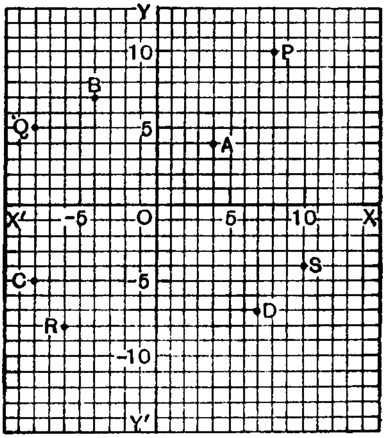
The co-ordinates x and y can have any numerical values, positive or negative, negative values of x being measured along OX', and negative values of y parallel to OY'.

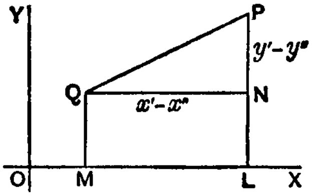
It is usual to speak of the point (x, y) meaning thereby the point whose co-ordinates are x and y.

1.21 In the accompanying figure the points marked A, B, C, D are the points (4, 4), (– 4, 7), (– 8, – 5), (7, – 7). As an exercise the student can write down the co-ordinates of the points P, Q, R, S.



1.3 Distance between two points. The distance between a pair of points P, Q of given co-ordinates (x', y') and (x'', y'') can be found by the theorem of Pythagoras.

Draw PL, QM, QN parallel to the axes, as in the figure, so that



$$OL = x', \quad LP = y',$$

$$OM = x'', \quad MQ = y'',$$

then $\quad QN = ML = x' - x''$,

and $\quad NP = LP - MQ = y' - y''$.

Then $PQ^2 = QN^2 + NP^2$

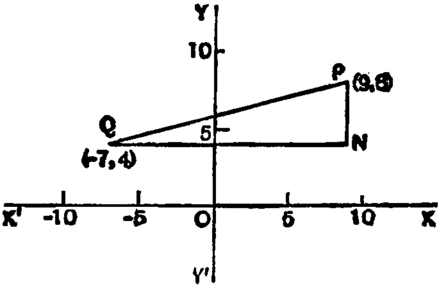
$$= (x' - x'')^2 + (y' - y'')^2,$$

and $$PQ = \sqrt{\{(x' - x'')^2 + (y' - y'')^2\}} \qquad \text{......(1).}$$

Cor. The distance of a point P, (x', y'), from the origin is given by $$OP = \sqrt{(x'^2 + y'^2)}.$$

1.31 Example. The distance between the points (9, 8) and (– 7, 4) is found directly from the formula 1.3 (1), by substituting $x' = 9$, $y' = 8$, $x'' = -7$, $y'' = 4$ and

$$= \sqrt{\{(9+7)^2 + (8-4)^2\}} = 4\sqrt{17}.$$



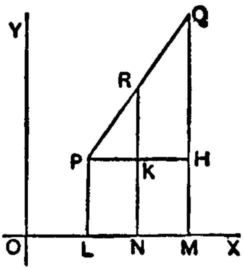
Or, without using the formula, we see from a figure that $QN = 16$ and $NP = 4$ and thus obtain PQ.

It is to be noted that the formulae of co-ordinate geometry are all *general* formulae applicable to points in all the quadrants provided that the proper signs are always affixed to the numerical values of the co-ordinates when they are introduced.

1.32 In 1.3 the distance PQ is found by extracting a square root, so that either sign might be given to it. Except in the case of lines parallel to either axis there is no convention about the directions to be considered positive or negative, and the distance between two points is measured as a positive number. But when we have three points A, B, C in a straight line it is necessary to adhere to the rules

$$AB = -BA \quad \text{and} \quad AB + BC = AC,$$

no matter in what order the points are placed.



1.4 Middle point of the line joining two given points. Let P be (x', y') and Q be (x'', y'') and let the co-ordinates of R the middle point of PQ be x, y. Drawing parallels to the axes as in the figure we have, by parallels, that since R is the middle point of PQ, therefore K is the middle point of PH and N is the middle point of LM or

$$LN = NM,$$

but $\quad LN = x - x'$ and $NM = x'' - x,$

therefore $\quad x - x' = x'' - x,$

and $\quad x = \frac{1}{2}(x' + x'').$

Similarly we can prove that

$$NR = \tfrac{1}{2}(LP + MQ),$$

or $\quad y = \frac{1}{2}(y' + y'').$

1.5 Point dividing a line in a given ratio. A like argument holds if the point R is to divide PQ in the ratio $m:n$, i.e. so that $\quad PR:RQ = m:n.$

For then by parallels

$$LN:NM = PK:KH = PR:RQ = m:n,$$

so that $\quad x - x' : x'' - x = m:n,$

or $\quad n(x - x') = m(x'' - x);$

and therefore $\quad x = \dfrac{nx' + mx''}{n + m},$

and similarly $\quad y = \dfrac{ny' + my''}{n + m}.$

We may use a single symbol to denote the ratio $m:n$, calling it the ratio $k:1$, say, then the co-ordinates of the point which divides the join of (x', y') and (x'', y'') in the ratio $k:1$ are

$$\frac{x' + kx''}{1 + k} \text{ and } \frac{y' + ky''}{1 + k}.$$

1.51 If we wish to divide a line *externally* in a given ratio the same formulae will hold good provided that the ratio $m:n$ or $k:1$ is a negative number.

Thus, if R is a point on PQ produced, we may say that R divides PQ externally in the ratio $PR:RQ$, and this is a negative number because RQ has the opposite sign to PR. Also, if the numerical value of the ratio $PR:RQ$ is greater than 1, R lies on PQ produced, as above; but, if the numerical value is less than 1, then R lies on QP produced.

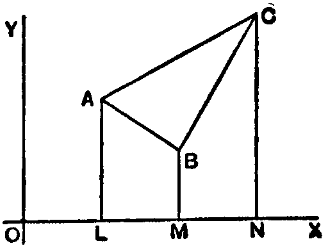
In numerical examples the word *externally* is to be taken as implying a negative sign for the ratio and when it is said that a line PQ is divided internally and externally at R and R' in the ratio $m:n$ we must understand this to mean that

$$PR:RQ=m:n \quad \text{and} \quad PR':R'Q=-m:n.$$

The co-ordinates of R are then given by 1.5, and the co-ordinates of R' are given by

$$x=\frac{nx'-mx''}{n-m}, \quad y=\frac{ny'-my''}{n-m}.$$

1.6 Area of a triangle. Let ABC be a triangle and let the co-ordinates of A, B, C be $x_1, y_1;\ x_2, y_2;\ x_3, y_3$ respectively.



Draw AL, BM, CN perpendicular to OX.

Then the area

$$ABC = ALNC - ALMB - BMNC.$$

But

$$\begin{aligned} ALNC &= \triangle ALN + \triangle ACN \\ &= \tfrac{1}{2}LA.LN + \tfrac{1}{2}LN.NC \\ &= \tfrac{1}{2}LN(LA+NC) \\ &= \tfrac{1}{2}(x_3-x_1)(y_1+y_3). \end{aligned}$$

Similarly $\qquad ALMB = \frac{1}{2}(x_2-x_1)(y_1+y_2),$

and $\qquad BMNC = \frac{1}{2}(x_3-x_2)(y_2+y_3).$

Therefore area

$$ABC = \tfrac{1}{2}\{(x_3-x_1)(y_1+y_3)-(x_2-x_1)(y_1+y_2) - (x_3-x_2)(y_2+y_3)\},$$

which reduces to

$$\tfrac{1}{2}\{x_2y_3-x_3y_2+x_3y_1-x_1y_3+x_1y_2-x_2y_1\}.$$

This will give a positive expression for the area if, as in the figure above, in going round the triangle ABC in the order of the letters we have the area on the left hand; a negative answer would result from interchanging the order of two of the points.

The result can also be expressed as a determinant in the form

$$\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

1.61 Examples.

1. Mark on a diagram the points (– 2, 4), (3, – 5) and find the distance between them.

2. Find the lengths of the lines joining the following pairs of points:

(i) (– 1, 1) and (2, – 1);
(ii) (4, 3) and (– 2, 2);
(iii) (a, b) and $(-b, a)$.

3. Prove that the line joining the points (2, 1) and (4, 7) has the same middle point as the line joining (5, 4) and (1, 4), and hence that the four points are the corners of a parallelogram.

4. Prove that the points (– 3, – 4), (2, 6) and (– 6, 10) are at the corners of a right-angled triangle.

5. Plot the points (0, 2), (1, 1), (4, 4) and (3, 5) and prove that they are at the corners of a rectangle.

6. Plot the points (– 2, – 2), (– 1, 2) and (3, 1) and prove that they are at the corners of an isosceles triangle.

7. In the last question, find the distance of the vertex of the triangle from the middle point of the base.

8. Prove that the points (– 1, 0), (0, 3), (3, 2) and (2, – 1) are at the corners of a square.

9. Prove that the points (– 1, 0), (3, 1), (2, 2) and (– 2, 1) are at the corners of a parallelogram. By finding the co-ordinates verify that the joins of the middle points of pairs of opposite sides have the same middle point.

10. Prove that the points (21, – 2), (15, 10), (– 5, 0) and (1, – 12) are at the corners of a rectangle, and find the co-ordinates of its centre.

11. Find the lengths of the medians of the triangle whose corners are at the points (1, 2), (0, 3) and (– 1, – 2).

12. Find the co-ordinates of the points that divide the line joining the points (– 35, – 20) and (5, – 10) into four equal parts.

13. Find the co-ordinates of the points of trisection of the line joining the points (– 5, – 5) and (25, 10).

14. Prove that the middle point of the line joining the points (– 5, 12) and (9, – 2) is a point of trisection of the line joining the points (– 8, – 5) and (7, 10).

15. The points (8, 5), (– 7, – 5) and (– 5, 5) are three of the corners of a parallelogram. Find the co-ordinates of the remaining corner which is to be taken as opposite to (– 7, – 5).

16. The point (2, 6) is the intersection of the diagonals of a parallelogram two of whose corners are at the points (7, 16) and (10, 2). Find the co-ordinates of the remaining corners.

17. Find the area of the triangle whose corners are the points (2, 3), (– 4, 7), (5, – 2).

18. Find the co-ordinates of points which divide the join of (2, 3), (– 4, 5) externally in the ratio 2 : 3, and also externally in the ratio 3 : 2.

19. Prove that if (x_1, y_1), (x_2, y_2), (x_3, y_3) are the corners of a triangle the co-ordinates of its centroid are

$$\tfrac{1}{3}(x_1+x_2+x_3),\quad \tfrac{1}{3}(y_1+y_2+y_3).$$

1.7 Loci. When a point moves in a plane in conformity with some geometrical law its path in the plane is called its *locus*. Thus if a point moves so as to keep at a fixed distance from a given point its locus is a circle; if it moves so that its distances from two fixed points are always equal to one another its locus is the perpendicular bisector of the join of the two points, and so on.

In co-ordinate geometry loci are represented by equations. The co-ordinates of the moving point are denoted by x and y, now called the "current co-ordinates," and the geometrical property in conformity with which the point moves is expressed symbolically as an equation between x and y, which is called the equation of the locus.

For example:

(i) *If a circle is to be described with the point* (2, 3) *as centre and with radius* 6, we have to express the fact that the point (x, y) is at a distance 6 from the point (2, 3), and from 1.3 it follows that

$$(x-2)^2+(y-3)^2=36,$$

or

$$x^2+y^2-4x-6y-23=0.$$

We call this the equation of the circle.

(ii) *A point is to move so that it is at a constant distance a from the y-axis.*

If nothing more is specified, the point may be on either side of the y-axis, and therefore all points for which $x = a$, or $x = -a$ satisfy the required condition, and the whole locus may be represented by the single equation

$$x^2 = a^2.$$

(iii) *A point moves so that its distances from the points* (1, 2) *and* (−2, −1) *are equal. Find the equation of the locus.*

If (x, y) denotes any position of the point, we have from **1.3**

$$(x-1)^2 + (y-2)^2 = (x+2)^2 + (y+1)^2,$$

which reduces to

$$x + y = 0.$$

1.8 We have seen in the foregoing examples that a single geometrical property about the motion of a point leads to an equation which represents the locus of the point. The method of co-ordinate geometry is to use some known fact about a curve in order to obtain its equation and then to deduce other properties of the curve from the equation so obtained.

1.9 Examples.

1. A point moves so that its distance from the point (2, 1) is double its distance from the point (1, 2). Find the equation of its locus.

2. Find the equation of the perpendicular bisector of the line joining the points (3, −4) and (−2, 3).

3. Find the equation of the circle of radius 5 with centre at (3, −4).

4. A point moves so that its distance from the y-axis is equal to its distance from the point (2, 1). Find the equation of its locus.

5. A point moves so that the sum of the squares of its distances from the points (3, 4) and (4, 3) is constant. Find the equation of the locus.

6. A point moves so that its distance from the axis of x is twice its distance from the point (0, 1). Find the equation of the locus.

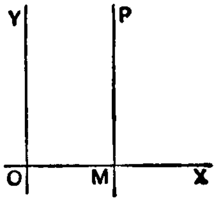
7. A point moves in such a way that with the points (2, 3) and (−3, 4) it forms a triangle of area 8·5. Show that its locus has an equation

$$(x + 5y)(x + 5y - 34) = 0.$$

CHAPTER II

THE STRAIGHT LINE

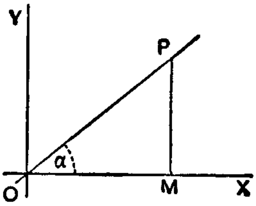
2.1 Straight lines parallel to an axis. If a straight line is parallel to the axis OY every point on it has the same abscissa, viz. OM, where M is the point in which the line cuts OX. Therefore if $OM = a$ the equation of the line is $x = a$.



Conversely any equation of the form $x = a$ represents a straight line parallel to OY, because every point on the locus is at the same distance from OY.

Similarly a straight line parallel to OX is represented by an equation $y = b$.

2.11 Straight lines through the origin. Let P be any point on a straight line through O which makes an angle α with OX. Draw PM perpendicular to OX and let x, y be the co-ordinates of P. Then



$$y = MP = OM \tan\alpha = x \tan\alpha.$$

Therefore $\quad y = x\tan\alpha$

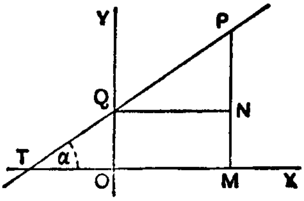
is the equation of the line.

We may also write m for $\tan\alpha$ and then the equation is

$$y = mx,$$

and conversely this is the equation of a straight line through the origin, m denoting the tangent of the angle that the line makes with OX.

2.12 Equation of any straight line. Let P be any point (x, y) on the straight line TQP which cuts OY in Q so that $OQ = c$ and makes an angle $PTX = \alpha$ with OX.



Draw PM perpendicular to OX and QN parallel to OX to meet PM in N.

Cambridge University Press
978-1-316-61267-5 — Elementary Co-ordinate Geometry
A. S. Ramsey
Excerpt
More Information

Then, by parallels, the angle

$$PQN = PTX = \alpha,$$

and $$y = MP = NP + MN = QN \tan\alpha + MN$$
$$= OM \tan\alpha + OQ$$
$$= x\tan\alpha + c.$$

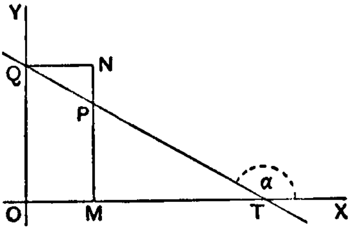
Therefore $$y = x\tan\alpha + c$$

is the equation of the straight line or

$$y = mx + c \qquad \text{......(1)},$$

if m denotes the tangent of the angle that the line makes with OX.

2.121 It must be noted that the angle α is measured from the positive direction of OX and that c is positive when OQ is positive.



The adjoining figure shows a case in which α is obtuse. Here we have

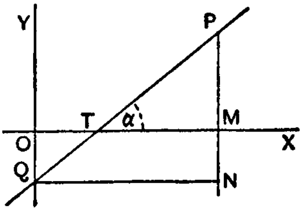
$$y = MP = MN - PN$$
$$= MN - QN\tan(\pi - \alpha)$$
$$= c + OM\tan\alpha = c + x\tan\alpha,$$

or $y = mx + c$, as before.

The next figure shows a case in which $OQ = c$ is negative. Here we have

$$y = MP = NP - NM$$
$$= QN\tan\alpha + OQ$$
$$= x\tan\alpha + c,$$

or $y = mx + c$, as before.



Cor. If the equations of two straight lines can be expressed in the forms

$$y = mx + c, \quad y = mx + c',$$

then the lines are parallel.

2.122 Examples.

(i) The equation of a straight line which makes an angle of 60° with OX and cuts OY at a distance of 2 units from O is

$$y = \sqrt{3}x + 2.$$

(ii) By drawing a figure to show the data it is easy to see that we can find the equation of a straight line when we know the point in