INTRODUCTION

1. *Historical*

The function with which these tables are mainly concerned seems to have been investigated first by Airy* (1838, 1849) and has been, in consequence, named after him. In calculating light-intensity in the neighbourhood of a caustic, Airy met with the integral

$$W = \int_0^\infty \cos \tfrac{1}{2}\pi(w^3 - mw)\,dw$$

He gave (1838, p. 390) a 5-decimal table of W for $m = -4{\cdot}0(0{\cdot}2)+4{\cdot}0$, and a table of W^2 to 5 or more decimals over the same range of m. These tables were derived from 7-decimal calculations of W, the values being given in full on page 402 of the same paper. Later (1849, p. 598) he gave a newly-calculated 5-decimal table of W for $m = -5{\cdot}6(0{\cdot}2)+5{\cdot}6$.

The 1838 table was calculated by quadratures supplemented by an expansion, asymptotic in character, for the 'tail' of the integral. For the 1849 table the ascending series was used. Airy's reasons for his choice of methods are not uninteresting (1849, p. 595):

"The computation by quadratures was exceedingly laborious, and I did not resort to it without trying other methods of a more refined nature. But in every attempt at expansion of the formula I was met by the integral of a sine or cosine with infinite limits. The reasonings upon which several mathematicians have attempted to establish the value of such an integral appeared to me so little conclusive, that I preferred at once to abandon the expansions which introduced them, and to rely only on the infallible but laborious method of quadratures.

"On my stating to Professor De Morgan, after terminating the calculations, the scruples which had led me to reject the expansions, he expressed himself so strongly confident of the correctness of the conclusions upon the point which I had considered doubtful, that I was induced to undertake the numerical computation of the series given by expansion of the formula."

That Airy's scruples were not entirely unjustified is apparent on examination of the various attempts to attach a meaning to sin ∞ and cos ∞, and to integrals of the type to which he refers.

Having performed the new calculations, he draws the conclusion (1849, p. 599):

"The agreement of the values of the integral, computed by methods so totally different, is not a little remarkable. On the one hand, it may be received by some persons as a proof of the correctness of that part of the theory of series which asserts the evanescence of the integral of a cosine when the limits are 0 and 1/0: on the other hand it may be considered to afford evidence of the great care with which the quadrature computations had been made."

A recalculation of Airy's values has been made, using the present tables, and noting that

$$W = 2\kappa\,\mathrm{Ai}(-\kappa m)$$

where $\kappa = (\pi^2/12)^{1/3}$. This constant κ is characteristic of relations between Airy's form of his integral and the definition adopted in this work; the values of κ and of its reciprocal are, to ten decimals,

$$\kappa = 0{\cdot}93692\ 78888 \qquad 1/\kappa = 1{\cdot}06731\ 79996$$

The recalculation has revealed that, in units of the seventh decimal, the greatest error in any of Airy's 7-decimal values for $-4{\cdot}0 \leqslant m \leqslant +2{\cdot}0$ is 52 units; the error then changes fairly steadily from -56 units at $m = 2{\cdot}2$ to -272 units at $m = 3{\cdot}4$; at $m = 3{\cdot}6$, $3{\cdot}8$ and $4{\cdot}0$ the errors are -23, $+858$ and $+2661$ of these units. Airy's revised table has only one error of more than 2 units of the fifth decimal; the value for $m = -3{\cdot}6$ is 3 units in error, the value in the original table being, however, correct to 5 decimals.

The ascending series, although an enormous improvement compared with quadratures, is still very laborious when m is large, and the desire to reduce this labour induced Stokes (1851, 1858, 1907) to develop asymptotic expansions and led to his remarkable discovery of the discontinuity of the

* Dated references are given in the Bibliography.

'arbitrary constants' appearing in asymptotic developments (1858). In the 1851 paper Stokes also developed asymptotic expansions for zeros of W and of its derivative, and tabulated, to 4 decimals, the first 50 zeros of W and the first 10 zeros of its derivative. These have been recomputed using the present tables; they are respectively $-a_s/\kappa$ and $-a'_s/\kappa$, where a_s and a'_s are the zeros of Ai(x) and Ai$'(x)$, see Table III, and $1/\kappa$ has the value already given. With one exception (the first zero of the derivative, which should be 1·0874 not 1·0845), all Stokes's values are correct within a unit of the fourth decimal.

Airy's and Stokes's tables have been reproduced several times (see Fletcher, Miller and Rosenhead, *Index of Mathematical Tables*, Sub-section 20·2, London, Scientific Computing Service), but, except for a small table of $\sqrt{\pi}\,\text{Ai}(-x)$ in Kramers (1926, p. 840), which has an unreliable third decimal, no new calculation seems to have been made until Jeffreys (1928, p. 107) announced that he had made a table of Ai(x) and Ai$'(x)$; this table is for $x = -2{\cdot}05(0{\cdot}05)+2{\cdot}05$, with 8 working decimals, and has been incorporated in the present tables. There are, however, tables of the closely related Bessel functions of order $\pm 1/3$ and $\pm 2/3$ and of their zeros (*Index*, Articles 17·221–17·232, 17·752–17·7536, 18·221–18·222); the most extensive of these yet published is in Watson (1922), which gives (pp. 714–729) $J_{1/3}(x)$, $Y_{1/3}(x)$, $|J_{1/3}(x)+iY_{1/3}(x)|$, $e^x K_{1/3}(x)$ to 7 decimals, and $\tan^{-1}\{Y_{1/3}(x)/J_{1/3}(x)\}$ to 0″·01, all for $x = 0{\cdot}00(0{\cdot}02)\,16{\cdot}00$, and (p. 751) the first 40 zeros of $J_{1/3}(x)$, $Y_{1/3}(x)$, $J_{-1/3}(x) \pm J_{1/3}(x)$ to 7 decimals. Watson also notes that to compute functions of order $-1/3$ the phase should be increased by 60°. A more extensive MS. table prepared by the Mathematical Tables Project of the New York W.P.A. has been announced (see *Mathematical Tables and other Aids to Computation*, **1**, 93, 1943); the main tables give $J_{\pm 1/3}(x)$, $J_{\pm 2/3}(x)$, $I_{1/3}(x)$ and $I_{2/3}(x)$ for $x = 0{\cdot}00(0{\cdot}01)25{\cdot}00$, and $I_{-1/3}(x)$ and $I_{-2/3}(x)$ for $x = 0{\cdot}00(0{\cdot}01)13{\cdot}00$, all to ten decimals or figures. The connections between these functions and the functions Ai(x) and Bi(x) are exhibited on pages B9 and B17. Another table of interest is a short (but apparently unique) table for pure imaginary argument given by Rayleigh (1915).

There have been several theoretical investigations of the properties of the Airy Integral and of allied functions; in particular those of Nicholson (1909), of Brillouin (1916) and of Kramers (1926) may be noted. A number of their results, and others, are given in Watson (1922, pp. 188–190, 248–252, 320–324).

Recently the demand for tables of the Airy Integral has revived; this revival is closely connected with the simplicity of the differential equation satisfied by the function. We readily verify that

$$\frac{d^2W}{dm^2} = -\frac{1}{12}\pi^2 mW = -\kappa^3 mW$$

Jeffreys (1928, p. 105, but with a later change in the sign of x) has introduced changes of scale in function and argument, defining*

$$\text{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{1}{3}t^3 + xt\right) dt \tag{1}$$

so that, using accents to denote differentiation with respect to the argument, Ai(x) satisfies the differential equation

$$y'' = xy \tag{2}$$

This differential equation arises naturally as an approximation to the general second order differential equation over a limited range of the argument, for, supposing such an equation to be reduced to the normal form (see, e.g., Ince, *Ordinary Differential Equations*, p. 394, 1927)

$$y'' + I(x)\,.\,y = 0$$

we may in general approximate to $I(x)$ in the neighbourhood of $x = x_0$ by an expression of the form

$$I(x) = a + b(x - x_0) = I(x_0) + (x - x_0)\,I'(x_0)$$

neglecting terms involving higher powers of $x - x_0$. A change of origin and scale for x now leads to the equation (2). This method of approximation is especially useful near a zero of $I(x)$, i.e. when $a = 0$; see Jeffreys (1942). It soon became apparent that tables of a suitable second and independent solution of the differential equation were also needed.

* Watson, in his discussions, deals with the functions $\int_0^\infty \cos(t^3 \pm xt)\,dt = 3^{-1/3}\pi\,\text{Ai}(\pm 3^{-1/3}x)$ which satisfy the differential equations $d^2y/dx^2 = \pm\tfrac{1}{3}xy$.

CONTENTS

2. *Description of the Tables*

2·1. The complete integral of the differential equation (2) may be written in the form

$$y = A\,\text{Ai}(x) + B\,\text{Bi}(x) \tag{3}$$

where A and B are constants, and Bi(x) is a suitably chosen independent solution of (2) that is defined in §4. The tables are concerned mainly with Ai(x) and its derivative; every solution of (2) which remains finite as $x \to \infty$ is a multiple of this integral, which itself tends to zero as $x \to \infty$.

The complete integral of (2) can also be written in the form

$$y = C\,F(x)\sin\{\chi(x) + \epsilon\} \tag{4}$$

in which C and ϵ are arbitrary constants. If we take

$$C^2 = A^2 + B^2 \qquad \tan\epsilon = B/A \tag{5}$$

where A and B are the same constants as in (3), then

$$\text{Ai}(x) = F(x)\sin\chi(x) \qquad \text{Bi}(x) = F(x)\cos\chi(x) \tag{6}$$

The corresponding derivative can be similarly expressed, with the *same* constants C and ϵ, as

$$y' = C\,G(x)\sin\{\psi(x) + \epsilon\} \tag{7}$$

where

$$\text{Ai}'(x) = G(x)\sin\psi(x) \qquad \text{Bi}'(x) = G(x)\cos\psi(x) \tag{8}$$

The tables aim at an 8-figure standard of accuracy throughout for Ai(x), Bi(x) and their derivatives; this corresponds to 6 decimals of a degree in the phases $\chi(x)$ and $\psi(x)$ in Table VII, and to 9 decimals for $\pm\tau = 0{\cdot}1\,\text{Bi}'(\pm x)$ in Table IV. Reduced derivatives of higher order in Table IV are given to the 10 decimals needed for interpolation of Bi′(x) to 8 decimals.

Provision for interpolation is made everywhere (except of course in Tables III and V); the desire to make this provision resulted in a decision to tabulate $\log_{10}\text{Ai}(x)$, Ai′(x)/Ai(x), $\log_{10}\text{Bi}(x)$ and Bi′(x)/Bi(x) in Tables II and VI. The details of interpolation are discussed in § 3; BRITISH ASSOCIATION *Auxiliary Table I* is available to assist in this process.

2·2. *Notation for Reduced Derivatives.* The operator τ is defined by

$$\tau\phi(x, h) = \int_0^h \frac{\partial}{\partial x}\phi(x, t)\,dt \tag{9}$$

If this operator is applied to the function $f(x)$, which is the same for all values of h, it is readily verified by repeated application that

$$\tau^n f(x) = (h^n/n!)\, f^{(n)}(x) \qquad (10)$$

that is, $\tau^n f(x)$ is the nth reduced derivative of $f(x)$, a term in the Taylor expansion of $f(x+h)$. For this reason τ may be called the *Taylor Operator*. It should also be noted that

$$(-\tau)^n f(-x) = (h^n/n!)\, f^{(n)}(-x) \qquad (11)$$

In these tables $\tau^n f(x)$ is denoted by τ^n when dealing with a specific function such as $\mathrm{Bi}(x)$ or $\mathrm{Bi}(-x)$, as in the headings of Table IV and in formulae (23) to (26). In order to emphasize the factor $(-1)^n$ in (11) and to enable the reduced derivatives to be given throughout with the correct sign (i.e. that of the derivative) $(-\tau)^n$ is given in Table IV when the argument is negative, that is, for $\mathrm{Bi}(-x)$ with $x > 0$. Since $h = 0{\cdot}1$ in this table, the columns headed τ and $-\tau$ thus give always $+\,0{\cdot}1\ \mathrm{Bi}'(\pm x)$.

2·3. *The Phases $\chi(x)$ and $\psi(x)$*. These angles are tabulated in degrees, to simplify determination of their sines and cosines. Decimals of a degree are given, and multiples of 360° have been subtracted from the angles; this multiple, although not usually needed, is also given.

The following tables give natural values of sines and cosines with argument in degrees and decimals; all include convenient provision for interpolation.

BUCKINGHAM, E. *Manual of Gear Design.* Section One, *Mathematical Tables.* New York, Machinery Publishing Co., 1935. This gives 8-decimal values for arguments 0°·00(0°·01)45°·00; see Comrie, *Mathematical Tables and other Aids to Computation*, **1**, 88, 1943 on errors (few and small in the tables of sines and cosines).

HERGET, P. *Astron. Journal*, **42**, 123–125, 196, 1933. This is a one-page table of 8-decimal values for arguments 0°(1°)45°.

PETERS, J. *Siebenstellige Werte der trigonometrischen Funktionen*... Berlin-Friedenau, Optische Anstalt C.P. Goerz, 1918. (Since 1930, Leipzig, Teubner; reprint, New York, Van Nostrand, 1942.) This gives 7-decimal values of sin θ for arguments 0°·000(0°·001)90°·000.

3. *Interpolation*

If full use is made of all differences, modified differences or reduced derivatives printed, the resulting interpolated value will be correct within $1\frac{1}{2}$ units of the last figure tabulated. Again, if this accuracy is desired, then, in general, all differences or modified differences that are given must be used; on the other hand, as many reduced derivatives are tabulated as are needed to obtain interpolated values of $\mathrm{Bi}'(x)$ from either end of the interval, and these will not all be needed if only the nearer tabular point is used, or if a value of $\mathrm{Bi}(x)$ is sought. Details of the interpolation methods suggested are set out below.

3·1. *Interpolation by Differences.* In Tables I, II, VI and VII second differences are given for interpolation with Everett's formula,

$$f_\theta = f(x+\theta h) = \phi f_0 + \theta f_1 + E_0^2\delta_0^2 + E_1^2\delta_1^2 \qquad (12)$$

where h is the tabular interval, $\phi = 1 - \theta$, and terms involving fourth and higher differences are omitted.

In some cases these omitted differences are not negligible, and the use of (12) may entail substantial end-figure error. In such cases slight modifications of Everett's formula are advocated, and means for their use provided.

If δ^4 is not greater than 1000 it may be allowed for by the "throw-back*", using the formula

$$f_\theta = \phi f_0 + \theta f_1 + E_0^2\delta_{m0}^2 + E_1^2\delta_{m1}^2 \qquad (13)$$

where δ_{m0}^2 and δ_{m1}^2 are the "modified second differences" of f_0 and f_1 respectively (see (16) below); (13) should be used where these, *and no other differences*, are provided. The greatest possible error, excluding rounding-off errors made during the interpolation, is less than a unit of the last figure tabulated.

* Comrie, *Interpolation and Allied Tables*, p. 928, 1936, London, H.M. Stationery Office. An elaboration of this idea is used below. See also BRITISH ASSOCIATION, *Mathematical Tables*, Vol. 1, p. xi, 1931.

There still remain short ranges of the argument where fourth and higher differences are of such magnitude that values given by (13) may be several units out in the end figure. The error can everywhere be reduced to less than $1\frac{1}{2}$ of these units by one or other of the formulae

$$f_\theta = \phi f_0 + \theta f_1 + E_0^2\delta_{m0}^2 + E_1^2\delta_{m1}^2 + M_0^4\gamma_0^4 + M_1^4\gamma_1^4 \tag{14}$$

$$f_\theta = \phi f_0 + \theta f_1 + E_0^2\delta_{m0}^2 + E_1^2\delta_{m1}^2 + T^4(\gamma_0^4 + \gamma_1^4) \tag{15}$$

where (the coefficient 0·184 being exact)

$$\left.\begin{aligned} \delta_m^2 &= \delta^2 - 0{\cdot}184\delta^4 + 0{\cdot}038082\delta^6 - 0{\cdot}00830\delta^8 + 0{\cdot}0019\delta^{10} - 0{\cdot}0004\delta^{12} + \dots \\ &= h^2 f'' - 0{\cdot}100667h^4 f^{iv} + 0{\cdot}010193h^6 f^{vi} - 0{\cdot}00103h^8 f^{viii} + 0{\cdot}00010h^{10} f^{x} - \dots \\ 1000\gamma^4 &= \delta^4 - 0{\cdot}27827\delta^6 + 0{\cdot}0685\delta^8 - 0{\cdot}0164\delta^{10} + 0{\cdot}004\delta^{12} - \dots \\ &= h^4 f^{iv} - 0{\cdot}111603h^6 f^{vi} + 0{\cdot}01142h^8 f^{viii} - 0{\cdot}0012h^{10} f^{x} + \dots \end{aligned}\right\} \tag{16}$$

and $$M^4 = 1000(E^4 + 0{\cdot}184E^2) \qquad 2T^4 = M_0^4 + M_1^4 \tag{17}$$

The necessity for the use of (14) or (15) is indicated by the tabulation of γ^4; if $|\gamma_1^4 - \gamma_0^4|$ averages more than a unit (14) must be used, otherwise (15) may be simpler and is subject (in such cases only) to a slightly smaller maximum error.

The coefficients E_0^2, E_1^2, M_0^4, M_1^4 and T^4 are given separately as BRITISH ASSOCIATION *Auxiliary Table I*, for arguments $\theta = 0{\cdot}00(0{\cdot}01)1{\cdot}00$. The following table of M^4 and T^4 may, however, be found helpful.

θ	M_0^4	M_1^4	T^4	θ	M_0^4	M_1^4	T^4
0·0	−0·000	+0·000	−0·000	0·5	+0·219	+0·219	+0·219
0·1	−0·698	+0·256	−0·221	0·6	+0·448	−0·128	+0·160
0·2	−0·768	+0·448	−0·160	0·7	+0·523	−0·506	+0·009
0·3	−0·506	+0·523	+0·009	0·8	+0·448	−0·768	−0·160
0·4	−0·128	+0·448	+0·160	0·9	+0·256	−0·698	−0·221
0·5	+0·219	+0·219	+0·219	1·0	+0·000	−0·000	−0·000

3·2. *Interpolation by Reduced Derivatives.* In terms of the operator τ, defined in (9), Taylor's expansion for $f(x + \theta h)$ may be written

$$f(x + \theta h) = (1 + \theta\tau + \theta^2\tau^2 + \dots + \theta^n\tau^n + \dots) f(x) \tag{18}$$

It is of interest to note, since

$$h\,\partial f(x + \theta h)/\partial x = \partial f(x + \theta h)/\partial\theta \tag{19}$$

that $$\theta\tau f(x + \theta h) = f(x + \theta h) - f(x)$$

whence $$(1 - \theta\tau) f(x + \theta h) = f(x) \tag{20}$$

in formal agreement with the expansion (18).

Differentiation and integration of (18) with respect to θ, with use of (19), give Taylor expansions for the derivative and integral of $f(x)$ at an arbitrary point:

$$hf'(x + \theta h) = (\tau + 2\theta\tau^2 + 3\theta^2\tau^3 + \dots + n\theta^{n-1}\tau^n + \dots) f(x) \tag{21}$$

$$\int_x^{x+\theta h} f(t)\,dt = h(\theta + \tfrac{1}{2}\theta^2\tau + \tfrac{1}{3}\theta^3\tau^2 + \dots + \frac{1}{n}\theta^n\tau^{n-1} + \dots) f(x) \tag{22}$$

The formulae (18) and (21) are to be used for interpolation in Table IV. Expressed in terms of the shortened notation mentioned in § 2·2 these become

$$f(x + \theta h) = f(x) + \theta\tau + \theta^2\tau^2 + \dots + \theta^n\tau^n + \dots \tag{23}$$

$$hf'(x + \theta h) = \tau + 2\theta\tau^2 + 3\theta^2\tau^3 + \dots + n\theta^{n-1}\tau^n + \dots \tag{24}$$

Similarly (cf. (11))

$$f\{-(x + \theta h)\} = f(-x) + \theta\tau + \theta^2\tau^2 + \dots + \theta^n\tau^n + \dots \tag{25}$$

$$-hf'\{-(x + \theta h)\} = \tau + 2\theta\tau^2 + 3\theta^2\tau^3 + \dots + n\theta^{n-1}\tau^n + \dots \tag{26}$$

where $f'(-x) = \{df(t)/dt\}_{t=-x} = -df(-x)/dx$. Again it must be noted that in Table IV the reduced derivatives are given with the true sign of the derivative, so that, when the argument $-x$ is negative, the quantity $(-\tau)^n = (h^n/n!)\{\mathrm{Bi}^{(n)}(t)\}_{t=-x}$ is tabulated.

In using, for example (24), proceed thus: Multiply $8\tau^8$ by θ and add $7\tau^7$; multiply by θ and add $6\tau^6$; continue in this way until τ has been added. The required derivative is then obtained by multiplication by $1/h$ This is the process indicated when (24) is written in the form

$$hf'(x+\theta h) = \tau + \theta\{2\tau^2 + \theta\{3\tau^3 + \theta\{4\tau^4 + \ldots\}\}\}$$

3·3. *Numerical Examples.* To find

$$\mathrm{Ai}'(1{\cdot}97) + \sqrt{3}\,\mathrm{Bi}'(1{\cdot}97) = 2G(1{\cdot}97)\sin\{\psi(1{\cdot}97)+60^\circ\} \quad \text{and} \quad \mathrm{Bi}'(-2{\cdot}57)$$

$\mathrm{Ai}'(1{\cdot}97) = -0{\cdot}05521\ 805$ is taken directly from Table I. $\mathrm{Bi}'(1{\cdot}97)$ is obtained from Table IV, using (i) $x = 2{\cdot}0$, $\theta = -0{\cdot}3$ and (ii) $x = 1{\cdot}9$, $\theta = 0{\cdot}7$ in (24). Individual terms are set out below, and a third column (iii) shows the application of (26); the comma indicates an extra decimal retained to minimize accumulation of error.

	(i) Bi′(1·97)	(ii) Bi′(1·97)		(iii) Bi′(− 2·57)
	$x = 2{\cdot}0,\ \theta = -0{\cdot}3$	$x = 1{\cdot}9,\ \theta = +0{\cdot}7$		$x = 2{\cdot}5,\ \theta = +0{\cdot}7$
τ	+0·41006 8205	+0·34951 6586	$(-\tau)$	−0·02204 2015
$2\theta\tau^2$	− 1978 8570,0	+ 3882 5052,3	$-2\theta\tau^2$	− 756 7393,3
$3\theta^2\tau^3$	+ 51 7475,7	+ 234 2198,0	$3\theta^2(-\tau^3)$	+ 2 9063,8
$4\theta^3\tau^4$	− 9627,2	+ 10 0205,0	$-4\theta^3\tau^4$	+ 1 7970,2
$5\theta^4\tau^5$	+ 144,4	+ 3481,8	$5\theta^4(-\tau^5)$	+ 294,7
$6\theta^5\tau^6$	− 1,8	+ 100,2	$-6\theta^5\tau^6$	− 11,7
$7\theta^6\tau^7$		+ 2,6	$7\theta^6(-\tau^7)$	− 4
$8\theta^7\tau^8$		+ 1		
Sum	0·39078 7626,1	0·39078 7626,0	Sum	−0·02956 2091,7
Thus	Bi′(1·97) =	3·90787 626	Bi′(− 2·57) =	−0·29562 092

It follows that

$$\mathrm{Ai}'(1{\cdot}97) + \sqrt{3}\,\mathrm{Bi}'(1{\cdot}97) = -0{\cdot}05521\ 805 + 6{\cdot}76864\ 023 = +\mathbf{6{\cdot}71342\ 218}$$

the eighth decimal being unreliable to the extent of 1 or 2 units.

$G(1{\cdot}97)$ and $\psi(1{\cdot}97)$ are obtained from Table VII, using (14) and (15) respectively, with $x = 1{\cdot}9$ and $\theta = 0{\cdot}7$.

	$G(1{\cdot}97)$		$\psi(1{\cdot}97)$
$0{\cdot}3G(1{\cdot}9)$	+1·04870 64,9	$0{\cdot}3\psi(1{\cdot}9)$	−0°·29718 9,9
$0{\cdot}7G(2{\cdot}0)$	+2·87071 79,9	$0{\cdot}7\psi(2{\cdot}0)$	−0 ·51922 5,7
$E_0^2\delta_{m0}^2$	− 433 77,2	$E_0^2\delta_{m0}^2$	+ 339 7,5
$E_1^2\delta_{m1}^2$	− 682 00,9	$E_1^2\delta_{m1}^2$	+ 348 6,5
$M_0^4\gamma_0^4$	+ 18,3	$T^4(\gamma_0^4+\gamma_1^4)$	+ 0,0
$M_1^4\gamma_1^4$	− 21,8		
$G(1{\cdot}97)$ =	3·90826 63,2	$\psi(1{\cdot}97)$ =	−0°·80953 1,6
$2G$ =	7·81653 26,4	$\psi + 60^\circ$ =	59°·19046 8,4

Thus $\sin(\psi + 60^\circ) = 0{\cdot}85887\ 470{,}7$ and $2G\sin(\psi + 60^\circ) = \mathbf{6{\cdot}71342\ 22}$

4. *Definitions and Properties of the Functions*

4·1. *Definitions of* Ai(x) *and* Bi(x). The most convenient starting point appears to be the solution of (2) by means of a Laplace contour integral, with complex variable, using the method described by Ince in his *Ordinary Differential Equations*, p. 187, 1927. This gives

$$y = \int_C \exp\left(\tfrac{1}{3}t^3 - xt\right)dt \qquad (27)$$

where C is an open contour such that the integrand vanishes at both ends. These ends must clearly be where $\frac{1}{3}t^3$ has infinitely negative real part, i.e. with phase between limits $(4n+1)\,\pi/6$ and $(4n+3)\,\pi/6$ for any integer n. Each end of C must thus lie at infinity in one of the sectors of the t-plane that are shaded in Fig. 1. A contour C beginning in the sector numbered r and ending in that numbered s may be denoted by L_{rs}. By Cauchy's theorem all contours L_{rs} for given r and s are equivalent, since the integrand of (27) has no singularity in the finite part of the plane.

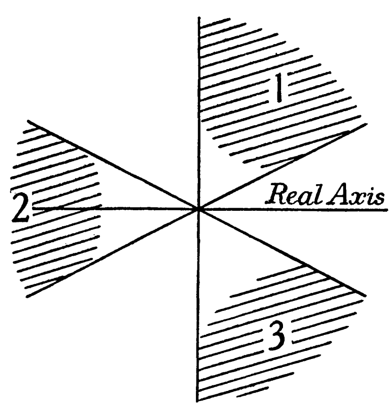

Fig. 1.

Two combinations which give real values of y may be taken as independent solutions of (2); these are

$$\mathrm{Ai}(x) = \frac{1}{2\pi i}\int_{L_{31}} \exp\left(\tfrac{1}{3}t^3 - xt\right) dt \tag{28}$$

$$\mathrm{Bi}(x) = \frac{1}{2\pi}\left\{\int_{L_{21}} \exp\left(\tfrac{1}{3}t^3 - xt\right) dt + \int_{L_{23}} \exp\left(\tfrac{1}{3}t^3 - xt\right) dt\right\} \tag{29}$$

To obtain from these the real integrals given on page B 17, take

$$L_{31}: \quad t = c + iu, \quad -\infty < u < \infty, \quad u \text{ real},$$

$$L_{21}, L_{23}: \quad -\infty < t \leqslant c, \quad t \text{ real}; \quad t = c \pm iu, \quad 0 \leqslant u < \infty, \quad u \text{ real}.$$

In these c is a real positive constant, which we ultimately make tend to zero.

Certain useful relations may be derived from (28) and (29) by means of the substitutions $t = \omega u$ and $t = \omega^2 u$, where $\omega = e^{2\pi i/3}$, a cube root of unity. Thus, for example,

$$\left.\begin{aligned} \mathrm{Ai}(x) + \omega\mathrm{Ai}(\omega x) + \omega^2\mathrm{Ai}(\omega^2 x) &= 0 \\ \mathrm{Bi}(x) + \omega\mathrm{Bi}(\omega x) + \omega^2\mathrm{Bi}(\omega^2 x) &= 0 \end{aligned}\right\} \tag{30}$$

$$\mathrm{Bi}(x) = i\{\omega^2\mathrm{Ai}(\omega^2 x) - \omega\mathrm{Ai}(\omega x)\} \tag{31}$$

It follows that

$$\left.\begin{aligned} \mathrm{Ai}(\omega x) &= -\tfrac{1}{2}\omega^2\{\mathrm{Ai}(x) - i\mathrm{Bi}(x)\} \\ \text{and} \quad \mathrm{Ai}(\omega^2 x) &= -\tfrac{1}{2}\omega\{\mathrm{Ai}(x) + i\mathrm{Bi}(x)\} \end{aligned}\right\} \tag{32}$$

may be obtained from the tables.

4·2. The solution in series of ascending powers of x by usual methods gives the general solution of (2) in the form

$$y = ay_1 + by_2 \tag{33}$$

where y_1 and y_2 are the series given on page B 17. Also, straight line contours bisecting the shaded sectors (Fig. 1) give

$$\left.\begin{aligned} \mathrm{Ai}(0) = 3^{-1/2}\mathrm{Bi}(0) = 3^{-2/3}/(-\tfrac{1}{3})! &= 0{\cdot}35502\ 80538\ 87817 \\ -\mathrm{Ai}'(0) = 3^{-1/2}\mathrm{Bi}'(0) = 3^{-1/3}/(-\tfrac{2}{3})! &= 0{\cdot}25881\ 94037\ 92807 \end{aligned}\right\} \tag{34}$$

4·3. By comparison of expansions in ascending powers of x the representations of Ai(x) and Bi(x) and of their derivatives in terms of Bessel functions of order $\pm 1/3$ and $\pm 2/3$ are readily derived. They are given on page B 17. Since Jeffreys (1942) remarks that "Bessel functions of order 1/3 seem to have no application except to provide an inconvenient way of expressing this function", i.e. the function Ai(x), inverted relations are given below, with $x = (\tfrac{3}{2}\xi)^{2/3}$.

$$\left.\begin{aligned} J_{1/3}(\xi) &= \tfrac{1}{2}x^{-1/2}\{3\mathrm{Ai}(-x) - \sqrt{3}\,\mathrm{Bi}(-x)\} & I_{1/3}(\xi) &= \tfrac{1}{2}x^{-1/2}\{\sqrt{3}\,\mathrm{Bi}(x) - 3\mathrm{Ai}(x)\} \\ J_{-1/3}(\xi) &= \tfrac{1}{2}x^{-1/2}\{3\mathrm{Ai}(-x) + \sqrt{3}\,\mathrm{Bi}(-x)\} & I_{-1/3}(\xi) &= \tfrac{1}{2}x^{-1/2}\{\sqrt{3}\,\mathrm{Bi}(x) + 3\mathrm{Ai}(x)\} \\ J_{2/3}(\xi) &= \tfrac{1}{2}x^{-1}\{\sqrt{3}\,\mathrm{Bi}'(-x) + 3\mathrm{Ai}'(-x)\} & I_{2/3}(\xi) &= \tfrac{1}{2}x^{-1}\{\sqrt{3}\,\mathrm{Bi}'(x) + 3\mathrm{Ai}'(x)\} \\ J_{-2/3}(\xi) &= \tfrac{1}{2}x^{-1}\{\sqrt{3}\,\mathrm{Bi}'(-x) - 3\mathrm{Ai}'(-x)\} & I_{-2/3}(\xi) &= \tfrac{1}{2}x^{-1}\{\sqrt{3}\,\mathrm{Bi}'(x) - 3\mathrm{Ai}'(x)\} \\ K_{1/3}(\xi) &= \sqrt{3}\,\pi x^{-1/2}\,\mathrm{Ai}(x) & K_{2/3}(\xi) &= -\sqrt{3}\,\pi x^{-1}\,\mathrm{Ai}'(x) \end{aligned}\right\} \tag{35}$$

4·4. The asymptotic expansions have been derived by the method of steepest descents. The contours L_{rs} in (28) and (29) are chosen to pass through the saddle points of the integrand, in such directions that the modulus may fall away as rapidly as possible from its maximum values. The saddle points are given by $t = \pm\sqrt{x}$ and the directions of steepest descent to and from these points are illustrated in Figs. 2 and 3 for the two cases $x > 0$ and $x < 0$.

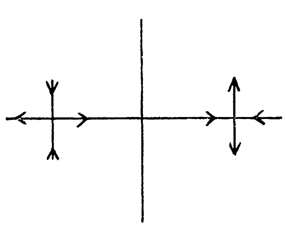

Fig. 2. $x > 0$.

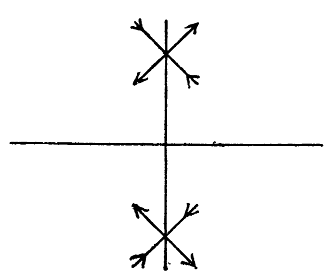

Fig. 3. $x < 0$.

Lack of space forbids the discussion of any but the simplest case, that of Ai(x) with $x > 0$. For this take

$$L_{31}: \qquad t = +\sqrt{x} + iu, \quad u \text{ real}, \quad -\infty < u < \infty. \tag{36}$$

Equation (28) now gives

$$\pi \exp\left(\tfrac{2}{3}x^{3/2}\right) \mathrm{Ai}(x) = \int_0^\infty e^{-u^2\sqrt{x}} \cos \tfrac{1}{3}u^3 du = \tfrac{1}{2}x^{-1/4}\int_{-\infty}^{\infty} e^{-v^2} \cos\left(\tfrac{1}{3}v^3 x^{-3/4}\right) dv$$

$$= \tfrac{1}{2}x^{-1/4}\int_{-\infty}^{\infty} e^{-v^2}\left(1 - \frac{v^6}{2!\,3^2x^{3/2}} + \frac{v^{12}}{4!\,3^4x^3} - \dots\right) dv$$

$$\sim \tfrac{1}{2}\pi^{1/2}x^{-1/4}\left(1 - \frac{3\cdot 5}{1!\,144x^{3/2}} + \frac{5\cdot 7\cdot 9\cdot 11}{2!\,144^2x^3} - \dots\right) \tag{37}$$

The other asymptotic expansions given on page B 17 may be derived similarly, although there are some troublesome variations which demand care. The asymptotic expansions for Ai$(-x)$ and Bi$(-x)$ were used to suggest a suitable ratio for the constants in the definitions (28) and (29).

4·5. *The Auxiliary Functions $F(x)$, $\chi(x)$, $G(x)$ and $\psi(x)$.* Combination of (2) and (4) gives

$$(F'' - F\chi'^2 - xF) \sin(\chi+\epsilon) + (2F'\chi' + F\chi'') \cos(\chi + \epsilon) = 0 \tag{38}$$

where, for brevity, the argument (x) has been dropped. This must be satisfied for all ϵ so that the coefficients of $\sin(\chi + \epsilon)$ and $\cos(\chi + \epsilon)$ must vanish separately. The vanishing of the latter coefficient leads to

$$F^2\chi' = \text{const.} = -1/\pi \tag{39}$$

where the value of the constant is determined by (6), which gives

$$z \equiv \{F(x)\}^2 = \mathrm{Ai}^2(x) + \mathrm{Bi}^2(x) \qquad \tan\chi(x) = \mathrm{Ai}(x)/\mathrm{Bi}(x) \tag{40}$$

Eliminating χ', (38) and (39) now give

$$F'' - 1/\pi^2F^3 = xF \tag{41}$$

or, in terms of z, after a further differentiation,

$$z''' - 4xz' - 2z = 0 \tag{42}$$

which is a linear equation. It may be verified that the complete solution of this last equation is

$$z = a\mathrm{Ai}^2(x) + b\mathrm{Ai}(x)\,\mathrm{Bi}(x) + c\mathrm{Bi}^2(x) \tag{43}$$

From (42) the asymptotic expansion for $\{F(x)\}^2$ on page B 48 may be derived. Some rather heavy algebra and use of (39) then leads to the expansion given for $\chi(x)$.

4·6. The expansions for $\{G(x)\}^2$ and $\psi(x)$ may be derived similarly from the equation

$$x\eta'' = \eta' + x^2\eta \tag{44}$$

satisfied by

$$\eta \equiv y' = A\,\mathrm{Ai}'(x) + B\,\mathrm{Bi}'(x) \tag{45}$$

or, more easily, by differentiating the relations (6). Thus

$$G^2\psi' = x/\pi \tag{46}$$

and

$$G^2 = F'^2 + F^2\chi'^2 = F'^2 + 1/\pi^2F^2 \tag{47}$$

Also, since $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$, etc.,

$$x^2F^2 = G'^2 + G^2\psi'^2 = G'^2 + x^2/\pi^2G^2 \tag{48}$$

and, again,

$$GG' = x\{\mathrm{Ai}(x)\,\mathrm{Ai}'(x) + \mathrm{Bi}(x)\,\mathrm{Bi}'(x)\} = xFF' \tag{49}$$

This enables the asymptotic expansion for $\{G(x)\}^2$ to be derived readily from that for $\{F(x)\}^2$ and leads somewhat laboriously to that for $\psi(x)$.

4·7. *Formulae for Zeros and Turning-Values.* The relations (4) and (7) provide a good approach to the determination of zeros and turning-values of the general solution of (2) and of its derivative. Zeros c, c', of y and y' respectively, satisfy equations

$$\chi(c) = s\pi - \epsilon \qquad \psi(c') = s\pi - \epsilon \tag{50}$$

Reversion of the series obtained by substituting for $\chi(c)$ and $\psi(c')$ their asymptotic expansions leads to the expressions (see also page B 48)

$$\left.\begin{aligned} c &\sim -\lambda^{2/3}\left(1 + \frac{5}{48}\frac{1}{\lambda^2} - \frac{5}{36}\frac{1}{\lambda^4} + \dots\right) & \lambda &= \tfrac{3}{8}\pi(4s-1) - \tfrac{3}{2}\epsilon \\ c' &\sim -\mu^{2/3}\left(1 - \frac{7}{48}\frac{1}{\mu^2} + \frac{35}{288}\frac{1}{\mu^4} - \dots\right) & \mu &= \tfrac{3}{8}\pi(4s+1) - \tfrac{3}{2}\epsilon \end{aligned}\right\} \quad (51)$$

where s is an integer, usually positive, but which may be zero for c' if ϵ (assumed positive) is between 0 and $\pi/6$.

Again, since by (4) and (7)

$$\left.\begin{aligned} y'/C &= G\sin(\psi+\epsilon) = F'\sin(\chi+\epsilon) + F\chi'\cos(\chi+\epsilon) \\ xy/C &= xF\sin(\chi+\epsilon) = G'\sin(\psi+\epsilon) + G\psi'\cos(\psi+\epsilon) \end{aligned}\right\} \quad (52)$$

it follows that

$$\left.\begin{aligned} y'(c) &= \pm CF\chi' = \mp C/\pi F(c) \\ y(c) &= \pm CG\psi'/x = \pm C/\pi G(c') \end{aligned}\right\} \quad (53)$$

giving the turning-values of y' and y.

4·8. Asymptotic expansions, such as (51), are satisfactory only when the zero is large, but an alternative possibility, that of inverse interpolation into Table VII, is available in many cases.

Values of y and y' (e.g., obtained from (3) or (45)) may also be used as a basis for interpolation, inverse or direct, to give zeros and turning-values. The inverse interpolation may be troublesome, but can be avoided as follows:

Suppose that $x = k$ is an approximation to a zero c of a solution y of (2). Write

$$c = k + h \quad (54)$$

Now it is known that an approximation to the value of h is $-y(k)/y'(k)$; write, therefore,

$$u(k) = y(k)/y'(k) \quad (55)$$

and develop h as a power series in u and k. Then starting with any suitable value of k the corresponding value of u is readily obtained and the value of c may be calculated; it is important that the unknown constant c should not occur explicitly in the expansion for h.

Total differentiation of (54) with respect to k, remembering that h is an explicit function of k and u, while u is itself a function of k, gives

$$0 = 1 + \frac{\partial h}{\partial k} + \frac{\partial h}{\partial u}\cdot\frac{du}{dk} \quad (56)$$

But from (55), using (2),

$$du/dk = 1 - ku^2 \quad (57)$$

whence

$$1 + \frac{\partial h}{\partial k} + (1-ku^2)\frac{\partial h}{\partial u} = 0 \quad (58)$$

Substituting $\qquad h = -u + a_2u^2 + a_3u^3 + \dots + a_nu^n + \dots$

where a_n may be a function of k, in (58) then leads (see page B 48) to

$$c = k - u - 2ku^3/3! + 2u^4/4! - 24k^2u^5/5! + \dots \quad (59)$$

In a similar fashion, writing

$$y'(c)/y'(k) = 1 + \lambda \quad (60)$$

leads to the equation

$$(1+\lambda)ku + \frac{\partial\lambda}{\partial k} + (1-ku^2)\frac{\partial\lambda}{\partial u} = 0 \quad (61)$$

whence

$$y'(c) = y'(k)(1 - ku^2/2! + u^3/3! - 3k^2u^4/4! + 14ku^5/5! - \dots) \quad (62)$$

Series for c' and $y(c')/y(k')$ have been derived similarly, and are given on page B 48.

4·9. *Numerical Application of* (59) *and* (62). To find the zero of $y = \mathrm{Ai}(x) - \mathrm{Bi}(x)$ near $x = k = -0{\cdot}4$.

	c	$y'(c)/y'(k)$
$y(-0{\cdot}4) = +0{\cdot}02420\ 467$	$-0{\cdot}40000\ 000$	$+1{\cdot}00000\ 000$
$y'(-0{\cdot}4) = -0{\cdot}71276\ 627$	$+\ {\cdot}03395\ 877{,}6$	$+0{\cdot}00023\ 064{,}0$
$u(-0{\cdot}4) = -0{\cdot}03395\ 877{,}6$	$-\ 522{,}1$	$-\ 652{,}7$
	$+\ 11{,}1$	$-\ 2{,}7$
	$+\ 1$	$+\ 2$
	Sum $-0{\cdot}36604\ 633{,}3$	$+1{\cdot}00022\ 408{,}8$

Thus $c = -0{\cdot}36604\ 633(3)$ and $y'(c) = -0{\cdot}71292\ 599(2)$

From $k = -0{\cdot}3$ $c = -0{\cdot}36604\ 632(1)$ and $y'(c) = -0{\cdot}71292\ 600(2)$

5. *Preparation of the Tables*

5·1. *Computation of Pivotal Values.* Basic or pivotal values of all functions were first calculated to at least three more decimals than are given in the final tables. The interval between successive arguments was chosen so that intermediate values could be found to the same accuracy, using not more than 8 or 10 terms of an appropriate interpolation formula. The required intermediate values for the functions $\mathrm{Ai}(x)$ and $\mathrm{Ai}'(x)$ were then derived to 10 decimals by subtabulation, using the Association's National machine, and the values in Table I were obtained. Apart from $\mathrm{Ai}'(x)/\mathrm{Ai}(x)$ with $20 \leqslant x \leqslant 25$ and the auxiliary functions F, χ, G and ψ with $-30 \leqslant x \leqslant -10$, for which subtabulation from unit interval was found to be practicable, other functions were obtained directly without subtabulation.

The calculation of the pivotal values of the various functions led the writer to make various investigations of method; it seems desirable to indicate here only those which have been found most effective. The main method used for obtaining these pivotal values was step-by-step application of the Taylor expansions (23) and (24) with $\theta = \pm 1$, the appropriate differential equation being used to give τ^n from $f(x)$ and $f'(x)$. As a rule h was taken as 0·1, although $h = 0{\cdot}05$ was found to be desirable for large values of $-x$.

For $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ repeated differentiation of (2) gives

$$(n+1)(n+2)\,\tau^{n+2} = h^2(x\tau^n + h\tau^{n-1}) \qquad n > 0$$

$$2\tau^2 = h^2 xy$$

These were found to give a most effective and rapid method of calculation. Further, the method is self-checking. Each step gives $f(x \pm h)$ and $f'(x \pm h)$, one of each pair of values is new, the other a check reproduction of a previous value. The effect of accumulation of error is discussed in § 5·2; it was found to be negligible.

The series for $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ in ascending powers of x and the asymptotic expansions (see page B 17) were used to give additional check values as follows:

$$\mathrm{Ai}(x), \quad \text{for} \quad \pm x = 0{\cdot}1(0{\cdot}1)\ 1, 2, 5, 10, 20; \qquad \mathrm{Bi}(x), \quad \text{for} \quad x = 1, 2, 5, 10. \tag{63}$$

For $\mathrm{Ai}'(x)/\mathrm{Ai}(x)$ in Table II and for $\mathrm{Bi}'(x)/\mathrm{Bi}(x)$ in Table VI the appropriate differential equation is

$$z' = x - z^2 \tag{64}$$

where $z = y'/y$, y being a solution of (2). This equation was used for $x \leqslant 20$. The values of $\log_{10} \mathrm{Ai}(x)$ and of $\log_{10} \mathrm{Bi}(x)$ were then obtained by numerical integration of z, partly by use of the Taylor expansion (22) with $\theta = 1$, and partly by use of the formula

$$h\int_0^1 f(x+\theta h)\,d\theta = \tfrac{1}{2}h\{(f_0 + f_1) - \tfrac{1}{12}(\delta_0^2 + \delta_1^2) + \tfrac{11}{720}(\delta_0^4 + \delta_1^4) - \tfrac{191}{60480}(\delta_0^6 + \delta_1^6) + \ldots\} \tag{65}$$

which was applied, with $h = 0{\cdot}1$, in the form

$$\Delta \log_{10} \mathrm{Ai}(x) = Z_0 + Z_1 \qquad Z_r = \frac{M}{14400}(720 - 60\delta^2 + 11\delta^4 - 2{\cdot}27\delta^6 + \ldots)\,z_r \tag{66}$$

where $M = \log_{10} e$. For $x \geqslant 20$, asymptotic expansions for $\mathrm{Ai}'(x)/\mathrm{Ai}(x)$ and $\log_{10} \mathrm{Ai}(x)$ were used at unit interval in x. These expansions were derived from that for $\mathrm{Ai}(x)$, (page B 17).