HISTORICAL SKETCH

THE arrangement of the subject-matter of Integral Calculus adopted in the text is almost exactly opposite to the order of the historical development of the subject. A little reflection will convince the student that this is the case in most branches of science. Attention is first drawn to some particular problem, insoluble by known methods. A method is discovered for this particular problem, and is gradually proved to be true for a large class of allied problems. Finally a series of rules governing the use of the method is drawn up. In teaching the subject, of course, the rules have to be learnt first and the application follows. This is exactly the history of the Integral Calculus.

It was quite inevitable that the attention of mathematicians should be drawn, sooner or later, to the problem of summation. As early as the third century B.C. we find the great geometer and physicist Archimedes determining the area bounded by a parabola, by splitting it up into rectangles and finding the sum of these rectangles. He arrived at the correct result, which shows that he knew the result, which we should express in the form

$$\int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1,$$

though, of course, he had not this notation. Very little advance was made for many years. The Hindu and Arab mathematicians who succeeded the Greek school concentrated their attention mainly on algebra and trigonometry and paid no great attention to questions of area and so forth. They were, however, responsible for the formulæ

$$\sum_1^n r^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_1^n r^3 = \frac{n^2(n+1)^2}{4},$$

which were of course used by later writers in performing summations.

But it is not till the time of Johann Kepler and Bonaventura Cavalieri that we find any definite traces of a general attempt to attack the problem of summation. In 1609 Kepler published his work "On the motion of the planet Mars" and it is clear that he was in possession of some method for finding the areas of focal sectors of an ellipse, whilst a few years later in another work he regards the volume of a cask as composed of thin circular discs, just as we should. Cavalieri, who was a Jesuit, published in 1635 the "Method of Indivisibles." This method, though his statement of it was not mathematically rigid, was in effect the method of elements that we use to-day. He stated, for instance, that a line is composed of a number of points without magnitude, a plane area of a number of lines without breadth, and a volume of a number of planes without thickness. He also arrived at the result

$$\int_0^a \frac{b^n x^n}{a^n}\,dx = \frac{ab^n}{n+1},$$

which is equivalent to the rule for integrating x^n.

The method of indivisibles found many exponents, and in the mathematical circles of correspondence* of which P. de Carcany and M. Mersenne were the intermediaries,

* These circles were the predecessors of the scientific societies of to-day, and were the normal means of spreading results at this period.

many definite integrations were exchanged. Notably we may mention Pascal (1623–62), who used the method in 1659 to find the area, volume of revolution and centre of gravity of the cycloid, and Wallis (1616–1703), a Cambridge mathematician who migrated to Oxford as Savilian Professor of Geometry. He expressed a complete set of rules for the summation of the series $\Sigma x^n dx$ for all cases except $n=-1$. His celebrated formulæ mentioned in Ch. V were obtained by regarding the area of a circle $\{y=(x-x^2)^{\frac{1}{2}}\}$ as intermediate between that of the curves $y=(x-x^2)^0$, $y=(x-x^2)^1$. A few years later he also gave some formulæ for the rectification of plane curves. Many other mathematicians of this period effected summations of this type, amongst whom we may mention Fermat, Edward Wright (who had a formula equivalent to $\int \sec\theta\, d\theta$), Roberval, Huyghens, and Torricelli.

We thus see that up to 1660 mathematicians had a good idea of the far-reaching results that might be effected by a general method of summation, and could perform many individual summations, but had no general rule for performing them. The discovery of the Theory of Fluxions by Newton and the Differential Calculus by Leibnitz at about the same date immediately linked up these results in the manner described in Ch. VI on the Fundamental Theorem. Almost immediately, therefore, a complete Integral Calculus sprang into being; the dry bones had long been there, this was the breath that gave them life. Space prohibits us from going into the long and bitter controversy which raged for over 100 years around the two principal figures in these discoveries, and which in some respects had a most unfortunate effect on English mathematics. Suffice it, however, to say that Newton had used his methods in the tract "de Quadratura Curvarum" many years before they were published—and that he knew the outline of the work at any rate as early as 1666. We do know definitely that Leibnitz saw some MSS of Newton's, during his visit to London in 1673, and quite possibly again in 1675 on a visit to Tschirnhausen, but we have no very clear evidence as to what ideas he derived from them. It is only fair to say that the controversy was not sought by the principal figures but by some of their followers, and it is unpleasant to have to add that many of Newton's supporters were more distinguished by their zeal than by their discretion or skill. Only one advantage arose out of this most unfortunate episode; I refer to the "challenge problems" sent out by both sides —at first merely as tests in the efficacy of their own methods, and later as annoyances to the other side. As these problems were often solved by both parties, they contributed but little to the controversy. They have however contributed many results, notably in Dynamics and Kinematics, which remain in the text-books to this day.

There is the other sad side to this subject. The main result was a standing aside of the English mathematicians from the results achieved on the Continent and a persistence in using the old notation and methods, which held English mathematics back at least 100 years, and from which we have hardly even now recovered. It was actually not till 1819 that Peacock, Herschel, and Babbage (the former of whom was moderator that year) were able to introduce the continental notation—or as they themselves cleverly put it* "to advocate the principles of pure

* In the Newtonian notation $\dot{x}$ is used where the continental writes $\frac{dx}{dt}$.

'd'-ism instead of the 'dot'-age of the University." So that finally the nett result was a grafting of the Leibnitzian notation on to Newtonian ideas.

However, four of Newton's contemporaries and immediate successors made some contributions to the theory. Brook Taylor (1685–1731) introduced the idea of changing the variable (Ch. III). Partial Fractions were studied by Cotes (1682–1716), whose early death robbed England of a second Newton, and his work was completed by de Moivre; whilst a standard treatise on the whole method was written by Maclaurin in 1742.

But in the main the development of the subject on the Continent was far in advance of that in this country, owing to the superiority of the notation employed. Leibnitz himself had invented the symbols $\int$ (at first he wrote "omnia y" instead of $\int y$) and $\frac{d}{dx}$, and the methods were enthusiastically supported by the brothers Bernoulli (James and John) who were successively professors at Bâle. James Bernoulli wrote in 1691 the first treatise on Integral Calculus, Count J. F. Riccati spread the new doctrines in Italy, and d'Alembert (1717–83) in his "Traité de dynamique" showed how the principles governing the motion of a body could be expressed by the new notation. The subject of Differential Equations naturally grew up at the same time, but its development has taken much longer and in fact is not yet finally completed. The theory of the simple equations given in this book is however mainly due to d'Alembert and Leonhard Euler (1707–83). The latter made so many valuable contributions to all branches of mathematical research that it is hardly possible to find any branch of the subject which has not a theorem bearing his name associated with it. He published a vast quantity of works, some of which are still read, including (in 1770) his "Institutiones Calculi Integralis," a very full account of the theory, including the method of successive reductions and the Beta and Gamma Functions. The properties of Resonance, which are mentioned in the last chapter, were studied by Lagrange, who published in 1772–85 a series of tracts on Differential Equations which practically forms the basis of the whole subject as read to-day.

The student who is desirous of further historical information is referred to the standard works, particularly those by the late W. W. R. Ball and Prof. F. Cajori, to which the author himself is mainly indebted. It is the author's opinion that far too little is learnt about those mighty men who laid the foundations of our knowledge, and that, if we studied their history more, we might catch perchance a little of the great enthusiasm and inspiration with which they made their immortal discoveries.

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CHAPTER I

DEFINITIONS AND STANDARD FORMS

1. Definition of Integration.

Integration is defined as the reversal of the process of differentiating. In the simpler cases one variable only is employed. Hence the process of "integrating" a given function of x is equivalent to that of finding a second function of x, which is such that the first function is the "differential" or "derivative" of the second. Or to put the matter into mathematical language—to integrate $F(x)$ is to find another function $\phi(x)$, such that $\frac{d}{dx}\phi(x) = F(x)$.

There is a standard notation for this, which is now almost universally adopted, viz. $\int F(x)\,dx$—which should be read "the integral of $F(x)$ with respect to x." The symbol $\int$ is in reality a conventionalized capital S and signifies the fact that integration is connected with the process of summation, as we shall see in Chapter VI.

2. Character of Integration.

A consideration which strikes us at the outset, is that in the final resort integration is a species of guesswork, i.e. we have to be able to see by intuition the kind of result which will, when differentiated, produce the function given. But it is not a process of illogical or unreasoning guesswork. The reader will find that there are definite things to try and definite rules to follow and to the total collection of these rules is given the name of "Integral Calculus."

We also remember that the differential of a constant is zero. It follows, then, that when we try to reverse the process of differentiating we do not know whether to add a constant to the result or not, or, in any case, what constant to add.

For instance $\frac{d}{dx}(x^4)$, $\frac{d}{dx}(x^4+4)$ and $\frac{d}{dx}(x^4+100)$ are all equal to $4x^3$. Which of these functions then is the "integral" of $4x^3$? We get over the difficulty by saying that $\int 4x^3\,dx$ is x^4+C, where C is a constant that remains to be determined or, as we usually put it, "an arbitrary constant." In any practical question, sufficient data are usually given to determine the value of this constant.

3. Indefinite Integrals. Necessity for Inclusion of Constant.

When an integral is given in this form it is known as an "indefinite" integral. The arbitrary constant should always be included unless it is quite clearly non-existent; its omission can often lead to serious error, as the following example will show.

We remember from Differential Calculus that

$\frac{d}{dx}(-\cos^{-1}x)$ and $\frac{d}{dx}(\sin^{-1}x)$ are both equal to $\frac{1}{\sqrt{1-x^2}}$.

It follows then (if we omit the arbitrary constants) that $\sin^{-1}x$ and $-\cos^{-1}x$ are equal, or that $\sin^{-1}x+\cos^{-1}x=0$, a result which is plainly ridiculous.

What we should have written is

$$\text{(i)}\quad \int\frac{1}{\sqrt{1-x^2}}=\sin^{-1}x+C,$$

or

$$\text{(ii)}\quad \int\frac{1}{\sqrt{1-x^2}}=-\cos^{-1}x+D;$$

whence $\sin^{-1}x + C = -\cos^{-1}x + D,$

or $\sin^{-1}x + \cos^{-1}x = D - C.$

This result is not impossible since we know that

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2},$$

and we have merely to choose the constants D and C so that they differ by $\frac{\pi}{2}$.

4. Since, as we have seen, integration and differentiation are reverse processes, it follows that the integral of the sum of a number of separate functions is the sum of the integrals of the separate functions,

viz. $$\int(u+v+w+\ldots)dx = \int u\,dx + \int v\,dx + \int w\,dx + \ldots.$$

It is also obvious that integration is commutative with respect to a constant multiplier,

i.e. if A is constant $\int Au\,dx = A\int u\,dx.$

A table of Standard Forms can be constructed by reversing well-known results of Differential Calculus. Such a table is given here but the reader will find it of far greater value to construct his own table and keep it for reference, adding to it from time to time any new results he may meet. The results are given here both in the Differential and Integral Forms but in all cases the arbitrary constant has, for considerations of space, been left out. It is, of course, much more essential that such a table should exist in the reader's memory than in the text-book, as a thorough knowledge of it is essential to his further progress.

5. Table of Standard Forms.

Differential Calculus	*Integral Calculus*
$\frac{d}{dx}x^n = nx^{n-1}.$	$\int x^n\,dx = \frac{x^{n+1}}{n+1}$ unless $n=-1$.
$\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1}.$	$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a}.$
$\frac{d}{dx}\log_e x = \frac{1}{x}.$	$\int \frac{1}{x}\,dx = \log_e x.$
$\frac{d}{dx}e^{ax} = ae^{ax}.$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax}.$
$\frac{d}{dx}\sin x = \cos x.$	$\int \cos x\,dx = \sin x.$
$\frac{d}{dx}\cos x = -\sin x.$	$\int \sin x\,dx = -\cos x.$
$\frac{d}{dx}\sin mx = m\cos mx.$	$\int \cos mx\,dx = \frac{1}{m}\sin mx.$
$\frac{d}{dx}\tan x = \sec^2 x.$	$\int \sec^2 x\,dx = \tan x.$
$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x.$	$\int \operatorname{cosec}^2 x\,dx = -\cot x.$
$\left.\begin{aligned}\frac{d}{dx}\sin^{-1}\frac{x}{a} &= \frac{1}{\sqrt{a^2-x^2}}\\ \frac{d}{dx}\cos^{-1}\frac{x}{a} &= -\frac{1}{\sqrt{a^2-x^2}}\end{aligned}\right\}.$	$\int \frac{dx}{\sqrt{a^2-x^2}} = \begin{cases}\sin^{-1}\frac{x}{a}, \\ -\cos^{-1}\frac{x}{a}.\end{cases}$
$\frac{d}{dx}\tan^{-1}\frac{x}{a} = \frac{a}{x^2+a^2}.$	$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$ $\left(\text{or } -\frac{1}{a}\cot^{-1}\frac{x}{a}\right).$
$\frac{d}{dx}a^x = a^x\log_e a.$	$\int a^x\,dx = \frac{a^x}{\log_e a}.$

N.B. (1) Notice that all integrals commencing with the prefix "co-" have a negative sign prefixed.

(2) In cases like $\int \frac{dx}{a^2+x^2}$ and $\int \frac{dx}{\sqrt{a^2-x^2}}$ it is useful to notice the