CAMBRIDGE

Cambridge University Press
978-1-316-60695-7 - Differential Geometry of Three Dimensions: Volume II
C. E. Weatherburn
Frontmatter
More information

DIFFERENTIAL GEOMETRY

OF THREE DIMENSIONS

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-316-60695-7 - Differential Geometry of Three Dimensions: Volume II
C. E. Weatherburn
Frontmatter
More information

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-316-60695-7 - Differential Geometry of Three Dimensions: Volume II
C. E. Weatherburn
Frontmatter
More information

DIFFERENTIAL GEOMETRY
OF THREE DIMENSIONS

By

C. E. WEATHERBURN, M.A., D.Sc.
PROFESSOR OF MATHEMATICS IN THE
UNIVERSITY OF WESTERN AUSTRALIA

VOLUME II

CAMBRIDGE
AT THE UNIVERSITY PRESS
1930

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-316-60695-7 - Differential Geometry of Three Dimensions: Volume II
C. E. Weatherburn
Frontmatter
More information

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781316606957

© Cambridge University Press 1930

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 1930
First paperback edition 2016

A catalogue record for this publication is available from the British Library

ISBN 978-1-316-60695-7 Paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-316-60695-7 - Differential Geometry of Three Dimensions: Volume II
C. E. Weatherburn
Frontmatter
More information

PREFACE

THE following pages continue the discussion of the subject along lines which are a natural extension of those followed in our earlier volume. Most of the chapters are based on recent papers by the author; but the scope of the book is widened by the treatment of other topics also. Chapter XII, for instance, contains a brief discussion of Flexion and Applicability of Surfaces, giving an account of some of the theorems that have already become classical; while Chapter IV supplements the treatment of Ruled Surfaces and W-surfaces contained in the first volume, and Chapter IX gives a simple exposition of Levi-Civita's theory of "parallel displacements" on a surface, with some applications to Tchebychef Systems of curves. Chapter XIII represents an extension of the theory of Curvilinear Congruences given in two recent papers by the author.

I have to thank many distant correspondents for their written appreciation of the preceding volume. I hope this one will prove equally helpful.

C. E. W.

THE UNIVERSITY,
PERTH, W.A.
May, 1929.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-316-60695-7 - Differential Geometry of Three Dimensions: Volume II
C. E. Weatherburn
Frontmatter
More information

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-316-60695-7 - Differential Geometry of Three Dimensions: Volume II
C. E. Weatherburn
Frontmatter
More information

CONTENTS

PAGE

PREFACE v

CHAPTER I

DIFFERENTIAL INVARIANTS FOR A SURFACE

ART.

1. Gradient of a scalar function. Derivatives 1
2. Divergence and rotation of a vector 2
3. Formulae of expansion 3
4. Differential invariants of the second order 5
5. Order of directional differentiations 6
6. Derivatives of the unit vectors **a**, **b**, **n** 7
7. Other differential invariants 8

EXAMPLES I 10

CHAPTER II

FAMILIES OF CURVES ON A SURFACE

8. Curvature properties 13
9. Rate of rotation of the trihedral 14
10. Moment of the family. Line of zero moment 15
11. Line of striction. Divergence of the family 16
12. Parallels and geodesics 17
13. Orthogonal systems of curves 19
14. Isometric orthogonal systems 21
15. Oblique trajectories of an isometric system 23
16. Alternative form of the condition for isometry 24
17. Family of parallel geodesics 24
18. Flux and circulation 25

EXAMPLES II 28

CHAPTER III

FAMILIES OF CURVES (*continued*).

OBLIQUE TRAJECTORIES

19. Two differential invariants of a scalar function 30
20. Related invariants of a vector function 31

FAMILY OF CURVES ($\phi = \text{const.}$)

21. Distance function. Line of striction. Moment 32
22. Normal curvature. Conjugate direction 34
23. Orthogonal trajectories of the family 36

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-316-60695-7 - Differential Geometry of Three Dimensions: Volume II
C. E. Weatherburn
Frontmatter
More information

OBLIQUE TRAJECTORIES

ART. PAGE

24. Line of striction. Parallels 37
25. Geodesic curvature. Various theorems 38
26. Moment of family. Lines of curvature 40
27. Normal curvature. Conjugate direction 41
EXAMPLES III 42

CHAPTER IV

RULED SURFACES. WEINGARTEN SURFACES

RULED SURFACES

28. Some general observations 43
29. Choice of coordinates. Curvatures 44
30. Explicit expressions 46
31. Geometrical illustration. Divergence of generators . . . 48
32. Isometric systems on a skew surface 49

WEINGARTEN SURFACES

33. Ruled Weingarten surfaces 50
34. Weingarten surfaces in general 52
35. Differential equation satisfied by the unit normal 54
36. Lines of constant curvature 55
37. Surface of constant first curvature 55
38. Surfaces of revolution 57
EXAMPLES IV. 58

CHAPTER V

CURVILINEAR COORDINATES IN SPACE. DIFFERENTIAL INVARIANTS

39. Notation. Fundamental magnitudes 62
40. Unit normals to the parametric surfaces 64
41. Gradient of a scalar function 65
42. The operator ∇ 67
43. Divergence and rotation of a vector 68
44. Formulae of expansion 70
45. Differential invariants of the second order 71
46. Orthogonal coordinates 72
47. Semi-orthogonal coordinates 74
48. A differential invariant 75

TRANSFORMATION OF INTEGRALS

49. Line and surface integrals. Stokes's Theorem 76
50. The Divergence Theorem of Gauss 78
51. Green's Theorem, and others 80
EXAMPLES V 81

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-316-60695-7 - Differential Geometry of Three Dimensions: Volume II
C. E. Weatherburn
Frontmatter
More information

CHAPTER VI

FAMILIES OF SURFACES

ART. PAGE

52. First curvature of a surface 85
53. Rotation of **n**. Parallel surfaces 87
54. Second curvature of a surface 88
55. Second curvature (*continued*) 90
56. Orthogonal trajectories of the surfaces 91
57. Lines of equidistance on any surface 92
58. Isometric system of surfaces 94
59. Family of Weingarten surfaces 95

LAMÉ FAMILIES OF SURFACES

60. Equation of condition 96
61. Orthogonal trajectories of the surfaces 98
62. The distance function for the family 99
63. Surfaces of constant first curvature 101
64. Lamé family of developable surfaces 102
EXAMPLES VI 103

CHAPTER VII

TENSORS OF THE SECOND ORDER.

DYADICS

65. Dyads and dyadics 106
66. Open products of vectors 107
67. Products of dyads and of dyadics 108
68. Nonion form. Scalar and vector of a dyadic 109
69. The unit dyadic. Reciprocal dyadics 110
70. Symmetric and anti-symmetric dyadics 112
71. Cross product of a dyadic and a vector 113
72. Double multiplication of dyads and dyadics 114
73. Second and third of a dyadic 115
74. Scalar invariants of a dyadic 116

DYADICS FORMED WITH THE OPERATOR ∇

75. The dyadics $\nabla\mathbf{s}$ and $\mathbf{s}\nabla$ 117
76. Formulae of expansion 118
77. The operator ∇ for a given surface 120
78. The dyadic $\nabla\mathbf{n}$ for a surface 121
79. Other geometrical illustrations 122
EXAMPLES VII 124

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-316-60695-7 - Differential Geometry of Three Dimensions: Volume II
C. E. Weatherburn
Frontmatter
More information

CHAPTER VIII

FAMILIES OF CURVES AND FUNCTIONS OF DIRECTION ON A SURFACE

Central Quadrics

ART. PAGE

80. Central quadric surfaces 128
81. Central conics 129

Family of Curves on a Surface

82. Tendency in any direction. First conic 131
83. Moment for any direction. Second conic 133
84. Swerve of the family. Third conic 136

Functions of Direction on a Surface

85. Mainardi-Codazzi relations. Gauss equation 137
86. Functions of Laguerre and Darboux 138
87. Codazzi function of three directions 141
Examples VIII 142

CHAPTER IX

LEVI-CIVITA'S PARALLEL DISPLACEMENTS. TCHEBYCHEF SYSTEMS

88. Levi-Civita's conception of parallelism 144
89. Condition for a parallel displacement 145
90. Two theorems on parallel displacement 148
91. Parallel displacement round a closed curve 149

Tchebychef Systems

92. Tchebychef nets 150
93. Lines of striction and normal curvature 152
94. Further properties of a Tchebychef system 153
Examples IX 155

CHAPTER X

REPRESENTATION OF SURFACES. CONICAL PROJECTION

95. Linear transformation. Extension ratio 156
96. Extension ellipses 157
97. Principal directions for the representation 158
98. Fundamental magnitudes and the operator ∇' on S' 160

Conical Projection

99. Conical projection in general 162
100. A particular case 163
Examples X 166

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-316-60695-7 - Differential Geometry of Three Dimensions: Volume II
C. E. Weatherburn
Frontmatter
More information

CHAPTER XI

SMALL DEFORMATIONS OF CURVES AND SURFACES

Small Deformations of Curves

ART. PAGE
101. Single twisted curve 167
102. Family of curves on a surface 169

Small Deformations of Surfaces

103. First order magnitudes. Modulus of dilation 171
104. Unit normal to the deformed surface 171
105. Curvatures of the deformed surface 172
106. Modulus of extension. Change of inclination 173
107. Rotation of element of surface 176
108. Alternative method 177
109. Differential invariants for the deformed surface 178
110. Small deformation by conical projection 179
111. Small inextensional deformation 181
112. Small deformation of a family of surfaces 182
Examples XI 185

CHAPTER XII

FLEXION OF SURFACES. APPLICABILITY

113. Applicability. Quantities unaltered by flexion 187
114. The problem of Minding 188
115. Exceptional case 190
116. Surfaces of constant Gaussian curvature 193
117. Three forms of ds^2 for pseudospherical surfaces 195
118. Surface deformable on itself. 197
119. Second problem of applicability 199
120. Assigned deform of a given curve on S 201

Flexion of Ruled Surfaces

121. Theorems of Bonnet and Beltrami 203
122. Second theorem of Bonnet 205
Examples XII 206

CHAPTER XIII

CURVILINEAR CONGRUENCES

123. Introduction 208

First Method

124. Notation 208
125. Foci. Focal surface 209
126. Limits. Limit surface 211
127. Surface of striction. Divergence of the congruence . . . 214

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-316-60695-7 - Differential Geometry of Three Dimensions: Volume II
C. E. Weatherburn
Frontmatter
More information

ART. PAGE

128. Surfaces of the congruence 215
129. Normal congruences 217
130. Rectilinear congruences 218

SECOND METHOD

131. First quadric. Cone of zero tendency 219
132. Surface of striction. Limit surface 221
133. Second quadric. Cone of zero moment 223
134. Surface of normality. Ultimate surface 225
135. Axes of normal sections 227
136. Rates of rotation 228
137. Where the cones are pairs of planes 229
138. Isotropic congruences 231
139. Three orthogonal congruences 232
140. Congruence of lines of equidistance 234
141. Small deformation of a congruence of curves 236

INDEX 237

NOTE. In the following pages articles marked with an asterisk may be omitted at the first reading.

© in this web service Cambridge University Press www.cambridge.org