\mathrm{Lt}} \frac{\sin\theta}{\theta}.$$

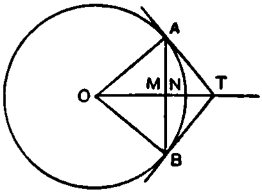

Fig. 1

In Fig. 1 let the angle AOB be θ radians, ON bisecting the angle AOB.

Then it is clear from elementary geometry that $\triangle AOB <$ sector $AOB <$ fig. $ATBO$ in area.

I.e. $\frac{1}{2}OA^2 \sin\theta < \frac{1}{2}OA^2 . \theta < OA^2 . \tan\frac{\theta}{2}$,

i.e. $$\sin\theta < \theta < 2\tan\frac{\theta}{2}.$$

Dividing both sides by $\sin\theta$,

$$1 < \frac{\theta}{\sin\theta} < \frac{2\tan\frac{\theta}{2}}{\sin\theta},$$

i.e. $$1 < \frac{\theta}{\sin\theta} < \sec^2\frac{\theta}{2},$$

or $$1 > \frac{\sin\theta}{\theta} > \cos^2\frac{\theta}{2}.$$

Hence $\frac{\sin\theta}{\theta}$ always lies between 1 and $\cos^2\frac{\theta}{2}$.

Now, by a proper choice of a small angle θ, $\cos^2\frac{\theta}{2}$ may be made to approach unity as closely as we please. Hence the ratio $\frac{\sin\theta}{\theta}$ may be made to differ from unity by as little as we please by a proper choice of θ.

Hence, by definition,

$$\underset{\theta\to 0}{\mathrm{Lt}} \frac{\sin\theta}{\theta} = 1.$$

This is an important result and should be carefully noted.

We have above

$$1 > \frac{\sin\theta}{\theta} > \cos^2\frac{\theta}{2}.$$

If $\theta = 2\phi$, then $$1 > \frac{\sin 2\phi}{2\phi} > \cos^2\phi.$$

Dividing by $\cos^2\phi$, $\sec^2\phi > \frac{\tan\phi}{\phi} > 1.$

Hence, as above, $$\underset{\phi\to 0}{\mathrm{Lt}} \frac{\tan\phi}{\phi} = 1.$$

It should be noted that while $\frac{\sin\theta}{\theta}$ and $\frac{\tan\theta}{\theta}$ tend to the same Limit, one tends to it from below, the other from above.

§ 9. Sum of a Series: the Exponential Series.

The idea of the "sum" of an infinite series is similar to that of a Limit. If s_n, the sum of n terms of the series, tends to a limit S as $n \to \infty$, the series is said to "Converge" to this "sum." Tests for convergency of infinite series are found in text-books on Algebra. The reader is probably familiar with this notion, possibly through the "Geometric" series.

Many important convergent series are in general use and it is assumed that the reader has met the Binomial Series.

Another important convergent series which occurs is the Exponential Series. The infinite series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

may be shown to be convergent for all values of x and is known as the "exponential" function of x. In particular the value assumed by this series when $x = 1$, viz.

$$1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots \text{ ad inf.},$$

may be shown to lie between 2 and 3 and is defined as the number e $(= 2{\cdot}7183\ldots)$.

If $E(x)$, $E(y)$ denote the exponential functions of x and y, it may be shown that $E(x) . E(y) = E(x+y)$ and thence that $E(x) = (E(1))^x = e^x$ for all values of x*.

* See text-books on Higher Algebra.