FINITE DIFFERENCES

CHAPTER I

DEFINITIONS AND FUNDAMENTAL FORMULAE

1. The function $y=a+bx+cx^2+\dots+kx^n$ is a rational integral function of the nth degree in x, where the indices are positive integers, n being the greatest, and $a, b, c \dots k$ are constants, of which $k \neq 0$ but the others are unrestricted.

A rational integral function is also called a polynomial, and it is convenient to represent a polynomial of the nth degree as $P_n(x)$.

Consider the simple polynomial $y=u_x=1+x+x^2$. It is quite easy to obtain the value of y corresponding to any value of x by substituting that value of x on the right-hand side of the equation. For example

x	0	1	2	3	4	5	6	7	8
y	1	3	7	13	21	31	43	57	73

It will be found that for successive integral values of x in the above table the values of y have interesting properties. If from each value of y the previous value of y be subtracted, we obtain a new set of figures:

$$(\alpha) \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \quad 16$$

and if the subtraction be performed on these figures in the same way the new differences are

$$(\beta) \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2$$

The sequence of 2's in (β) is not a mere coincidence: it will be shown later that when y has the value supposed all the terms in (β) have the same value, 2, however far the series extends.

This leads us to another method of obtaining values of y. Suppose that we write down the original table in a different form, and include in the table the two sets of figures (α) and (β) thus:

x	y	(α)	(β)
0	1		
		2	
1	3		2
		4	
2	7		2
		6	
3	13		2
		8	
4	21		2
		10	
5	31		2
		12	
6	43		2
		14	
7	57		2
		16	
8	73		

We can now find any further value of y by extending the columns (α) and (β). We must however work from (β) to (α) and then to y instead of from y to (α) and then to (β) as has already been done. For example, to obtain the value of y when x has the value 9, i.e. to obtain u_9, a new 2 must be inserted in the (β) column: the new value in the (α) column will be $16+2=18$, and the required value of y will be $73+18=91$. To find u_{10} the process is continued. Any value of y corresponding to an integral value of x can be obtained in a similar manner.

2. The above is a particular instance of a far more general set of operations. We have used the simplest possible numerical values of x, namely the natural numbers, and we have evolved our example from a known quadratic function $y=u_x=1+x+x^2$. As a general rule the form of the function is not known and the given values of x are not necessarily consecutive integers.

3. Now suppose that instead of numerical values of x differing by unity we have the following consecutive values of x:

$$a,\ a+h,\ a+2h,\ a+3h,\ \ldots,$$

where the values of x differ by a quantity h instead of by unity.

Then if the function be still $y=u_x$ the values of y corresponding to the above values of x will be

$$u_a,\ u_{a+h},\ u_{a+2h},\ u_{a+3h},\ \ldots.$$

In order to form a column similar to column (α) above we shall have to write down

$$u_{a+h}-u_a,\ u_{a+2h}-u_{a+h},\ u_{a+3h}-u_{a+2h},\ \ldots.$$

These are called the *first differences* of the function $y=u_x$ and are denoted by

$$\Delta u_a,\ \Delta u_{a+h},\ \Delta u_{a+2h},\ \ldots,$$

where Δ is not a quantity but a symbol representing an "operation".

Column (β), being the differences of column (α), will be

$$(u_{a+2h}-u_{a+h})-(u_{a+h}-u_a),$$
$$(u_{a+3h}-u_{a+2h})-(u_{a+2h}-u_{a+h}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

or, more shortly,

$$\Delta u_{a+h}-\Delta u_a,$$
$$\Delta u_{a+2h}-\Delta u_{a+h},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

These are called the *second differences* of u_x and are denoted by

$$\Delta^2 u_a,\ \Delta^2 u_{a+h},\ \Delta^2 u_{a+2h}\ \ldots,$$

where, it must be emphasized, the symbol Δ^2 does not represent the square of a quantity but denotes the repetition of the operation Δ.

Similarly, third, fourth, ... nth differences, formed in exactly the same way, are denoted by

$$\Delta^3 u_a,\ \Delta^4 u_a\ \ldots\ \Delta^n u_a.$$

4. Before forming a *difference table* similar to that in paragraph 1, it is convenient to introduce alternative names for x and y in our equation $y=u_x$. Where our ultimate object is to obtain numerical values of x or y, the independent variable is often termed the *argument*, and the corresponding value of y the *entry*.

In a table of logarithms the number itself is the argument and the logarithm the entry. The converse holds in a table of antilogarithms, where the logarithm is the argument. Similarly in a table of sin α, α is the argument and the sine the entry, whereas α is the entry in a table of $\sin^{-1}\alpha$.

5. Our new difference table is therefore

Argument	Entry	First differences	Second differences	Third differences
a	u_a			
		Δu_a		
$a+h$	u_{a+h}		$\Delta^2 u_a$	
		Δu_{a+h}		$\Delta^3 u_a$
$a+2h$	u_{a+2h}		$\Delta^2 u_{a+h}$	
		Δu_{a+2h}		$\Delta^3 u_{a+h}$
$a+3h$	u_{a+3h}		$\Delta^2 u_{a+2h}$	
		Δu_{a+3h}		$\Delta^3 u_{a+2h}$
$a+4h$	u_{a+4h}		$\Delta^2 u_{a+3h}$	
		Δu_{a+4h}		
$a+5h$	u_{a+5h}			

The first term in the table (u_a) is called the *leading term*, and the differences which stand at the head of the respective columns, namely Δu_a, $\Delta^2 u_a$, $\Delta^3 u_a$ ..., are called the *leading differences*.

6. Although we have expressed the terms in the difference table by the use of Δ symbols, it is quite easy to obtain any difference in terms of the function alone.

For example, $\Delta^3 u_a$ is the difference between $\Delta^2 u_{a+h}$ and $\Delta^2 u_a$, or $\Delta^3 u_a = \Delta^2 u_{a+h} - \Delta^2 u_a$.

Again, $\Delta^2 u_a$ is the difference between Δu_{a+h} and Δu_a, or

$$\Delta^2 u_a = \Delta u_{a+h} - \Delta u_a,$$

and as $\Delta u_a = u_{a+h} - u_a,$

we have

$$\begin{aligned}\Delta^3 u_a &= \Delta^2 u_{a+h} - \Delta^2 u_a \\ &= (\Delta u_{a+2h} - \Delta u_{a+h}) - (\Delta u_{a+h} - \Delta u_a) \\ &= \Delta u_{a+2h} - 2\Delta u_{a+h} + \Delta u_a \\ &= (u_{a+3h} - u_{a+2h}) - 2(u_{a+2h} - u_{a+h}) + (u_{a+h} - u_a) \\ &= u_{a+3h} - 3u_{a+2h} + 3u_{a+h} - u_a.\end{aligned}$$

7. It is a simple matter to construct a difference table from a given set of data.

Consider the following examples:

Example 1.

Construct a difference table from the following values, where y is a function of x:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		6
5	125		30	
		91		6
6	216		36	
		127		
7	343			

Example 2.

Show that, in the following table of annuity-values, third differences are practically constant:

Argument x	Entry a_x	Δa_x	$\Delta^2 a_x$	$\Delta^3 a_x$
35	14·298			
		−·154		
36	14·144		−·004	
		−·158		+·001
37	13·986		−·003	
		−·161		·000
38	13·825		−·003	
		−·164		+·001
39	13·661		−·002	
		−·166		+·001
40	13·495		−·001	
		−·167		
41	13·328			

It will be observed that in Ex. 1 third differences are invariably the same. In the second example, however, third differences are not quite constant, although the error in assuming them to be so is very small.

The difference in the two examples lies in the fact that, while the first function is $y=x^3$, the table of annuity-values from which the data in the second example have been taken does not conform to a simple mathematical law and, further, the values do not naturally terminate with the third decimal place, but are rounded off at that place.

Example 3.

Assuming third differences constant, find the values of u_2 and u_3 from the data:

x	4	5	6	7	8
u_x	·35	·88	1·71	2·90	4·51

Construct the difference table from the given values, and fill in the vacant spaces in the $\Delta^3 u_x$ column with the constant third difference, thus:

x	u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$
2	− ·05			
		·11		
3	+ ·06		·18	
		·29		·06
4	+ ·35		·24	
		·53		·06
5	+ ·88		·30	
		·83		·06
6	+ 1·71		·36	
		1·19		·06
7	+ 2·90		·42	
		1·61		
8	+ 4·51			

8. Now it has been stated above that a convenient method for expressing the difference between two successive values of a function u_{a+h} and u_a is by the symbol Δ prefixed to u_a, so that $\Delta u_a = u_{a+h} - u_a$. It will be seen therefore that to find Δu_a we perform two operations: we change u_a to u_{a+h} and subtract u_a from it. The new function u_{a+h} resulting from the first of these operations is denoted symbolically by Eu_a, and the double operation may be written

$$\Delta u_a = Eu_a - u_a.$$

This gives

$$Eu_a = u_a + \Delta u_a.$$

Eu_a may therefore otherwise be expressed as the sum of u_a and its first difference.

Suppose that in either of the above relations the u_a which occurs in each of the terms be omitted. Then we can state that the two operations denoted by "E" and "Δ" are connected by the symbolic identity

$$E \equiv 1 + \Delta.$$

It must be distinctly understood that we have not "factorized out" u_a in the relation $Eu_a = u_a + \Delta u_a$, and that we must relate the symbols to the functions on which they operate. If, therefore, we were using the equivalence $\Delta \equiv E - 1$, and we operated on the function $\sin x$, it would be wrong to say that $\Delta \sin x = E \sin x - 1$. The correct statement is $\Delta \sin x = E \sin x - \sin x$. When we are dealing with symbols of operation we cannot treat any of them as quantities, and on forming the algebraic or trigonometrical identity the function must be included in all three terms. In other words, in the identity $E \equiv 1 + \Delta$ the 1 is a symbol of operation just as are E and Δ, and its meaning is that the function on which it operates is to be taken once without alteration.

9. In the same way as Δ^2 denotes, when operating on a function, the difference of the difference of the function, i.e. the second difference, so E^2 denotes the operation of repeating E. That is to say

$$E^2 u_x = E.Eu_x = Eu_{x+h} = u_{x+2h},$$

$$E^3 u_x = u_{x+3h},$$

$$\dots\dots\dots\dots\dots\dots$$

and, generally, $$E^n u_x = u_{x+nh}.$$

Care must be taken not to confuse the expression $E^2 u_x$ with $(Eu_x)^2$. For example,

$$E^2(x^2) = (x+2h)^2 = x^2 + 4hx + 4h^2,$$

but $$(Ex)^2 = (x+h)^2 = x^2 + 2hx + h^2.$$

10. It is evident that the first difference of a function of the form cx, where c is a constant, is constant: for $\Delta cx = c(x+h) - cx = ch$, which is constant.

Let us consider the effect of differencing a function of x of higher degree than the first.

Example 4.

Difference successively the functions (i) $y=bx^2$ and (ii) $y=ax^3$.

(i) $\Delta bx^2=b\,(x+h)^2-bx^2=2bhx+bh^2$,

$$\Delta^2 bx^2=\Delta\,(2bhx+bh^2)=2bh\,(x+h)+bh^2-2bhx-bh^2=2bh^2,$$

and since $2bh^2$ is constant all higher differences will be zero.

(ii) $\Delta ax^3=a\,(x+h)^3-ax^3=3ahx^2+3ah^2x+ah^3$,

$$\Delta^2 ax^3=6ah^2x+6ah^3,$$

and $\Delta^3 ax^3=6ah^3$, higher differences vanishing.

Collating the above results, we have that

the first differences of functions of the form				cx	are constant,
the second	,,	,,	,,	bx^2	,,
the third	,,	,,	,,	ax^3	,,

It follows therefore that third differences of ax^3+bx^2+cx+d are constant, for before we reach the third differences the terms bx^2, cx and d will have been eliminated.

11. The above considerations lead us to the following important proposition:

If u_x be a polynomial of the nth degree in x, then the nth difference of the function is constant.

Let the function be

$$u_x=ax^n+bx^{n-1}+cx^{n-2}+\ldots+s;$$

then
$$\Delta u_x=a\,(x+h)^n+b\,(x+h)^{n-1}+c\,(x+h)^{n-2}+\ldots+s$$
$$-ax^n-bx^{n-1}-cx^{n-2}-\ldots-s$$
$$=anx^{n-1}h+b'x^{n-2}+c'x^{n-3}+\ldots+r',$$

where b', c', ... r' are coefficients involving h but not x.

Similarly,

$$\Delta^2 u_x=an\,(n-1)\,x^{n-2}h^2+b''x^{n-3}+c''x^{n-4}+\ldots+q'',$$

and so on.

Each time that we difference we lower the degree of the function by unity. After differencing n times no terms after the first will appear, and we shall be left with

$$\Delta^n u_x = an(n-1)(n-2)(n-3)\ldots 2.1.h^n \text{ or } an!\,h^n,$$

which is independent of x and is therefore constant.

As a corollary we may note that $\Delta^{n+1}u_x = 0$, a property of a polynomial of the nth degree which is of value in the practical application of the work.

The converse proposition is of importance: if the $(n+1)$th difference of a function is the first to become zero, the function is a polynomial of not more than the nth degree.

12. It should be remembered that we are dealing here with a particular form of function. Should the function be other than a polynomial the nth difference will not vanish however great n may be. Thus, we have

Example 5.

Find the nth difference of e^x.

$$\Delta e^x = e^{x+h} - e^x = e^x(e^h - 1),$$

$$\Delta^2 e^x = (e^h - 1)(e^{x+h} - e^x) = e^x(e^h - 1)^2.$$

Similarly, $\Delta^3 e^x = e^x(e^h - 1)^3,$

....

Generally, $\Delta^n e^x = e^x(e^h - 1)^n$, which is still a function of x, and is therefore not constant.

13. Although it has been said that the symbols Δ and E are in no sense algebraic quantities, our definitions, namely that Δ^n denotes the operation of differencing the function n times, and that E^n denotes the operation of obtaining a new function when the argument is increased by n unit differences, enable us to apply to these symbols the ordinary algebraic laws. For example,

$$\Delta(u_x + u_y) = u_{x+h} + u_{y+h} - u_x - u_y \quad \text{or} \quad u_{x+h} - u_x + u_{y+h} - u_y,$$

which is $\Delta u_x + \Delta u_y$. This relation is exactly similar to the ordinary algebraic identity $3(x+y) = 3x + 3y$.

The three simple algebraic laws are the laws of (i) distribution, (ii) commutation, (iii) indices.

(i) $\Delta\,(u_x+v_x+w_x+\ldots)$

$$=(u_{x+h}+v_{x+h}+w_{x+h}+\ldots)-(u_x+v_x+w_x+\ldots)$$
$$=(u_{x+h}-u_x)+(v_{x+h}-v_x)+(w_{x+h}-w_x)$$
$$=\Delta u_x+\Delta v_x+\Delta w_x+\ldots.$$

Similarly,

$$E\,(u_x+v_x+w_x+\ldots)=Eu_x+Ev_x+Ew_x+\ldots.$$

(ii) The symbols Δ and E are commutative in their operation as regards constants. For if c be a constant,

$$\Delta cu_x=cu_{x+h}-cu_x=c\,(u_{x+h}-u_x)=c\Delta u_x,$$

and $$Ecu_x=cu_{x+h}=cEu_x.$$

(iii) The application of indices to the symbols Δ and E may be shown thus:

If m be a positive integer, then Δ^m represents the operation of differencing u_x m times.

$$\Delta^m u_x=(\Delta\Delta\Delta\Delta\ \ldots\ m \text{ times})\ u_x,$$
$$\Delta^n\,(\Delta^m u_x)=(\Delta\Delta\Delta\Delta\ \ldots\ n \text{ times})\ (\Delta\Delta\Delta\Delta\ \ldots\ m \text{ times})\ u_x$$
$$=(\Delta\Delta\Delta\Delta\ \ldots\ \overline{m+n} \text{ times})\ u_x$$
$$=\Delta^{m+n}u_x.$$

Similarly, $$E^m u_x=u_{x+mh},$$
$$E^n\,(E^m u_x)=E^n u_{x+mh}=u_{x+mh+nh}=E^{m+n}u_x.$$

14. In connection with the law of indices we must be careful to define Δ^m, Δ^n, E^m, ... when m and n are not positive integers. So far, the symbols Δ^m and E^m are intelligible only when we can actually perform the operations defined above and obtain the values of the new functions. We have not yet defined these symbols when the indices are negative. Consider for example the symbol Δ^{-1}. Since we have assumed that the symbol Δ obeys the ordinary algebraic laws, Δ^{-1} must be such that $\Delta\,(\Delta^{-1}u_x)$ gives $\Delta^0 u_x$, i.e. u_x.

Let m be a positive integer. Then we define $\Delta^{-m}u_x$ as a function such that if it be operated on by Δ^m the result will be $\Delta^{m-m}u_x$,