CHAPTER I

ELEMENTARY TRIGONOMETRY

1. A knowledge of trigonometrical functions is essential for the proper understanding of various formulae of the differential and integral calculus. The present chapter is therefore devoted to the development of the elementary functions and their properties. The account is short, and for the purpose of studying the functions generally recourse should be had to a recognized textbook. The chapter has been included only with the object of enabling those who have not studied trigonometry to obtain sufficient knowledge to follow the remainder of the book.

2. Definitions.

Consider a straight line X_1OX of indefinite length fixed in a plane. At a point O in the straight line is hinged another straight line OA, also of indefinite length, capable of being revolved about the hinge at O, but only in an anti-clockwise direction. Then, as OA revolves, it sweeps out an angle XOA.

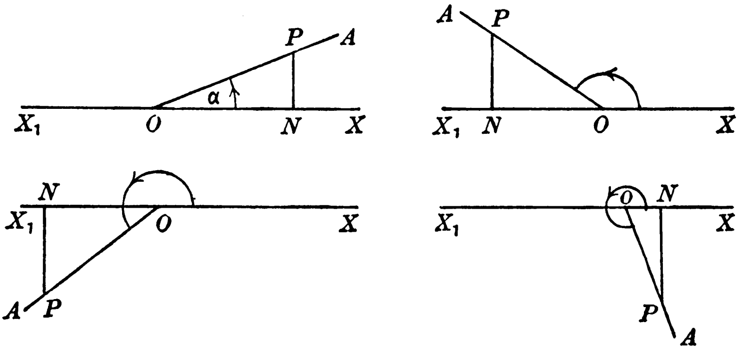


Fig. 1.

Take any point P on the moving line OA and drop a perpendicular PN on to the fixed line X_1OX. Then, by the properties of similar triangles, the ratios between the sides of the right-angled triangle PON will be the same for all positions of P, for any one position of the line OA. These constant ratios are the trigonometrical ratios of the angle XOA, and are defined thus, where α stands for the angle XOA:

$\frac{NP}{OP}$ is the sine of the angle XOA and is written $\sin\alpha$,

$\frac{ON}{OP}$ „ cosine „ „ XOA „ „ $\cos\alpha$,

$\frac{NP}{ON}$ „ tangent „ „ XOA „ „ $\tan\alpha$.

These are the principal ratios, and most trigonometrical problems can be solved by the use of these three ratios only. It is often convenient, however, to use the reciprocals of the ratios: the respective reciprocals are

$\frac{OP}{NP}$, the cosecant of the angle XOA, written as $\operatorname{cosec}\alpha$,

$\frac{OP}{ON}$, the secant „ „ XOA, „ „ $\sec\alpha$,

$\frac{ON}{NP}$, the cotangent „ „ XOA, „ „ $\cot\alpha$.

3. It is important to note that, even if the two triangles PON in the first two diagrams of Fig. 1 are geometrically equal, it does not follow that the trigonometrical ratios of the two angles given by the positions of OP are the same. In elementary plane geometry, the straight line joining any two points L and M may be indifferently denoted by LM or ML. On the other hand, the straight lines which enter into the definitions of the trigonometrical ratios have *sign* as well as magnitude, and the direction of the straight line determines the sign. To ascertain the correct sign to be given to a straight line, we proceed in the following manner. Imagine the plane in which the fixed line has been drawn to be divided into four sections by the straight line X_1OX and a straight line YOY_1 through O perpendicular to X_1OX.

If OP be any position of the revolving line, we can arrive at the point P from O either by proceeding along OP or by the double journey ON, NP. In order to develop a logical system we must adopt a convention based on the direction to be taken to arrive at

P from O, and on the particular quadrant in which the point P lies. The convention is that lines drawn from O in the directions OX or OY are positive and that those drawn from O in the directions

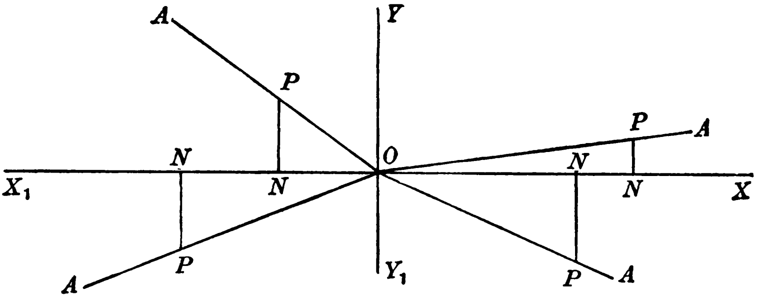


Fig. 2.

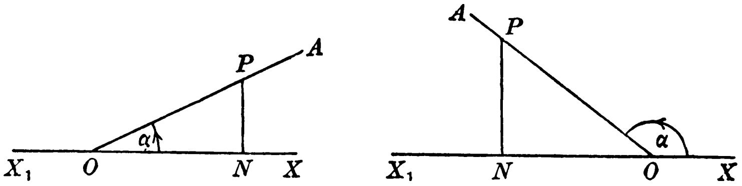
OX_1 and OY_1 are negative. The line OA is called the radius and is always to be considered as positive. The perpendicular line NP is to be regarded as drawn in the direction $N \rightarrow P$, i.e. from the line OX to the radius OA, and not from P to N.

In Fig. 2, therefore, we have

1. Quadrant XOY: ON positive and NP positive;
2. ,, X_1OY: ON negative and NP positive;
3. ,, X_1OY_1: ON negative and NP negative;
4. ,, XOY_1: ON positive and NP negative.

These quadrants are called the first, second, third and fourth quadrants respectively.

4. It is evident that the trigonometrical ratios, being derived from the ratios between ON, NP, OP, will have sign as well as magni-



(i) (ii)

Fig. 3.

tude. For example, for the angle α in Fig. 3 (i) all the sides of the triangle ONP are positive in direction and as a result all the trigonometrical ratios of the angle will be positive.

On the other hand, in Fig. 3 (ii) we shall have

$\sin\alpha = NP/OP$ (positive)/(positive) i.e. *positive*,
$\cos\alpha = ON/OP$ (negative)/(positive) i.e. *negative*,
$\tan\alpha = NP/ON$ (positive)/(negative) i.e. *negative*,

and similarly for the reciprocal ratios.

5. Negative angles.

If the revolving line be constrained to move in a *clockwise* direction, it is said to trace out a negative angle. For example, let the straight line OA_1 take up the position indicated, not by a

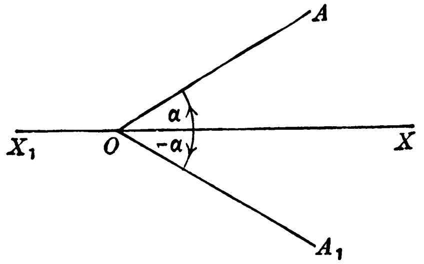


Fig. 4.

revolution passing first through the position OA, but by passing in the opposite direction direct to OA_1; then the angle XOA_1 is a negative angle.

In the figure the angle XOA is α, and the angle XOA_1 is $-\alpha$.

6. Relations between the ratios.

From the definitions of the ratios we have at once

$$\tan\alpha = NP/ON = \frac{\dfrac{NP}{OP}}{\dfrac{ON}{OP}} = \frac{\sin\alpha}{\cos\alpha} \qquad \text{......(i).}$$

Similarly,
$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} \qquad \text{......(ii).}$$

Again, from any of the diagrams in Fig. 1,

$$NP^2 + ON^2 = OP^2.$$

$$\therefore\ (NP/OP)^2 + (ON/OP)^2 = 1,$$

i.e.
$$(\sin\alpha)^2 + (\cos\alpha)^2 = 1.$$

A more convenient method of writing $(\sin \alpha)^2$, $(\cos \alpha)^2$, etc. is by omitting the brackets and denoting the squares of the ratios by $\sin^2 \alpha$, $\cos^2 \alpha$, etc. The above relation is therefore

$$\sin^2 \alpha + \cos^2 \alpha = 1 \qquad \text{......(iii).}$$

Similarly, by dividing both sides of the identity

$$ON^2 + NP^2 = OP^2$$

by ON^2 we shall have

$$1 + (NP/ON)^2 = (OP/ON)^2$$

or

$$1 + \tan^2 \alpha = \sec^2 \alpha \qquad \text{......(iv).}$$

Again, dividing by NP^2, we shall obtain

$$1 + \cot^2 \alpha = \operatorname{cosec}^2 \alpha \qquad \text{......(v).}$$

7. Identities.

Just as algebraic identities can be proved by the application of various fundamental rules, so the relations between the trigonometrical ratios can be applied to the proof of trigonometrical identities.

Example 1.

Prove that $\tan \alpha + \cot \alpha = \sec \alpha \operatorname{cosec} \alpha$.

$$\begin{aligned} \tan \alpha + \cot \alpha &= \sin \alpha/\cos \alpha + \cos \alpha/\sin \alpha \\ &= (\sin^2 \alpha + \cos^2 \alpha)/\cos \alpha \sin \alpha \\ &= 1/\cos \alpha \sin \alpha \\ &= (1/\cos \alpha)(1/\sin \alpha) \\ &= \sec \alpha \operatorname{cosec} \alpha. \end{aligned}$$

Example 2.

Prove that $\sec^2 \alpha - \operatorname{cosec}^2 \alpha = \tan^2 \alpha - \cot^2 \alpha$.

$$\begin{aligned} \sec^2 \alpha - \operatorname{cosec}^2 \alpha &= (\tan^2 \alpha + 1) - (\cot^2 \alpha + 1) \\ &= \tan^2 \alpha - \cot^2 \alpha. \end{aligned}$$

Example 3.

Prove that $$\frac{\tan \alpha - \tan \beta}{\cot \alpha - \cot \beta} + \tan \alpha \tan \beta = 0.$$

Multiply through by $\cot \alpha - \cot \beta$ and the expression becomes

$$\tan \alpha - \tan \beta + \cot \alpha \tan \alpha \tan \beta - \tan \alpha \tan \beta \cot \beta$$

or $\tan \alpha - \tan \beta + \tan \beta - \tan \alpha$, which is zero.

Alternatively:

$$\cot\alpha - \cot\beta = \frac{1}{\tan\alpha} - \frac{1}{\tan\beta} = -\frac{\tan\alpha - \tan\beta}{\tan\alpha\tan\beta}.$$

$$\therefore \frac{\tan\alpha - \tan\beta}{\cot\alpha - \cot\beta} = -\tan\alpha\tan\beta.$$

8. Magnitude of angles. Degrees.

The unit angle in elementary geometry is the *degree*. An angle of x degrees is denoted by x°. The degree is defined as the angle subtended at the centre of a circle by an arc equal in length to 1/360 of the circumference. For arithmetical calculation the degree is a convenient unit, and we can obtain the values of the trigonometrical ratios of many angles by reference to simple geometrical figures.

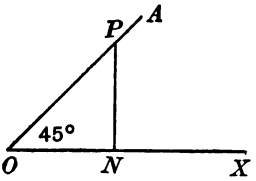
Example 4.

Find the sine, cosine and tangent of (i) 45°, (ii) 30°, (iii) 60°.

(i) Let ONP be an isosceles triangle, right-angled at N, so that $ON = NP$. Then, if ON be of unit length,

$$OP^2 = ON^2 + NP^2 = 1 + 1 = 2,$$

so that $OP = \sqrt{2}$.



Therefore, easily,

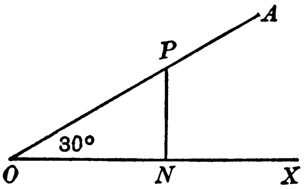
$$\sin 45^\circ = 1/\sqrt{2},\quad \cos 45^\circ = 1/\sqrt{2},\quad \tan 45^\circ = 1.$$

(ii) Take the angle XOA to be 30°. Then the angle NPO is 60° and the figure ONP is one-half of an equilateral triangle of side equal in length to OP.

If, therefore, NP be of unit length, $OP = 2$ and $ON = \sqrt{3}$, so that

$$\sin 30^\circ = \tfrac{1}{2},\quad \cos 30^\circ = \sqrt{3}/2,$$

$$\tan 30^\circ = 1/\sqrt{3}.$$



(iii) From a consideration of the above figure it is evident that

$$\sin 60^\circ = \sqrt{3}/2,\quad \cos 60^\circ = \tfrac{1}{2},\quad \tan 60^\circ = \sqrt{3}.$$

9. Magnitude of angles. Radians.

A more convenient unit for analytical purposes is the angle subtended at the centre of a circle by an arc equal in length to the radius: this angle is called a *radian*. Since the ratio between the angle at the centre and the arc on which it stands is constant for all circles, it follows that the radian is the same whatever the radius of the circle: the radian may therefore be taken as a unit of measurement.

To obtain the number of radians corresponding to the number of degrees in an angle, all that is necessary is to multiply the number of degrees by $\frac{\pi}{180}$. This is easily seen to be so, for if x be the number of degrees corresponding to a radian, we have

$$\frac{\text{angle subtended by the arc equal in length to the radius}}{\text{angle subtended by half the circumference}} = \frac{\text{radian}}{180^\circ}$$

i.e. $r/\pi r = x/180,$

so that $x = 180/\pi$, or π radians $= 180°$.

In applying the calculus to trigonometrical functions it is essential that angles should be expressed in terms of an absolute unit of measurement. Consequently, in all the work that follows, unless otherwise stated, angles must be taken to be measured in radians.

10. Periodicity of the trigonometrical ratios.

If we consider the definitions of the ratios, taking into account the signs as well as the magnitudes, it can easily be shown that there will be more than one angle having the same particular ratio. To take a simple example: in the following figure, let the radius

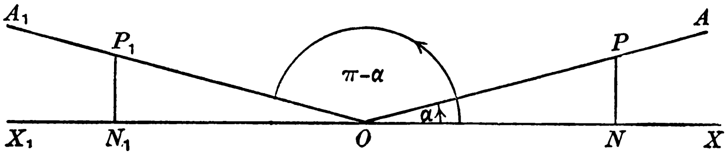


Fig. 5.

take up the positions OA and OA_1, where the angle XOA is the angle α and the angle XOA_1 is the supplement of XOA, i.e. $\pi - \alpha$.

Then, attending to the directions of the lines involved, we shall have

$$\sin\alpha = NP/OP = N_1P_1/OP_1 = +\sin(\pi-\alpha),$$

$$\cos\alpha = ON/OP = -ON_1/OP_1 = -\cos(\pi-\alpha),$$

and $$\tan\alpha = NP/ON = N_1P_1/-ON_1 = -\tan(\pi-\alpha).$$

Again, from the figure it can be shown similarly that

$$\sin \alpha = -\sin(\pi + \alpha) = -\sin(2\pi - \alpha),$$
$$\cos \alpha = -\cos(\pi + \alpha) = +\cos(2\pi - \alpha),$$
$$\tan \alpha = +\tan(\pi + \alpha) = -\tan(2\pi - \alpha).$$

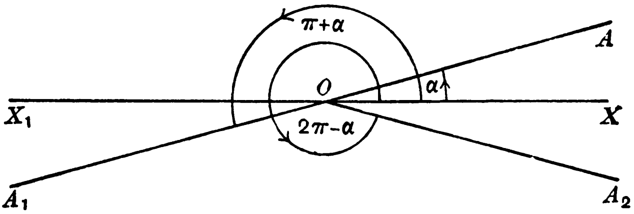


Fig. 6.

If now the radius make a complete revolution, so that, starting from the position OX it takes up the position OA after first tracing out the angle 2π, then it is evident that $\sin \alpha = \sin(2\pi + \alpha)$; $\cos \alpha = \cos(2\pi + \alpha)$; $\tan \alpha = \tan(2\pi + \alpha)$.

We have, therefore, that

$$\sin \alpha = \sin(\pi - \alpha) = \sin(2\pi + \alpha) = \sin(3\pi - \alpha) = \ldots\ldots,$$
$$\cos \alpha = \cos(2\pi - \alpha) = \cos(2\pi + \alpha) = \cos(4\pi - \alpha) = \ldots\ldots,$$
$$\tan \alpha = \tan(\pi + \alpha) = \tan(2\pi + \alpha) = \tan(3\pi + \alpha) = \ldots\ldots.$$

These relations may be generalised in the forms:

all angles having the same sine as α are the values of $n\pi + (-1)^n \alpha$,
,, ,, cosine ,, ,, $2n\pi \pm \alpha$,
,, ,, tangent ,, ,, $n\pi + \alpha$,

where n is a positive integer.

For example, it has been proved above that $\sin 30° = \frac{1}{2}$. In absolute measure this is $\sin \pi/6 = \frac{1}{2}$, so that all angles whose sine is $\frac{1}{2}$ are the successive values of $\{n\pi + (-1)^n \pi/6\}$, i.e. $\pi/6$, $5\pi/6$, $13\pi/6$, $17\pi/6$, and so on.

It will be seen that if we replace n by $2m$, so that only even values of the positive integers are taken into account, the general angle for $\sin \alpha$ is $2m\pi + \alpha$; this brings the property of the sine into line with those of the other ratios. For all the trigonometrical functions, therefore, we may say that

$$f(x + 2m\pi) = f(x).$$

If a function have this property it is said to be a *periodic function* with period 2π.

A graphical representation (Fig. 7) shows quite clearly the periodic property of the sine, cosine and tangent.

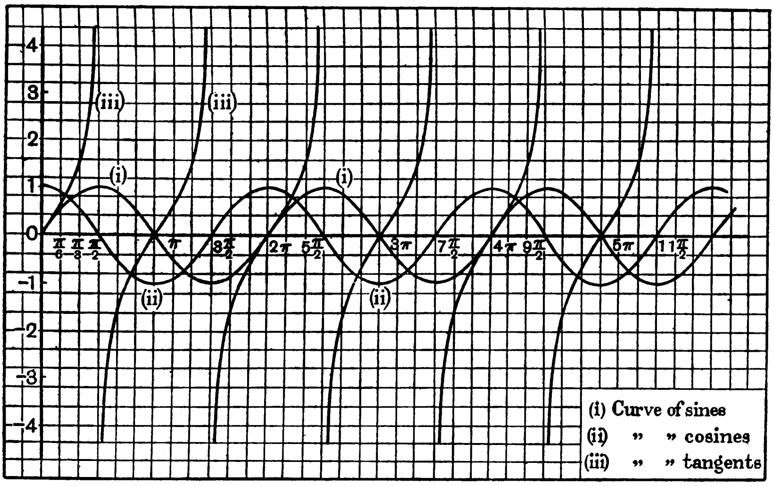


Fig. 7.

Notes: (i) The tangent and cotangent are periodic with period π.

(ii) It can be shown quite easily by the consideration of a diagram similar to Fig. 6 that the generalised forms hold equally for negative integral values of n.

11. Ratios of ($\frac{1}{2}\pi \pm \alpha$).

(i) If in Fig. 8 the angles XOP and XOP_1 are α and $\frac{1}{2}\pi - \alpha$ respectively, and we make $OP_1 = OP$, then by considering the geometry of the two triangles ONP and ON_1P_1 it is easily seen that

$$ON/OP = N_1P_1/OP_1,$$
$$NP/OP = ON_1/OP_1,$$
$$NP/ON = ON_1/N_1P_1;$$

so that

$$\cos\alpha = \sin(\tfrac{1}{2}\pi - \alpha),$$
$$\sin\alpha = \cos(\tfrac{1}{2}\pi - \alpha),$$
$$\tan\alpha = \cot(\tfrac{1}{2}\pi - \alpha).$$

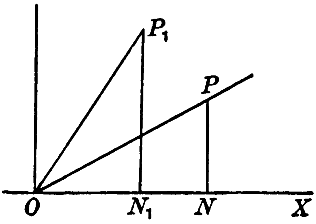


Fig. 8.

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Similarly, $\cot \alpha = \tan(\frac{1}{2}\pi - \alpha)$ and $\operatorname{cosec} \alpha = \sec(\frac{1}{2}\pi - \alpha)$.

The angles α and $\frac{1}{2}\pi - \alpha$ are called *complementary* angles, the "co" in cosine, cotangent and cosecant corresponding to the *com*plementary angle.

(ii) If the angle $XOP_1 = \frac{1}{2}\pi + \alpha$ as in Fig. 9

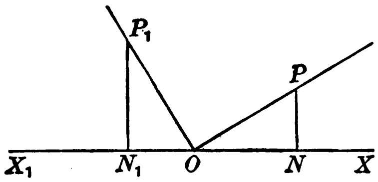

Fig. 9.

we shall have $ON/OP = N_1P_1/OP_1$, $\therefore \cos \alpha = \sin(\frac{1}{2}\pi + \alpha)$;

also $NP/OP = -ON_1/OP_1$, $\therefore \sin \alpha = -\cos(\frac{1}{2}\pi + \alpha)$;

and $\tan \alpha = -\cot(\frac{1}{2}\pi + \alpha)$.

(iii) When the angle XOP is so small that ON and OP coincide,

$$\sin 0 = 0, \quad \cos 0 = 1, \quad \tan 0 = 0;$$

and from the above

$$\sin \tfrac{1}{2}\pi = \cos 0 = 1, \quad \cos \tfrac{1}{2}\pi = \sin 0 = 0, \quad \tan \tfrac{1}{2}\pi = \infty.$$

12. Inverse functions.

From the identity $\sin \pi/6 = \frac{1}{2}$ we can obtain the inverse relation, namely, that $\pi/6$ is the angle whose sine is $\frac{1}{2}$. The notation adopted for this is

$$\sin^{-1} \tfrac{1}{2} = \pi/6.$$

This inverse notation is not to be confused with the algebraic notation for negative indices. Although a^{-1} is equivalent to $1/a$, $\sin^{-1} x$ is not $1/\sin x$, but the angle whose sine is x. We have generally from the above, that if $\sin \alpha = x$, then

$$\sin^{-1} x = n\pi + (-1)^n \alpha.$$

As a general rule it is convenient to take the inverse function as the numerically smallest angle (with the proper sign) giving the required value of the direct function.