# CHAPTER I

## DIFFERENTIAL INVARIANTS FOR A SURFACE

**1. Gradient of a scalar function. Derivatives.** The differential invariants discussed in Chapter XII of the first volume of this work play an important part in the following pages. We shall therefore remind the reader of their chief properties, and collect for reference the most important formulae, of which constant use will be made.

The *gradient* of a scalar point-function $\phi$ on a given surface $S$ is denoted by $\nabla\phi$. It is a vector quantity whose direction at any point $P$ is that direction on the surface which gives the maximum arc-rate of increase of $\phi$, and whose magnitude is this maximum rate of increase. It is thus a vector point-function on the surface. The *derivative*, or rate of increase, of $\phi$ in any direction on the surface is the resolved part of $\nabla\phi$ in this direction. Thus if $\mathbf{c}$ is a unit surface vector at $P$, that is to say a unit vector parallel to the tangent plane at $P$, the derivative of $\phi$ in the direction of $\mathbf{c}$ has the value $\mathbf{c}\cdot\nabla\phi$. In terms of any convenient parameters $u$, $v$ on the surface, and the corresponding first order magnitudes $E$, $F$, $G$, the gradient of $\phi$ may be expressed*

$$\nabla\phi = H^{-2}[(G\phi_1 - F\phi_2)\,\mathbf{r}_1 + (E\phi_2 - F\phi_1)\,\mathbf{r}_2] \quad \text{......(1)},$$

where $\mathbf{r}$ is the position vector of the current point $P$ on the surface, and suffixes 1, 2 denote partial differentiations with respect to $u$ and $v$ respectively. Thus $\nabla\phi$ may be regarded as the result obtained by operating on the function $\phi$ with the vectorial differential operator

$$\nabla = \frac{1}{H^2}\left[\mathbf{r}_1\left(G\frac{\partial}{\partial u} - F\frac{\partial}{\partial v}\right) + \mathbf{r}_2\left(E\frac{\partial}{\partial v} - F\frac{\partial}{\partial u}\right)\right] \text{......(2)}$$

obeying the distributive law. When the parametric curves are orthogonal, $F = 0$ and $H^2 = EG$; so that (2) takes the simpler form

$$\nabla = \frac{1}{E}\mathbf{r}_1\frac{\partial}{\partial u} + \frac{1}{G}\mathbf{r}_2\frac{\partial}{\partial v} \text{.....................(3)}.$$

* *Loc. cit.*, Art. 114.

For any choice of parametric curves the operator $\nabla$ and the function $\nabla\phi$ may be expressed very simply by means of the vectors of the system reciprocal to $\mathbf{r}_1, \mathbf{r}_2, \mathbf{n}$. This system $\mathbf{l}, \mathbf{m}, \mathbf{n}$ is defined by

$$H\mathbf{l} = \mathbf{r}_2 \times \mathbf{n}, \quad H\mathbf{m} = \mathbf{n} \times \mathbf{r}_1, \quad H\mathbf{n} = \mathbf{r}_1 \times \mathbf{r}_2.$$

From these relations it follows that

$$\left.\begin{aligned} H^2\mathbf{l} &= \mathbf{r}_2 \times (\mathbf{r}_1 \times \mathbf{r}_2) = G\mathbf{r}_1 - F\mathbf{r}_2 \\ H^2\mathbf{m} &= (\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{r}_1 = E\mathbf{r}_2 - F\mathbf{r}_1 \end{aligned}\right\},$$

and consequently the gradient of $\phi$ is given by

$$\nabla\phi = \mathbf{l}\frac{\partial\phi}{\partial u} + \mathbf{m}\frac{\partial\phi}{\partial v} \quad \text{.....................(4)}$$

and the operator $\nabla$ by

$$\nabla = \mathbf{l}\frac{\partial}{\partial u} + \mathbf{m}\frac{\partial}{\partial v} \quad \text{........................(5).}$$

If $\theta$ is a point-function on $S$, and $f(\theta)$ a function of $\theta$, it follows immediately from the definition by (1) or (4) that

$$\nabla f(\theta) = f'(\theta)\nabla\theta \quad \text{.....................(6).}$$

Similarly, if $f(\theta, \phi, \psi, \ldots)$ is a function of several point-functions $\theta, \phi, \psi, \ldots$,

$$\nabla f(\theta, \phi, \ldots) = \frac{\partial f}{\partial\theta}\nabla\theta + \frac{\partial f}{\partial\phi}\nabla\phi + \ldots \quad \text{.........(7).}$$

The operator $\mathbf{c}\cdot\nabla$ may also be applied to a vector point-function $\mathbf{u}$, giving the derivative of $\mathbf{u}$ in the direction of the unit vector $\mathbf{c}$. Thus by (5)

$$\mathbf{c}\cdot\nabla\mathbf{u} = (\mathbf{c}\cdot\mathbf{l})\frac{\partial\mathbf{u}}{\partial u} + (\mathbf{c}\cdot\mathbf{m})\frac{\partial\mathbf{u}}{\partial v}.$$

And, though the interpretation of $\mathbf{c}\cdot\nabla\mathbf{u}$ as the rate of change of $\mathbf{u}$ in the direction of $\mathbf{c}$ is applicable only when $\mathbf{c}$ is tangential to the surface, and a unit vector, we define the function $\mathbf{c}\cdot\nabla\mathbf{u}$ for all values of $\mathbf{c}$ by the above equation. In particular, if $\mathbf{c}$ is normal to the surface, $\mathbf{c}\cdot\nabla\mathbf{u}$ is equal to zero.

**2. Divergence and rotation of a vector.** The operator $\nabla$ may be applied to a vector function $\mathbf{u}$ in different ways. One of these gives a scalar function called the *divergence* of $\mathbf{u}$, and denoted by div $\mathbf{u}$ or $\nabla\cdot\mathbf{u}$. It is defined by

$$\begin{aligned} \operatorname{div}\mathbf{u} &= \mathbf{l}\cdot\mathbf{u}_1 + \mathbf{m}\cdot\mathbf{u}_2 \\ &= H^{-2}[\mathbf{r}_1\cdot(G\mathbf{u}_1 - F\mathbf{u}_2) + \mathbf{r}_2\cdot(E\mathbf{u}_2 - F\mathbf{u}_1)] \quad \text{......(8),} \end{aligned}$$

and is invariant with respect to the choice of parameters $u$, $v$. Similarly $\nabla$ may be applied to $\mathbf{u}$ in such a way as to give a vector differential invariant, called the *rotation* or *curl* of $\mathbf{u}$, and denoted by rot $\mathbf{u}$, curl $\mathbf{u}$ or $\nabla \times \mathbf{u}$. It is defined by

$$\begin{aligned}\text{rot}\, \mathbf{u} &= \mathbf{l} \times \mathbf{u}_1 + \mathbf{m} \times \mathbf{u}_2 \\ &= H^{-2}[\mathbf{r}_1 \times (G\mathbf{u}_1 - F\mathbf{u}_2) + \mathbf{r}_2 \times (E\mathbf{u}_2 - F\mathbf{u}_1)] \quad \text{......(9).}\end{aligned}$$

We shall use the abbreviation rot $\mathbf{u}$, rather than curl $\mathbf{u}$, throughout this volume.

The unit vector $\mathbf{n}$, normal to the surface, is a point-function which plays an important part. From the above formulae it is easily verified, as we have already shown*, that

$$\text{div}\, \mathbf{n} = -J, \quad \text{rot}\, \mathbf{n} = 0 \quad \text{..................(10),}$$

$J$ being the first curvature (or mean curvature) of the surface. Similarly for any vector $\phi\mathbf{n}$, normal to the surface, we find

$$\text{div}\, \phi\mathbf{n} = -J\phi, \quad \text{rot}\, \phi\mathbf{n} = \nabla\phi \times \mathbf{n} \quad \text{...........(11),}$$

while, for any tangential vector $P\mathbf{r}_1 + Q\mathbf{r}_2$, we have

$$\text{div}\,(P\mathbf{r}_1 + Q\mathbf{r}_2) = \frac{1}{H}\left[\frac{\partial}{\partial u}(HP) + \frac{\partial}{\partial v}(HQ)\right] \quad \text{......(12).}$$

In particular, if $\mathbf{a}$ and $\mathbf{b}$ are the unit tangents to the orthogonal parametric curves $v =$ const. and $u =$ const. respectively,

$$\text{div}\, \mathbf{a} = \frac{G_1}{2G\sqrt{E}}, \quad \text{div}\, \mathbf{b} = \frac{E_2}{2E\sqrt{G}} \quad \text{.........(13).}$$

These two quantities will be met very frequently in the following pages.

**3. Formulae of expansion.** Let $\phi$, $\psi$ be scalar functions and $\mathbf{u}$, $\mathbf{v}$ vector functions on the given surface. We frequently require expansions for the gradients of the products $\phi\psi$ and $\mathbf{u}\cdot\mathbf{v}$, and for the divergence and rotation of the vectors $\phi\mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$, in terms of differential invariants of the separate functions. From (1) or (4), and the rule for differentiating a product, it follows immediately that

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi \quad \text{..................(14).}$$

* *Loc. cit.*, Arts. 116, 118.

Similarly, as we have already shown*, it follows from (8) and (9) that

$$\operatorname{div}(\phi\mathbf{u}) = \nabla\phi\cdot\mathbf{u} + \phi\operatorname{div}\mathbf{u} \quad\ldots\ldots\ldots\ldots\ldots(15),$$

$$\operatorname{rot}(\phi\mathbf{u}) = \nabla\phi\times\mathbf{u} + \phi\operatorname{rot}\mathbf{u} \quad\ldots\ldots\ldots\ldots\ldots(16),$$

$$\operatorname{div}(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\operatorname{rot}\mathbf{u} - \mathbf{u}\cdot\operatorname{rot}\mathbf{v} \quad\ldots\ldots\ldots\ldots(17).$$

There are two other relations, not previously considered, which we shall have frequent occasion to use. These are expressed by the formulae

$$\operatorname{rot}(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{u}\operatorname{div}\mathbf{v} - \mathbf{v}\operatorname{div}\mathbf{u} \;\ldots\ldots(18),$$

$$\nabla(\mathbf{u}\cdot\mathbf{v}) = \mathbf{v}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{v}\times\operatorname{rot}\mathbf{u} + \mathbf{u}\times\operatorname{rot}\mathbf{v} \quad (19).$$

To prove the first of these we observe that, in virtue of (9),

$$\begin{aligned}\operatorname{rot}(\mathbf{u}\times\mathbf{v}) &= \mathbf{l}\times(\mathbf{u}_1\times\mathbf{v} + \mathbf{u}\times\mathbf{v}_1) + \mathbf{m}\times(\mathbf{u}_2\times\mathbf{v} + \mathbf{u}\times\mathbf{v}_2)\\ &= (\mathbf{l}\cdot\mathbf{v})\,\mathbf{u}_1 - (\mathbf{l}\cdot\mathbf{u}_1)\,\mathbf{v} + (\mathbf{l}\cdot\mathbf{v}_1)\,\mathbf{u} - (\mathbf{l}\cdot\mathbf{u})\,\mathbf{v}_1\\ &\quad + (\mathbf{m}\cdot\mathbf{v})\,\mathbf{u}_2 - (\mathbf{m}\cdot\mathbf{u}_2)\,\mathbf{v} + (\mathbf{m}\cdot\mathbf{v}_2)\,\mathbf{u} - (\mathbf{m}\cdot\mathbf{u})\,\mathbf{v}_2\\ &= \mathbf{v}\cdot\nabla\mathbf{u} - \mathbf{v}\operatorname{div}\mathbf{u} + \mathbf{u}\operatorname{div}\mathbf{v} - \mathbf{u}\cdot\nabla\mathbf{v},\end{aligned}$$

which is the required formula (18). To demonstrate (19) we have similarly

$$\nabla(\mathbf{u}\cdot\mathbf{v}) = \mathbf{l}\,(\mathbf{u}_1\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v}_1) + \mathbf{m}\,(\mathbf{u}_2\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v}_2) \;\ldots\ldots(\text{i}).$$

Also, in virtue of (9),

$$\begin{aligned}\mathbf{v}\times\operatorname{rot}\mathbf{u} &= \mathbf{v}\times(\mathbf{l}\times\mathbf{u}_1 + \mathbf{m}\times\mathbf{u}_2)\\ &= (\mathbf{v}\cdot\mathbf{u}_1)\,\mathbf{l} - (\mathbf{v}\cdot\mathbf{l})\,\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\,\mathbf{m} - (\mathbf{v}\cdot\mathbf{m})\,\mathbf{u}_2\\ &= (\mathbf{v}\cdot\mathbf{u}_1)\,\mathbf{l} + (\mathbf{v}\cdot\mathbf{u}_2)\,\mathbf{m} - \mathbf{v}\cdot\nabla\mathbf{u} \quad\ldots\ldots\ldots\ldots\ldots(\text{ii}),\end{aligned}$$

and similarly

$$\mathbf{u}\times\operatorname{rot}\mathbf{v} = (\mathbf{u}\cdot\mathbf{v}_1)\,\mathbf{l} + (\mathbf{u}\cdot\mathbf{v}_2)\,\mathbf{m} - \mathbf{u}\cdot\nabla\mathbf{v} \;\ldots\ldots(\text{iii}).$$

From (i), (ii) and (iii) the formula (19) follows at once.

An important particular case of (19) is that in which the two vectors **u**, **v** are equal and of constant length. Then

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^2 = \text{const.},$$

and the first member of (19) vanishes. Hence, *if* **u** *is a vector of constant length*,

$$\mathbf{u}\cdot\nabla\mathbf{u} = -\,\mathbf{u}\times\operatorname{rot}\mathbf{u} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20).$$

* *Loc. cit.*, Art. 120.

**Example.** Deduce the Circulation Theorem from the Divergence Theorem (Vol. I, Arts. 122, 124).

With the notation of those theorems let $C$ be a closed curve on the given surface, $ds$ the length of an element of arc, $\mathbf{t}$ the unit tangent and $\mathbf{m}$ the unit surface vector perpendicular to $\mathbf{t}$ and drawn outward, so that $\mathbf{m}$, $\mathbf{t}$, $\mathbf{n}$ form a right-handed system. Apply the Divergence Theorem to the vector $\mathbf{u}\times\mathbf{n}$, where $\mathbf{u}$ is a vector point-function for the surface. Then

$$\iint \operatorname{div}(\mathbf{u}\times\mathbf{n})\,dS=\int_{\circ}(\mathbf{u}\times\mathbf{n})\cdot\mathbf{m}\,ds \quad \text{.................(i)},$$

the line integral being taken round the curve $C$, and the surface integral over the region enclosed. Now

$$(\mathbf{u}\times\mathbf{n})\cdot\mathbf{m}=\mathbf{u}\cdot(\mathbf{n}\times\mathbf{m})=\mathbf{u}\cdot\mathbf{t}.$$

Also, in virtue of (17), $\quad \operatorname{div}(\mathbf{u}\times\mathbf{n})=\mathbf{n}\cdot\operatorname{rot}\mathbf{u},$

since rot $\mathbf{n}$ vanishes identically. Hence we may write (i)

$$\iint \mathbf{n}\cdot\operatorname{rot}\mathbf{u}\,dS=\int_{\circ}\mathbf{u}\cdot\mathbf{t}\,ds=\int_{\circ}\mathbf{u}\cdot d\mathbf{r} \quad \text{.................(ii)},$$

which expresses the Circulation Theorem for the function $\mathbf{u}$ and the curve $C$, the line integral in (ii) being the circulation of the vector $\mathbf{u}$ round the curve $C$.

**4. Differential invariants of the second order.** Certain differential invariants of the second order are of frequent occurrence. The most important of these is the divergence of the gradient of a scalar function. This function div grad $\phi$ or $\nabla\cdot\nabla\phi$ will be denoted by $\nabla^2\phi$. It is identical with Beltrami's "differential parameter of the second order." From (1) and (12) it follows that

$$\nabla^2\phi=\frac{1}{H}\left[\frac{\partial}{\partial u}\left(\frac{G\phi_1-F\phi_2}{H}\right)+\frac{\partial}{\partial v}\left(\frac{E\phi_2-F\phi_1}{H}\right)\right] \quad \text{...(21)},$$

or, when the parametric curves are orthogonal,

$$\nabla^2\phi=\frac{1}{\sqrt{(EG)}}\left[\frac{\partial}{\partial u}\left(\phi_1\sqrt{\frac{G}{E}}\right)+\frac{\partial}{\partial v}\left(\phi_2\sqrt{\frac{E}{G}}\right)\right] \quad \text{...(22)}.$$

The operator $\nabla^2$, defined by these equations, may also be applied to a vector point-function, giving a vector differential invariant of the second order. We have already shown that*, if $\mathbf{r}$ is the position vector of the current point on the surface, and $\mathbf{n}$ the unit normal,

$$\nabla^2\mathbf{r}=J\mathbf{n},$$

and
$$\nabla^2\mathbf{n}=-(J^2-2K)\,\mathbf{n}-\nabla J \quad \text{..............(23)},$$

* *Loc. cit.*, Art. 119.

$K$ being the second curvature (or Gaussian curvature) of the surface. From the last equation we deduce the important formula

$$2K = \mathbf{n} \cdot \nabla^2 \mathbf{n} + (\operatorname{div} \mathbf{n})^2 \quad \text{.................(24)},$$

*which expresses* $K$ *as a differential invariant of* $\mathbf{n}$ *of the second order.*

The rotation of the gradient of $\phi$, or rot $\nabla\phi$, is frequently met with; and we have shown that this vector is tangential to the surface*. Similarly the divergence of the rotation of $\mathbf{u}$, or div rot $\mathbf{u}$, also presents itself. If $\mathbf{u}$ is normal to the surface this function vanishes everywhere. For any normal vector is of the form $\phi\mathbf{n}$, and by (16) and (17), since rot $\mathbf{n}$ is zero, we have

$$\operatorname{div} \operatorname{rot} \phi\mathbf{n} = \operatorname{div} (\nabla\phi \times \mathbf{n}) = \mathbf{n} \cdot (\operatorname{rot} \nabla\phi),$$

which vanishes, since rot $\nabla\phi$ is a surface vector.

**Ex. 1.** Prove the formulae

$$\nabla^2 (\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi \cdot \nabla\psi + \psi\nabla^2\phi,$$
$$\operatorname{rot} \nabla (\phi\psi) = \phi \operatorname{rot} \nabla\psi + \psi \operatorname{rot} \nabla\phi.$$

**Ex. 2.** If $\theta$ is a point-function, and $f(\theta)$ a function of $\theta$,

$$\nabla^2 f(\theta) = f''(\theta)(\nabla\theta)^2 + f'(\theta)\nabla^2\theta.$$

**5. Order of directional differentiations.** If a function is differentiated successively in two different directions, the order of the differentiations is not in general commutative. Let $\phi$ be the function considered, and let the directions of differentiation be those of the parametric curves. If $\dfrac{d}{ds}$ and $\dfrac{d}{ds'}$ be used for the moment as symbols of differentiation in the directions of $v = \text{const.}$ and $u = \text{const.}$ respectively, we have

$$\frac{d\phi}{ds} = \frac{1}{\sqrt{E}} \frac{\partial\phi}{\partial u},$$

and therefore

$$\frac{d^2\phi}{ds'ds} = \frac{1}{\sqrt{G}} \left( \frac{1}{\sqrt{E}} \frac{\partial^2\phi}{\partial v \partial u} - \frac{E_2}{2E\sqrt{E}} \frac{\partial\phi}{\partial u} \right).$$

Similarly

$$\frac{d^2\phi}{ds\,ds'} = \frac{1}{\sqrt{E}} \left( \frac{1}{\sqrt{G}} \frac{\partial^2\phi}{\partial u \partial v} - \frac{G_1}{2G\sqrt{G}} \frac{\partial\phi}{\partial v} \right).$$

* *Loc. cit.*, Art. 118.

Consequently the difference of these two derivatives has a value given by

$$\frac{d^2\phi}{ds\,ds'}-\frac{d^2\phi}{ds'ds}=\frac{E_2}{2E\sqrt{G}}\frac{d\phi}{ds}-\frac{G_1}{2G\sqrt{E}}\frac{d\phi}{ds'}\quad\text{......(25).}$$

A case of particular importance is that in which the two directions are at right angles. Suppose then that the parametric curves are orthogonal, with **a** and **b** as their unit tangents at any point. Then, in virtue of (13), we may write the above equation

$$\mathbf{a}\cdot\nabla(\mathbf{b}\cdot\nabla\phi)-\mathbf{b}\cdot\nabla(\mathbf{a}\cdot\nabla\phi)=(\mathbf{a}\cdot\nabla\phi)\operatorname{div}\mathbf{b}-(\mathbf{b}\cdot\nabla\phi)\operatorname{div}\mathbf{a}.$$

Now the geodesic curvatures $\gamma$, $\gamma'$ of the orthogonal curves $v=$ const., $u=$ const. are given by*

$$\gamma=-\operatorname{div}\mathbf{b},\quad \gamma'=\operatorname{div}\mathbf{a},$$

in virtue of (13). Consequently the difference of the two derivatives given above may be expressed

$$\frac{d^2\phi}{ds'ds}-\frac{d^2\phi}{ds\,ds'}=\gamma\mathbf{a}\cdot\nabla\phi+\gamma'\mathbf{b}\cdot\nabla\phi$$
$$=(\gamma\mathbf{a}+\gamma'\mathbf{b})\cdot\nabla\phi\quad\text{...............(26).}$$

**6. Derivatives of the unit vectors a, b, n.** As above let **a**, **b** be the unit tangents to the orthogonal parametric curves. It will be found convenient to record for reference the derivatives of **a**, **b**, **n** in the directions of **a** and **b**. Using the values of the partial derivatives of **a**, **b** with respect to $u$ and $v$, as found in Vol. I, Art. 41, we have

$$\frac{1}{\sqrt{E}}\frac{\partial\mathbf{a}}{\partial u}=\frac{L}{E}\mathbf{n}-\frac{E_2}{2E\sqrt{G}}\mathbf{b},$$

$$\frac{1}{\sqrt{G}}\frac{\partial\mathbf{a}}{\partial v}=\frac{M}{\sqrt{(EG)}}\mathbf{n}+\frac{G_1}{2G\sqrt{E}}\mathbf{b},$$

which may be expressed

$$\left.\begin{aligned}\mathbf{a}\cdot\nabla\mathbf{a}&=\kappa_n\mathbf{n}+\gamma\mathbf{b}\\ \mathbf{b}\cdot\nabla\mathbf{a}&=\tau\mathbf{n}\;+\gamma'\mathbf{b}\end{aligned}\right\}\quad\text{.....................(27),}$$

where $\gamma$, $\gamma'$ are the geodesic curvatures of the curves $v=$ const. and $u=$ const. respectively, $\kappa_n$ the normal curvature in the direction

* See also Vol. I, Art. 121.

of the former, and $\tau$ the torsion of the geodesic tangent in the same direction. Similarly we find for the derivatives of **b**

$$\left.\begin{aligned}\mathbf{a}\cdot\nabla\mathbf{b} &= \tau\mathbf{n} - \gamma\mathbf{a}\\ \mathbf{b}\cdot\nabla\mathbf{b} &= \kappa_n'\mathbf{n} - \gamma'\mathbf{a}\end{aligned}\right\} \qquad \text{..................(28)},$$

$\kappa_n'$ being the normal curvature in the direction of **b**.

Again, when the parametric curves are orthogonal, the derivatives of **n** found in Vol. I, Art. 27, become

$$\frac{\partial\mathbf{n}}{\partial u} = -\frac{L}{E}\frac{\partial\mathbf{r}}{\partial u} - \frac{M}{G}\frac{\partial\mathbf{r}}{\partial v},$$

$$\frac{\partial\mathbf{n}}{\partial v} = -\frac{M}{E}\frac{\partial\mathbf{r}}{\partial u} - \frac{N}{G}\frac{\partial\mathbf{r}}{\partial v}.$$

And from these it follows immediately that

$$\left.\begin{aligned}\mathbf{a}\cdot\nabla\mathbf{n} &= -\kappa_n\mathbf{a} - \tau\mathbf{b}\\ \mathbf{b}\cdot\nabla\mathbf{n} &= -\tau\mathbf{a} - \kappa_n'\mathbf{b}\end{aligned}\right\} \qquad \text{..................(29)}.$$

The above formulae will be used very frequently in the following chapters.

***7. Other differential invariants.** In closing this chapter we may draw attention to certain other differential invariants. We shall first prove the theorem:

*For any vector point-function on a given surface, the vector product of its derivatives in any two surface directions, divided by the scalar triple product of the unit normal and the unit vectors in those two directions, is an invariant*†.

Let **s** be a vector function, and **c** and **d** unit vectors tangential to the surface. Then if **l**, **m**, **n** are the reciprocal system of vectors to $\mathbf{r}_1$, $\mathbf{r}_2$, **n** as defined in Art. 1, the derivatives of **s** in the directions of **c** and **d** are

$$\mathbf{c}\cdot\nabla\mathbf{s} = (\mathbf{c}\cdot\mathbf{l})\,\mathbf{s}_1 + (\mathbf{c}\cdot\mathbf{m})\,\mathbf{s}_2,$$

$$\mathbf{d}\cdot\nabla\mathbf{s} = (\mathbf{d}\cdot\mathbf{l})\,\mathbf{s}_1 + (\mathbf{d}\cdot\mathbf{m})\,\mathbf{s}_2,$$

and their vector product has the value

$$\begin{aligned}&\{(\mathbf{c}\cdot\mathbf{l})(\mathbf{d}\cdot\mathbf{m}) - (\mathbf{c}\cdot\mathbf{m})(\mathbf{d}\cdot\mathbf{l})\}\,\mathbf{s}_1\times\mathbf{s}_2\\ &\qquad = (\mathbf{c}\times\mathbf{d})\cdot(\mathbf{l}\times\mathbf{m})\,\mathbf{s}_1\times\mathbf{s}_2 = \frac{1}{H}(\mathbf{c}\times\mathbf{d}\cdot\mathbf{n})\,\mathbf{s}_1\times\mathbf{s}_2.\end{aligned}$$

† See Art. 4 of a paper by the author "On Families of Curves and Surfaces," *Quarterly Journal*, Vol. 50 (1927), pp. 350–361.

Consequently the quotient of this vector product by $(\mathbf{c}\times\mathbf{d})\cdot\mathbf{n}$ has the value $\mathbf{s}_1\times\mathbf{s}_2/H$. It is therefore independent of the directions of $\mathbf{c}$ and $\mathbf{d}$, and is thus an invariant. The result may be expressed in the form:

*The function $\mathbf{s}_1\times\mathbf{s}_2/H$ is independent of the choice of parametric curves.*

Let us denote this differential invariant of $\mathbf{s}$ by $\Lambda\mathbf{s}$. A simple case already met with is that in which $\mathbf{s}=\mathbf{r}$. For

$$\Lambda\mathbf{r}=\mathbf{r}_1\times\mathbf{r}_2/H=\mathbf{n}.$$

Also, the same invariant of the unit normal $\mathbf{n}$ is equal to $K\mathbf{n}$, where $K$ is the second curvature of the surface. For, taking the lines of curvature as parametric curves, we have

$$\Lambda\mathbf{n}=\frac{\mathbf{n}_1\times\mathbf{n}_2}{H}=\frac{1}{H}\left(-\frac{L}{E}\mathbf{r}_1\right)\times\left(-\frac{N}{G}\mathbf{r}_2\right)$$

$$=K\mathbf{r}_1\times\mathbf{r}_2/H=K\mathbf{n}\quad\text{........................(30).}$$

Consequently the second curvature of the surface may be expressed as a differential invariant of the unit normal in the form

$$K=\mathbf{n}\cdot(\Lambda\mathbf{n})\quad\text{........................(31).}$$

And this may be written in terms of $\mathbf{r}$,

$$K=(\Lambda\mathbf{r})\cdot(\Lambda\Lambda\mathbf{r})\quad\text{.....................(32).}$$

Or again, if $\phi$ is a point-function for the surface, so also is $\phi\mathbf{n}$. Then

$$\Lambda(\phi\mathbf{n})=\frac{1}{H}(\phi_1\mathbf{n}+\phi\mathbf{n}_1)\times(\phi_2\mathbf{n}+\phi\mathbf{n}_2)$$

$$=\phi^2K\mathbf{n}+\frac{1}{H}\phi\mathbf{n}\times(\phi_1\mathbf{n}_2-\phi_2\mathbf{n}_1).$$

The first term is itself an invariant; and, since $\mathbf{n}$ is perpendicular to $\mathbf{n}_1$ and $\mathbf{n}_2$, we have the result:

*The function $(\phi_1\mathbf{n}_2-\phi_2\mathbf{n}_1)/H$ is independent of the choice of parametric curves.*

Similarly by considering $\Lambda(\phi\mathbf{s})$ we may show that

$$\mathbf{s}\times(\phi_1\mathbf{s}_2-\phi_2\mathbf{s}_1)/H$$

is a differential invariant of $\phi$ and $\mathbf{s}$.

## EXAMPLES I

**1.** Show that, for orthogonal parametric curves,

$$\operatorname{rot}\mathbf{a} = \frac{M}{\sqrt{(EG)}}\mathbf{a} - \frac{L}{E}\mathbf{b} - \frac{E_2}{2E\sqrt{G}}\mathbf{n}$$
$$= \tau\mathbf{a} - \kappa_n\mathbf{b} + \gamma\mathbf{n},$$
$$\operatorname{rot}\mathbf{b} = \frac{N}{G}\mathbf{a} - \frac{M}{\sqrt{(EG)}}\mathbf{b} + \frac{G_1}{2G\sqrt{E}}\mathbf{n}$$
$$= \kappa_n'\mathbf{a} - \tau\mathbf{b} + \gamma'\mathbf{n},$$

and deduce the formula

$$K = \mathbf{n}\cdot(\operatorname{rot}\mathbf{a}\times\operatorname{rot}\mathbf{b}) = [\mathbf{n}, \operatorname{rot}\mathbf{a}, \operatorname{rot}\mathbf{b}].$$

**2.** Show that, for any choice of parametric curves,

$$\operatorname{rot}(P\mathbf{r}_1 + Q\mathbf{r}_2) = \frac{1}{H}\left[\frac{\partial}{\partial u}(FP+GQ) - \frac{\partial}{\partial v}(EP+FQ)\right]\mathbf{n}$$
$$+\frac{1}{H}[(PM+QN)\,\mathbf{r}_1 - (PL+QM)\,\mathbf{r}_2],$$

and deduce that rot $\nabla\phi$ is tangential to the surface.

**3.** If $f(\theta, \phi)$ is a function of the two point-functions $\theta$, $\phi$, show that

$$\nabla^2 f = \frac{\partial f}{\partial\theta}\nabla^2\theta + \frac{\partial f}{\partial\phi}\nabla^2\phi + \frac{\partial^2 f}{\partial\theta^2}(\nabla\theta)^2 + 2\frac{\partial^2 f}{\partial\theta\partial\phi}\nabla\theta\cdot\nabla\phi + \frac{\partial^2 f}{\partial\phi^2}(\nabla\phi)^2,$$

and that

$$(\nabla f)^2 = \left(\frac{\partial f}{\partial\theta}\right)^2(\nabla\theta)^2 + 2\frac{\partial f}{\partial\theta}\frac{\partial f}{\partial\phi}\nabla\theta\cdot\nabla\phi + \left(\frac{\partial f}{\partial\phi}\right)^2(\nabla\phi)^2.$$

**4.** Prove the relations

$$\operatorname{div}\operatorname{rot}\phi\mathbf{s} = \mathbf{s}\cdot(\operatorname{rot}\nabla\phi) + \phi\operatorname{div}\operatorname{rot}\mathbf{s},$$
$$\nabla^2(\phi\mathbf{s}) = \phi\nabla^2\mathbf{s} + 2\nabla\phi\cdot\nabla\mathbf{s} + \mathbf{s}\nabla^2\phi.$$

**5.** If $\mathbf{a}$ and $\mathbf{b}$ have the usual significance, show that

$$K = -\operatorname{div}(\mathbf{a}\operatorname{div}\mathbf{a} + \mathbf{b}\operatorname{div}\mathbf{b}),$$

and that

$$K = (\mathbf{a}\cdot\nabla\mathbf{a})\cdot(\mathbf{b}\cdot\nabla\mathbf{b}) - (\mathbf{b}\cdot\nabla\mathbf{a})\cdot(\mathbf{a}\cdot\nabla\mathbf{b}).$$

**6.** With the same notation, show that

$$\mathbf{n}\cdot\operatorname{rot}(\mathbf{a}\operatorname{div}\mathbf{a} + \mathbf{b}\operatorname{div}\mathbf{b}) = \operatorname{div}\{(\mathbf{a}\operatorname{div}\mathbf{a} + \mathbf{b}\operatorname{div}\mathbf{b})\times\mathbf{n}\}$$
$$= \operatorname{div}(\mathbf{a}\operatorname{div}\mathbf{b} - \mathbf{b}\operatorname{div}\mathbf{a})$$
$$= \frac{1}{2\sqrt{(EG)}}\frac{\partial^2}{\partial u\partial v}\log\frac{E}{G}.$$

Hence deduce that, if the orthogonal parametric curves constitute an isometric system, the vector $\mathbf{a}\operatorname{div}\mathbf{a} + \mathbf{b}\operatorname{div}\mathbf{b}$ is the gradient of some scalar function.

Also show that $\mathbf{a}\cdot\nabla\operatorname{div}\mathbf{b} - \mathbf{b}\cdot\nabla\operatorname{div}\mathbf{a} = 0$

is a necessary and sufficient condition that the parametric curves be isometric.