CHAPTER 1

BASIC CONCEPTIONS AND GENERAL IDEAS

1·1. The payment of interest as a reward for the use of capital is an established part of our economic life. In a sense interest may be regarded as a reward paid by a person, who is given the use of a sum of capital, to the owner of the capital. In theory the two items, the capital which is being used and the interest which is being paid, need not be expressed in terms of the same commodity. For example, if a farmer lends his neighbour a tractor for his harvest in return for a proportion of the corn reaped, the tractor could be considered as the capital lent and the corn as the interest paid. In financial and actuarial theory, however, it is necessary to consider only the case where both capital and interest are expressed in terms of money. The theory of compound interest is concerned with the continued growth of a sum of money under the operation of interest.

1·2. Rate of interest

In practice the interest which it has been agreed will be paid for the use of the capital is payable at stated intervals of time. The rate of interest which operates during one of these intervals is found by considering the amount of interest agreed to be paid in relation to the capital invested. Thus the strict definition of a rate of interest is

The amount contracted to be paid in one unit interval of time for each unit of capital invested.

The general financial practice is to make the unit interval of time a year, and this unfortunately tends to induce the preconceived idea that rates of interest must be annual rates, and still more unfortunately it is customary, as will be seen below, to describe certain rates of interest as 'rates per annum' when in fact they are not. For example, one of the best known British Government

Securities is referred to as '$3\frac{1}{2}$% War Loan'. The actual rate of interest paid on this security is not, however, $3\frac{1}{2}$% per annum because it is paid twice yearly in June and December, and in terms of the definition above the rate of interest is $1\frac{3}{4}$% per half-year. As will be seen later, this is not the same as $3\frac{1}{2}$% per annum, and it is important, especially in considering first principles, that this distinction should be clearly grasped at the outset.

1·3. Compound interest

By definition, if a stated rate of interest is to be paid it means that at the end of a stated interval of time the lender or investor will receive a fixed sum of money for the use of his capital during that period. In the theory of compound interest it is assumed that when the lender receives this sum he can immediately use it as capital and invest it so that it earns interest in the same way as the original loan. The accumulated amount of the original capital invested or 'principal', plus the interest paid on it and similarly invested, is called the 'amount' of the principal. In the theory of compound interest it is a fundamental hypothesis that the amount of a given principal is a continuous function of time. In actual practice, of course, interest is added only at stated intervals of time, with the result that the amount of the principal displays sudden jumps at the end of each interval. But in the theoretical aspect of the subject interest is regarded as accruing continuously throughout the interval so that the amount of the principal is subject to a process of continuous growth. In other words if, in Fig. 1, AA_1, BB_1, CC_1, etc., represent the amounts of interest paid after 1, 2, 3, ... intervals, the graph of the amount of principal invested is represented by the continuous curve and not the dotted line.

1·4. Analogy with natural processes

The accumulation of a sum of money at compound interest may in fact be compared with the natural process of a plant or tree growing. When a tree grows the stem puts forth branches. These branches are like interest being added to the principal. They in turn throw off twigs, which also grow and are like the interest, which it has been postulated can be earned on the original interest, once it has been invested and thus become capital. The total amount by which the tree has increased over its original size in a given period is equivalent to the total interest which the principal has earned in that period. The obvious difference is that in natural growth there is no doubt about the continuity of the process, and it may therefore help to give a clearer picture if the analogy is considered in greater detail.

For the sake of simplicity let it be assumed that there is a tree whose branches grow only lengthwise and that it is desired to measure the rate of longitudinal growth of one of its branches. The

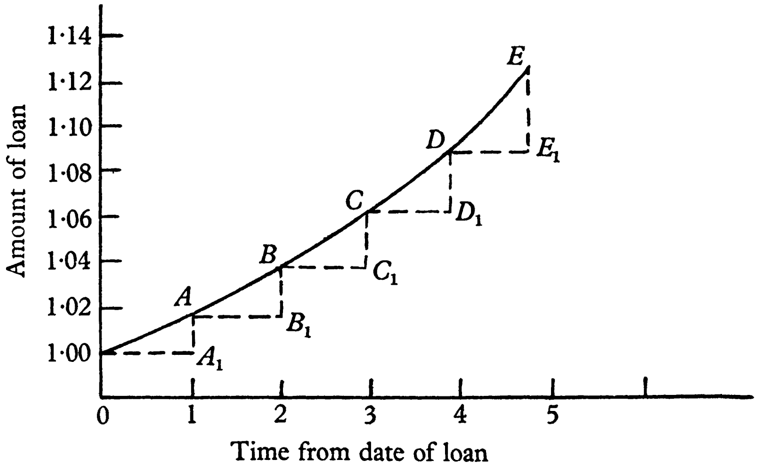

Fig. 1.

obvious method would be to measure its length at the start of some interval of time and to measure it again at the end of the interval. The difference expressed per unit length of the original branch would then be the average rate of growth of the branch for that given period. It is clear that this rate is the result of the continuous process by which the branch grows, and that by reducing the time interval over which the length of the branch is measured it would be possible to approximate more and more closely to the instantaneous rate of growth. It should also be clear that the instantaneous rate is the fundamental rate, and that a rate found by measurement for any other period of time is only a way of expressing the results of the instantaneous rate by relating the effect of its operation to a given interval of time. The same is true of principal accumulating at compound interest.

1·5. Application of the calculus

Still considering the natural process of a branch of a tree growing, let the initial length be $f(0)$ and the length after time t be $f(t)$. Then the amount by which the length of the branch increases between time t and time $t+h$ is $f(t+h)-f(t)$. The rate of growth per unit interval of time is therefore $f(t+h)-f(t)/h$, and the rate of growth for each unit length of $f(t)$ is $\dfrac{f(t+h)-f(t)}{hf(t)}$. The instantaneous rate of growth of each unit length at time t is therefore

$$\begin{aligned}\lim_{h\to 0}\frac{f(t+h)-f(t)}{hf(t)} &= \frac{1}{f(t)}\frac{d}{dt}f(t)\\ &= \frac{d}{dt}\log_e f(t)\\ &= \delta_t \text{ say.}\end{aligned}$$

Hence by integrating between the limits 0 and t,

$$\log_e f(t) = \int_0^t \delta_t dt + \log f(0)$$

or

$$f(t) = f(0)\exp\left[\int_0^t \delta_t dt\right].$$

1·6. Numerical example based on natural growth

Suppose that the initial length of the branch is 10 ft. and that it is found to be 10 ft. 6 in. after 1 year. The average rate of growth is therefore 6 in. per 10 ft. per annum, i.e. 5% per annum. If the branch grows continuously then the continuous rate of growth per annum is δ_t, where

$$f(1) = 10{\cdot}5 = f(0)\exp\left[\int_0^1 \delta_t dt\right].$$

If it is assumed that the continuous rate of growth is δ, where δ is constant throughout the period, then

$$\begin{aligned}\delta &= \log_e 1{\cdot}05\\ &= {\cdot}04879,\end{aligned}$$

i.e. the statement that the branch grows 6 in. in 1 year is merely expressing the result in 1 year of the effect of a constant continuous rate of growth of 4·879 % per annum. For this reason the statement might be made that the effective rate of growth is 5 % per annum, and this statement gives without further calculation the information that the increase which would be found in the branch if it were measured after 1 year would be 5 % of 10 ft., or 6 in.

1·7. Consideration of intermediate points

If it were desired, given that the constant instantaneous rate of growth is 4·879 % per annum, to find how much the branch would grow in 6 months, the process would be

$$\begin{aligned} f(\tfrac{1}{2}) &= f(0)\exp\left[\int_0^{\frac{1}{2}} \cdot 04879\,dt\right] \\ &= 10\,e^{\cdot 02439} \\ &= 10\cdot 247 \text{ ft.} \end{aligned}$$

The increase would be ·247 ft., or 2·964 in. This is of course less than half the increase for the year because the 2·964 in. which have grown in the first 6 months will themselves grow further in the second 6 months and will make up the total growth to 6 in. To investigate the result of this let it be assumed:

(*a*) that, if the increase in length at any time is cut off, the original branch will grow at the same rate as before;

(*b*) that, if the increase is planted, it will grow at the same rate as the parent stem, but, if it is not planted, it is unable to grow.

Thus if, after 6 months, the increase in length of 2·964 in. is removed, the original 10 ft. of branch will grow a further 2·964 in. as before in the next 6 months. If the portion cut off is replanted it will increase, as before, to

$$\begin{aligned} 2\cdot 964\exp\left[\int_0^{\frac{1}{2}} \cdot 04879\,dt\right] &= 2\cdot 964 \times 1\cdot 0247 \\ &= 3\cdot 036 \text{ in.} \end{aligned}$$

The total growth is therefore $2\cdot 964 + 3\cdot 036 = 6$ in., as before.

In other words the effect of a continuous rate of growth of 4·879 % per annum could be stated either as an actual increase of 5 % per annum, or as 2·47 % per 6 months. Now if the amount of the increase after 6 months had not been replanted, the increase in the second 6 months would have been only 2·964 in., and the total growth would have been 5·928 in., or at the rate of 4·94 %. This assumption, however, implies that at the end of 6 months the increase in the length of the branch is cut off and prevented from growing further. If it is left to follow its own natural processes it will grow and the effect of the given rate will still be to add 6 in. to the length of the branch in 1 year.

1·8. Application to compound interest

Exactly the same arguments hold good for a sum of money accumulating at interest. It so happens that because most financial transactions are viewed on a year-to-year basis, the most common way of stating the rate of growth is as a percentage per annum, i.e. it is more common to state the effect in the course of a year of a continuous rate of growth than it is to state the continuous rate of growth itself. But this should not obscure the fact that the basic force giving rise to the effect is the continuous rate of growth. It is this function which lies at the root of all the theory of compound interest and the natural emphasis laid in practice on the expression of rates of interest as yearly rates must not be allowed to obscure this fundamental fact. In actuarial language the continuous rate at which a sum of money grows under the operation of interest is called the 'force of interest' and the increase per unit due to the effect of the operation of this force of interest in any given period is called the 'effective rate of interest' for that period. The relations between these, and other rates of interest, will be discussed in the following chapter.

CHAPTER 2

DEFINITIONS AND ELEMENTARY PROPOSITIONS

2·1. In Chapter 1 the general theory of compound interest was stated as the continuous growth of money under the operation of a force of interest. It is necessary now to consider the application of this theory to the problems which arise in practice. Normally, rates of interest are stated in terms of an interval of one year, or sometimes a submultiple of a year. They may be true effective rates for the period stated, or, as has been mentioned in § 1·2, they may be rates per annum in name only. There will be no difficulty in understanding these differences if the stated rate is always related to the underlying force of interest.

2·2. Effective rate of interest

The effective rate of interest during any period has already been defined as the actual rate of increase per unit invested during that time. In Chapter 1 it was seen that the effect in 1 year of a continuous rate of increase of 4·879 % during the period was to increase the length of the branch by 5 %, or in other words to add 5 % to the value of a unit invested at the beginning of the period. In this case the effective rate of interest is 5 % per annum. It was also seen that the effect of the same rate of continuous increase was to add 2·47 % after 6 months to the value of a unit invested at the start of the 6 months. The effective rate of interest might be stated in this case as 2·47 % per half-year.

2·3. Nominal rate of interest

As has been mentioned in Chapter 1, the common financial practice is to express rates of interest as rates per annum, and sometimes a rate is stated as so much per annum, even though the interest is paid more frequently than once per year. Thus an effective rate of 2·47 % per half-year might be expressed as a rate of 4·94 % per annum, payable half-yearly. But in §2·2 it was shown that the effect in 1 year of an effective rate of interest of 2·47 % per half-year was to add 5 % to the value of a unit invested. The rate stated as 4·94 % per annum is therefore not an effective rate. It is a rate of interest in name only, and for this reason it is called a 'nominal' rate of interest. It merely expresses *the total interest which is payable in a year on a unit invested at the beginning of the year assuming that any interest paid during the year is not reinvested.* To complete the definition the number of times the interest is payable during the year must be stated. This is done by employing one of the following phrases which have identical meanings:

payable half-yearly, quarterly, monthly or, generally, m times per annum;

or convertible half-yearly, quarterly, monthly or, generally, m times per annum;

or payable with half-yearly, quarterly, monthly or, generally, $1/m$thly rests.

If the number of times a year that interest is convertible is increased indefinitely the nominal rate becomes the annual rate of continuous growth, which, as already mentioned, is known as the 'force of interest'.

2·4. Analogy with Chapter 1

It is now possible to translate the results of Chapter 1 into terms of rates of interest. The continuous rate of increase corresponds to the force of interest. The amount by which the branch grows in a given period under the operation of the continuous rate for that period corresponds to the effective rate of interest for the period concerned. A nominal rate of interest payable more frequently than once during the given period corresponds to the result of cutting off the amount by which the branch has grown after part of the period and not replanting it. This process is repeated after each part of the period, and the sum of the measurements of the portions thus detached and not replanted would correspond to the nominal rate of interest for the period concerned. If these portions had been free to grow, the total at the end of the period would have been the same as the effective rate of interest for that period.

2·5. Symbolic representation of rates of interest

The generally accepted symbols for the rates of interest are

(*a*) The force of interest per annum is represented by δ.

(*b*) An effective rate of interest per annum is represented by i.

(*c*) A nominal rate of interest per annum payable m times a year is represented by $i^{(m)}$.*

2·6. Relations between i, $i^{(m)}$ and δ

By definition

$$\delta = \lim_{h\to 0} \frac{1}{h} \frac{f(t+h)-f(t)}{f(t)},$$

where $f(t)$ is the amount of a unit of principal at time t.

$$\therefore\ \delta = \frac{1}{f(t)} \frac{d}{dt} f(t)$$

$$= \frac{d}{dt} \log_e f(t),$$

$$\therefore\ \log_e f(1) = \int_0^1 \delta\, dt$$

$$= \delta.$$

But the amount of a unit after one year at an effective rate of i per annum is clearly $(1+i)$, since the original unit is intact and the interest paid at the end of a year is i.

$$\therefore\ \delta = \log_e(1+i) \qquad (2·1)$$

or

$$i = e^{\delta} - 1. \qquad (2·2)$$

Also

$$\log_e f\left(\frac{1}{m}\right) = \int_0^{1/m} \delta\, dt$$

$$= \frac{1}{m}\delta$$

$$= \frac{1}{m}\log_e(1+i).$$

$$\therefore\ f\left(\frac{1}{m}\right) = (1+i)^{(1/m)}.$$

* In older actuarial works the symbol used is $j_{(m)}$.

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But the amount of 1 after $1/m$th of a year at a nominal rate of $i^{(m)}$ is also $1+\frac{i^{(m)}}{m}$, since the effective rate of interest is $\frac{i^{(m)}}{m}$ per interval.

$$\therefore \quad \left(1+\frac{i^{(m)}}{m}\right)=(1+i)^{1/m}, \tag{2·3}$$

or

$$i=\left(1+\frac{i^{(m)}}{m}\right)^m-1. \tag{2·4}$$

Equation (2·4) merely expresses symbolically what should be apparent from general reasoning. If a unit is invested at a nominal rate $i^{(m)}$ per annum the interest received after $1/m$th of a year is $\frac{i^{(m)}}{m}$ and, if this is invested also, the capital becomes $\left(1+\frac{i^{(m)}}{m}\right)$. In the next interval the interest is $\frac{i^{(m)}}{m}\left(1+\frac{i^{(m)}}{m}\right)$ and the amount of the capital becomes

$$\left(1+\frac{i^{(m)}}{m}\right)+\frac{i^{(m)}}{m}\left(1+\frac{i^{(m)}}{m}\right)=\left(1+\frac{i^{(m)}}{m}\right)^2.$$

In the next interval the interest will be $\frac{i^{(m)}}{m}\left(1+\frac{i^{(m)}}{m}\right)^2$, and the amount of the capital at the end of the interval will be

$$\left(1+\frac{i^{(m)}}{m}\right)^2+\frac{i^{(m)}}{m}\left(1+\frac{i^{(m)}}{m}\right)^2=\left(1+\frac{i^{(m)}}{m}\right)^3.$$

After m intervals, i.e. after 1 year, the amount of the capital is similarly $\left(1+\frac{i^{(m)}}{m}\right)^m$, and the total interest earned in the year, which by definition is the effective rate of interest per annum, is therefore

$$\left(1+\frac{i^{(m)}}{m}\right)^m-1.$$

Equation (2·2) could also be obtained by algebraic means as follows:

$$\begin{aligned} i &= \left(1+\frac{i^{(m)}}{m}\right)^m-1 \\ &= 1+i^{(m)}+\frac{m(m-1)}{2!}\left(\frac{i^{(m)}}{m}\right)^2+\ldots-1. \end{aligned}$$