CHAPTER I.

PLANES AND LINES.

The student will be familiar with the notion of a **plane**, of which we shall not offer a definition. The ordinary test for planeness is to apply a straight-edge to the surface in every direction, and see if it always fits. This test depends on the obvious fact that **if two points lie in a plane, the straight line joining them lies entirely in that plane.**

Ex. 1. How would the above fact be of use in testing a surface that is slightly convex?

Determination of a plane. It will be remembered that a straight line is fixed, or determined, by 2 suitable conditions. Thus it may be required to pass through two points, or to pass through a point and lie in a given direction.

We shall see that 3 suitable conditions are needed to determine a plane.

First, fix one point A of a plane (e.g. by placing the plane in contact with the corner of a table). The plane can now take up any direction, or **orientation**, in space. Fix a second point B: the plane can now turn about AB as a line of hinges. Fix a third point C, and the plane is deprived of all freedom of motion. The plane has thus been fixed by 3 conditions.

By experimenting with a sheet of cardboard, it will become clear that the above is only one of many ways of fixing a plane by suitable conditions, e.g.:

A plane is determined uniquely if it is required

(i) **to pass through three given points;**

(ii) **to contain a given straight line and pass through a given point not on the line;**

(iii) **to contain two intersecting straight lines (or lines which intersect if produced)*;**

(iv) **to contain two parallel straight lines (i.e. two straight lines intersecting at infinity).**

Ex. 2. Try to make a plane contain two straight lines not intersecting nor parallel.

Ex. 3. Why does a three-legged table stand more steadily on uneven ground than a four-legged one?

Ex. 4. If each of three straight lines meets the other two, the three lines are either concurrent or coplanar.

Generation of a plane. A plane may be swept out, or 'generated'

(i) by a straight line passing through a fixed point and sliding on a fixed straight line, not containing the point;

(ii) by a straight line sliding on two fixed intersecting straight lines;

(iii) by a straight line sliding on two fixed parallel straight lines;

(iv) by a straight line sliding on a fixed straight line, and remaining in a fixed direction (or, moving parallel to itself). (This is really a particular case of (i), the fixed point being at infinity.)

Two planes intersect, in general, in a straight line; in other words, they determine a straight line.

Associate this with the fact that two points determine a straight line.

* In future it will be assumed that lines and planes are infinite, unless it is stated that a finite portion is under consideration.

If two planes have no finite point in common, they are **parallel**. Mathematically, their line of intersection is a **line at infinity**; a set of parallel planes determine one line at infinity.

Straight line and plane. In general, a straight line intersects a plane in a point (illustrate with a stick and the floor, etc.); that is to say, a straight line and a plane determine a point.

Compare this with the fact that a straight line and a point, in general, determine a plane.

If a straight line and a plane have no finite point in common, they are **parallel**. Mathematically, they have in common the point at infinity on the line.

Ex. 5. Hold a pencil parallel to a sloping desk, or book. Note that it need not be horizontal. Can it be horizontal? Can it be vertical?

Ex. 6. How many lines can there be through a given point parallel to a given plane? What can be asserted about the whole set of such lines?

Ex. 7. What is generated by a line passing through a given point, and moving so as to remain parallel to a given plane?

*Ex. 8. What is the locus of the intersection of a fixed plane, and a line through a fixed point intersecting a fixed line?

Ex. 9. What is the locus of the intersection of a fixed plane, and a line constant in direction and intersecting a fixed line?

Two straight lines. In general, two straight lines in space are not in the same plane, and do not intersect; such lines are said to be **skew** to one another.

If they happen to be in the same plane, they either intersect or are parallel.

Note carefully the distinction between skew and parallel lines. It has been pointed out in the course of plane geometry that parallel lines must be in the same plane. Parallel lines must also be in the same direction; skew lines cannot be so.

* In dealing with a mathematical problem, the student is strongly recommended not to pass on till he has examined the particular cases that may arise: e.g. in the present problem he should consider the case of the line being parallel to the plane.

Ex. 10. Find examples of (i) a pair of skew lines, (ii) a pair of parallel lines (not drawn on paper).

Ex. 11. How many lines are there in space through a given point and parallel to a given line?

Ex. 12. What can be asserted about the whole set of lines passing through a given point and intersecting a given line?

Ex. 13. Are *any* two horizontal lines necessarily parallel? Are *any* two vertical lines necessarily parallel?

Ex. 14. Find two horizontal lines skew to one another.

Ex. 15. Prove that, among the lines meeting two skew lines, it is impossible to find a pair which intersect or are parallel.

Ex. 16. Determine a line intersecting two skew lines and parallel to a third line.

Ex. 17. Given four straight lines of which two are parallel, one straight line can be drawn to intersect the four.

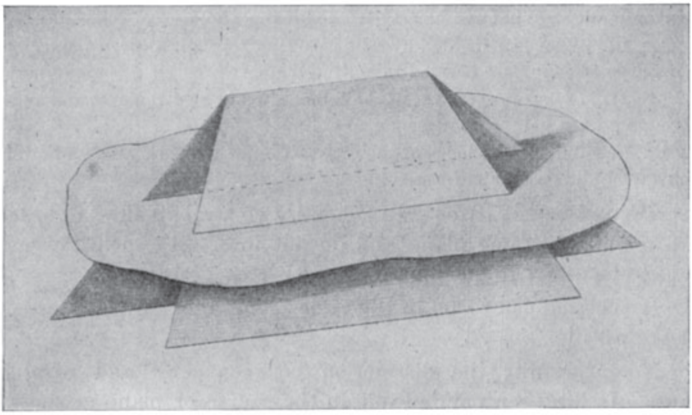

fig. 1.

Three planes. In general, any two planes have a line in common; this line intersects a third plane in a point. Three planes therefore determine a point (e.g. two meeting walls of a room, and the ceiling).

Compare these two facts—in general 3 planes determine 1 point; 3 points determine 1 plane.

Particular case (i). It may happen that the line of intersection of two of the planes is parallel to the third plane. The three planes will then have no point in common; see fig. 1. (Compare the three side-faces of a triangular prism; the three planes of a folding screen; the two slopes of a roof, and the ground plane.)

Notice that in this case the line common to *any* two of the planes is parallel to the third; the whole system will consist of three planes, and three parallel lines.

This case is derived from the general case by making the common point of the three planes retreat to infinity.

Particular case (ii). Two of the planes may be parallel. The third plane will cut these in two parallel lines, and there will be no point common to the three planes; see fig. 2. (Compare two parallel walls and the floor.)

This is a particular case of (i).

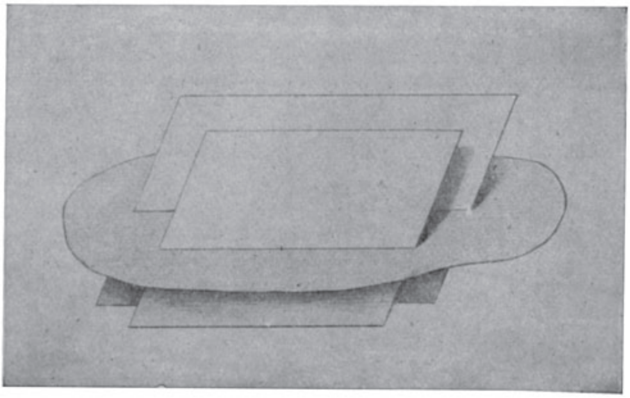

fig. 2.

Particular case (iii). The common lines of each pair of planes may coincide; we shall then have three planes passing through the same line (like three leaves of a partly open book). Here the three planes have an infinite number of common points, lying on a line.

Just as 3 points *may* lie in a line, so 3 planes *may* pass through a line.

Particular case (iv). The three planes may be parallel (like the various floors of a building).

This is a particular case of (ii); also of (iii), for the three planes have in common a line at infinity.

Ex. **18.** How are the planes disposed that are drawn through a point to contain a set (i) of concurrent coplanar lines? (ii) of concurrent lines, not coplanar? (iii) of parallel coplanar lines? (iv) of parallel lines, not coplanar?

CHAPTER II.

PARALLEL POSITIONS OF PLANES AND LINES.

The reader should convince himself of the truth of the following statements, by placing books, pencils, etc. to represent planes and lines. The truth of a proposition is usually more evident if one of the planes in question is taken to be horizontal; most of the planes that we are concerned with in experience are horizontal or vertical.

(i) Planes parallel to the same plane are parallel to one another.

Ex. 19. What can be asserted about planes perpendicular to the same plane? about planes perpendicular to the same line? about planes parallel to the same line?

(ii) Straight lines parallel to the same straight line are parallel to one another.

Here note that the three lines need not be in the same plane, e.g. three edges of a triangular file. This property will be recognised as an extension of a property in plane geometry.

Ex. **20.** What can be asserted about lines parallel to the same plane? about lines perpendicular to the same plane?

Ex. **21.** Prove that the shadows of a number of vertical sticks, thrown on the ceiling by a candle, are a set of concurrent lines.

(iii) **A straight line parallel to a straight line lying in a certain plane is parallel to the plane.**

Ex. **22.** How many straight lines can there be through a point parallel to each of two intersecting planes?

Ex. **23.** In what case can there be one line parallel to each of three planes?

(iv) **If a plane α pass through a line l which is parallel to a plane β, the line of intersection of α and β will be parallel to l.**

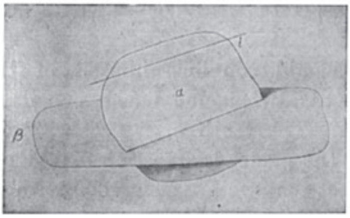

fig. 3.

Think of α as the sloping top of a desk, l the line of hinges parallel to β the ground-plane.

†Ex. **24.** Prove this by *reductio ad absurdum.*

Ex. **25.** In what case is a stick parallel to its shadow on the ground?

Ex. **26.** If a plane α pass through a line l which is perpendicular to a plane β, what is the line of intersection of α and β?

Ex. **27.** Let a line l meet a plane β obliquely, let α be a moving plane passing through l, and let m be the line of intersection of α and β. Prove that the different positions of m form a set of lines passing through a fixed point. Can any position of m be (i) parallel, (ii) perpendicular, to l?

Ex. **28.** A line moves parallel to a given line so as always to intersect a given line; what does it generate?

Ex. **29.** A set of planes are parallel to a given line. What is the arrangement of the intersections of these planes?

Ex. **30.** In a given plane, and through a given point in that plane, determine a line (i) to meet a given line (not in the plane), (ii) parallel to a given plane.

Ex. **31.** Through a given point determine a line parallel to a given plane and meeting a given line.

Ex. **32.** Through a moving point on a fixed line are constructed pairs of planes parallel to two fixed planes. What is the locus of the intersection of the planes so constructed?

(v) **Two parallel planes are cut by a third plane in parallel lines.**

(Geological strata generally lie in planes, over small areas. In such cases, their sections are seen as parallel lines on the sides of a railway cutting, the face of a cliff, etc.)

†Ex. **33.** Prove this by *reductio ad absurdum.*

Ex. **34.** What would be the shape of the section exposed by a slanting cut across a wooden plank?

(vi) **If intersecting lines l', m' are drawn respectively parallel to intersecting lines l, m, the plane containing l', m' is parallel to the plane containing l, m.**

(Draw l, m on a sheet of paper; and hold pencils to represent l', m'.)

(vii) **If intersecting lines l', m' are drawn respectively parallel to intersecting lines l, m, the angles between l', m' are equal to the angles between l, m.**

This will be recognised as an extension of a theorem in plane geometry.

(viii) **If straight lines are cut by parallel planes, they are cut in the same ratio.**

Data The straight lines AEB, CFD (fig. 4) are cut by the parallel planes α, β, γ (note that AEB, CFD are not necessarily in the same plane).

To prove that AE : EB = CF : FD.

Proof Let AD meet the plane β in X.

Join AC, BD, EX, XF.

The plane ABD cuts || planes β, γ in || lines EX, BD

$\therefore$ AE : EB = AX : XD.

Similarly, CF : FD = AX : XD.

$\therefore$ AE : EB = CF : FD.

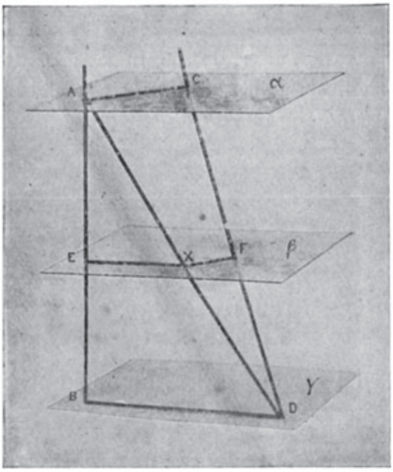

fig. 4.

Ex. **35.** In what case are E, X, F in a straight line? (fig. 4.)

Ex. **36.** In what case are AC and BD parallel? (fig. 4.)

†Ex. **37.** Points A, B, C, D in a plane are joined to a point O not in the plane. OA, OB, OC, OD are divided in the same ratio at A′, B′, C′, D′. Prove that A′B′C′D′ is a plane quadrilateral similar to ABCD. (Let plane A′B′C′ cut OD in X; shew that X coincides with D′.)

Ex. **38.** A stick 1 foot long is placed inside a box 10 inches high, touching the top and bottom. Prove that the locus of any point on the stick is a plane.