INTRODUCTION

VECTOR NOTATION AND FORMULAE

SINCE elementary vector methods are freely employed throughout this book, some space may be given at the outset to an explanation of the notation used and the formulae required*. Vectors are denoted by **Clarendon** symbols†. The *position vector* $\mathbf{r}$, of a point P relative to the origin O, is the vector whose magnitude is the length OP, and whose direction is from O to P. If x, y, z are the coordinates of P relative to rectangular axes through O, it is frequently convenient to write

$$\mathbf{r} = (x, y, z),$$

x, y, z being the resolved parts of $\mathbf{r}$ in the directions of the coordinate axes. The point, whose position vector is $\mathbf{r}$, is referred to as "the point $\mathbf{r}$." If $\mathbf{n}$ is a *unit* vector, that is to say a vector of unit length, and if

$$\mathbf{n} = (l, m, n),$$

then l, m, n are the *direction cosines* of $\mathbf{n}$. The *module* or *modulus* of a vector is the positive number which is the measure of its length.

The law of **vector addition** is a matter of common knowledge. If three points O, P, Q are such that the vectors OP and PQ are equal respectively to $\mathbf{a}$ and $\mathbf{b}$, the vector OQ is called the *sum* of $\mathbf{a}$ and $\mathbf{b}$, and is denoted by $\mathbf{a} + \mathbf{b}$. The negative of the vector $\mathbf{b}$ is a vector with the same modulus but the opposite direction. It is denoted by $-\mathbf{b}$. The *difference* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the sum of $\mathbf{a}$ and $-\mathbf{b}$. We write it

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

* For proofs of the various formulae the reader is referred to the author's *Elementary Vector Analysis* (G. Bell & Sons), of which Arts. 1—8, 12, 15—17, 23—29, 42—46, 49—51, 55—57 would constitute a helpful companion course of reading. (References are to the old edition.)

† In MS. work Greek letters and script capitals will be found convenient.

The commutative and associative laws hold for the addition of any number of vectors. Also the general laws of association and distribution for scalar multipliers hold as in ordinary algebra. Thus if p and q are scalar multipliers,

$$p\,(q\mathbf{r}) = pq\mathbf{r} = q\,(p\mathbf{r}),$$

$$(p+q)\,\mathbf{r} = p\mathbf{r} + q\mathbf{r},$$

$$p\,(\mathbf{r}+\mathbf{s}) = p\mathbf{r} + p\mathbf{s}.$$

If $\mathbf{r}$ is the position vector of any point on the straight line through the point $\mathbf{a}$ parallel to the vector $\mathbf{b}$, then

$$\mathbf{r} = \mathbf{a} + t\mathbf{b},$$

where t is a number, positive or negative. This equation is called the vector equation of the straight line.

Products of Vectors

If $\mathbf{a}$, $\mathbf{b}$ are two vectors whose moduli are a, b and whose directions are inclined at an angle θ, the **scalar product** of the vectors is the number $ab\cos\theta$. It is written $\mathbf{a}\cdot\mathbf{b}$. Thus

$$\mathbf{a}\cdot\mathbf{b} = ab\cos\theta = \mathbf{b}\cdot\mathbf{a}.$$

Hence *the necessary and sufficient condition that two vectors be perpendicular is that their scalar product vanish.*

If the two factors of a scalar product are equal, the product is called the *square* of either factor. Thus $\mathbf{a}\cdot\mathbf{a}$ is the square of $\mathbf{a}$, and is written $\mathbf{a}^2$. Hence

$$\mathbf{a}^2 = \mathbf{a}\cdot\mathbf{a} = a^2,$$

so that the square of a vector is equal to the square of its modulus.

If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then $\mathbf{a}\cdot\mathbf{b} = \cos\theta$. Also the resolved part of any vector $\mathbf{r}$, in the direction of the unit vector $\mathbf{a}$, is equal to $\mathbf{r}\cdot\mathbf{a}$.

The *distributive law* holds* for scalar products. Thus

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}+\ldots) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} + \ldots,$$

and so on. Hence, in particular,

$$(\mathbf{a}+\mathbf{b})^2 = \mathbf{a}^2 + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}^2,$$

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2.$$

* *Elem. Vect. Anal.*, Art. 26.

Also, if we write

$$\mathbf{a} = (a_1, a_2, a_3),$$
$$\mathbf{b} = (b_1, b_2, b_3),$$

the coordinate axes being rectangular, we have

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and

$$\mathbf{a}^2 = a_1^2 + a_2^2 + a_3^2.$$

The last two formulae are of constant application.

The unit vector $\mathbf{n}$ perpendicular to a given plane is called its *unit normal.* If $\mathbf{r}$ is any point on the plane, $\mathbf{r} \cdot \mathbf{n}$ is the projection of $\mathbf{r}$ on the normal, and is therefore equal to the perpendicular p from the origin to the plane. The equation

$$\mathbf{r} \cdot \mathbf{n} = p$$

is therefore one form of the *equation of the plane.* If $\mathbf{a}$ is any other point on the plane, then $\mathbf{a} \cdot \mathbf{n} = p$, and therefore

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0.$$

This is another form of the equation of the plane, putting in evidence the fact that the line joining two points $\mathbf{r}$ and $\mathbf{a}$ in the plane is perpendicular to the normal.

The positive sense for a *rotation* about a vector is that which bears to the direction of the vector the same relation that the sense of the rotation of a *right-handed screw* bears to the direction of its translation. This convention of the right-handed screw plays an important part in the following pages.

Let OA, OB be two intersecting straight lines whose directions

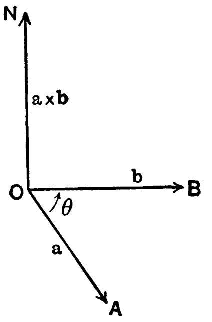

Fig. A.

are those of the two vectors $\mathbf{a}$, $\mathbf{b}$, and let ON be normal to the plane OAB. By choosing one direction along this normal as positive we fix the sense of the rotation about ON which must be regarded as positive. Let θ be the angle of rotation from OA to OB in this positive sense. Then if a, b are the moduli of $\mathbf{a}$, $\mathbf{b}$ the **vector product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector $ab \sin\theta\,\mathbf{n}$, where $\mathbf{n}$ is the unit vector in the positive direction along the normal. This is denoted by $\mathbf{a}\times\mathbf{b}$, and is often called the *cross product* of $\mathbf{a}$ and $\mathbf{b}$. Thus

$$\mathbf{a}\times\mathbf{b} = ab\sin\theta\,\mathbf{n}.$$

It should be noticed that the result is independent of the choice of positive direction along the normal. For, if the opposite direction is taken as positive, the direction of $\mathbf{n}$ is reversed, and at the same time θ is replaced by $-\theta$ or $2\pi-\theta$, so that $ab\sin\theta\,\mathbf{n}$ remains unaltered. Hence the vector product $\mathbf{a}\times\mathbf{b}$ is a definite vector.

It is important, however, to notice that $\mathbf{b}\times\mathbf{a}$ is the negative of $\mathbf{a}\times\mathbf{b}$. For, with the above notation, the angle of rotation from OB to OA in the positive sense is $2\pi-\theta$, so that

$$\mathbf{b}\times\mathbf{a} = ab\sin(2\pi-\theta)\,\mathbf{n} = -\mathbf{a}\times\mathbf{b}.$$

Thus the order of the factors in a cross product cannot be changed without altering the sign of the product.

If $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\sin\theta = 0$, and the cross product vanishes. Hence *the necessary and sufficient condition for parallelism of two vectors is that their cross product vanish.*

A *right-handed system* of mutually perpendicular *unit* vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ (Fig. 3, Art. 3) is such that

$$\mathbf{t} = \mathbf{n}\times\mathbf{b}, \quad \mathbf{n} = \mathbf{b}\times\mathbf{t}, \quad \mathbf{b} = \mathbf{t}\times\mathbf{n},$$

the cyclic order of the factors being preserved throughout. We shall always choose a right-handed system of rectangular coordinate axes, so that unit vectors in the directions OX, OY, OZ satisfy the above relations.

The *distributive law* holds* also for vector products; but the order of the factors in any term must not be altered. Thus

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}+\dots) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} + \dots$$

and $$(\mathbf{b}+\mathbf{c}+\dots)\times\mathbf{a} = \mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{a} + \dots.$$

And if we write $$\mathbf{a} = (a_1, a_2, a_3),$$
$$\mathbf{b} = (b_1, b_2, b_3),$$

* *Elem. Vect. Anal.*, Art. 28.

then, in virtue of the distributive law, and the fact that the coordinate axes form a right-handed system, we have

$$\mathbf{a}\times\mathbf{b}=(a_2b_3-a_3b_2,\ a_3b_1-a_1b_3,\ a_1b_2-a_2b_1).$$

This formula should be carefully remembered.

If a vector **d** is localised in a line through the point whose position vector is **r** relative to O, the *moment* of **d** about O is the vector $\mathbf{r}\times\mathbf{d}$. Thus the moment of a vector *about a point* is a vector, sometimes called its "vector moment." It will, however, be seen shortly that the moment of **d** about an axis is a scalar quantity.

The **scalar triple product** $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$ is the scalar product of **a** and $\mathbf{b}\times\mathbf{c}$. Except as to sign it is numerically equal to the volume of the parallelepiped whose edges are determined by the three vectors*. Its value is unaltered by interchanging the dot and the cross, or by altering the order of the factors, provided the same cyclic order is maintained. Thus

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\times\mathbf{a}\cdot\mathbf{b},$$

and so on. The product is generally denoted by

$$[\mathbf{a},\mathbf{b},\mathbf{c}],$$

a notation which indicates the three vectors involved as well as their cyclic order If the cyclic order of the factors is altered, the sign of the product is changed. Thus

$$[\mathbf{a},\mathbf{c},\mathbf{b}]=-[\mathbf{a},\mathbf{b},\mathbf{c}].$$

In terms of the resolved parts of the three vectors, the scalar triple product is given by the determinant

$$[\mathbf{a},\mathbf{b}\ \mathbf{c}]=\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.$$

It is also clear that, if the three vectors **a**, **b**, **c** are coplanar, $[\mathbf{a},\mathbf{b},\mathbf{c}]=0$; and conversely. Thus *the necessary and sufficient condition that three vectors be coplanar is that their scalar triple product vanish.*

If one of the factors consists of a sum of vectors, the product may be expanded according to the distributive law. Thus

$$[\mathbf{a},\mathbf{b},\mathbf{c}+\mathbf{d}]=[\mathbf{a},\mathbf{b},\mathbf{c}]+[\mathbf{a},\mathbf{b},\mathbf{d}],$$

and similarly if two or all of the factors consist of vector sums.

* *Elem. Vect. Anal.*, Art. 43.

The **vector triple product** $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is the vector product of $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$. It is a vector parallel to the plane of $\mathbf{b}$ and $\mathbf{c}$, and its value is given by*

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \mathbf{c}\ \mathbf{b} - \mathbf{a} \cdot \mathbf{b}\ \mathbf{c}.$$

Similarly
$$(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = \mathbf{b} \cdot \mathbf{a}\ \mathbf{c} - \mathbf{c} \cdot \mathbf{a}\ \mathbf{b}.$$

Both of these expansions are written down by the same rule. Each scalar product in the expansion contains the factor outside the brackets, and the first is the scalar product of the extremes.

The *scalar* product of **four vectors**, $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$, is the scalar product of $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$. It may be expanded† as

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{a} \cdot \mathbf{c}\ \mathbf{b} \cdot \mathbf{d} - \mathbf{a} \cdot \mathbf{d}\ \mathbf{b} \cdot \mathbf{c}.$$

The *vector* product of four vectors, $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$, may be expanded in terms either of $\mathbf{a}$ and $\mathbf{b}$ or of $\mathbf{c}$ and $\mathbf{d}$. Thus‡

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= [\mathbf{a}, \mathbf{c}, \mathbf{d}]\ \mathbf{b} - [\mathbf{b}, \mathbf{c}, \mathbf{d}]\ \mathbf{a} \\ &= [\mathbf{a}, \mathbf{b}, \mathbf{d}]\ \mathbf{c} - [\mathbf{a}, \mathbf{b}, \mathbf{c}]\ \mathbf{d}.\end{aligned}$$

On equating these two expressions for the product we see that *any vector* $\mathbf{d}$ *is expressible in terms of any three non-coplanar vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ by the formula

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]\ \mathbf{d} = [\mathbf{d}, \mathbf{b}, \mathbf{c}]\ \mathbf{a} + [\mathbf{d}, \mathbf{c}, \mathbf{a}]\ \mathbf{b} + [\mathbf{d}, \mathbf{a}, \mathbf{b}]\ \mathbf{c}.$$

If a vector $\mathbf{d}$ is localised in a line through the point $\mathbf{r}$, its *moment about an axis* through the origin O, parallel to the *unit* vector $\mathbf{a}$, is the resolved part in this direction of its vector moment about O. It is therefore equal to

$$M = \mathbf{a} \cdot \mathbf{r} \times \mathbf{d} = [\mathbf{a}, \mathbf{r}, \mathbf{d}].$$

Thus the moment of a vector about an axis is a scalar quantity.

The *mutual moment* of the two straight lines

$$\mathbf{r} = \mathbf{a} + t\mathbf{b},$$
$$\mathbf{r} = \mathbf{a}' + t\mathbf{b}',$$

with the positive senses of the *unit* vectors $\mathbf{b}$ and $\mathbf{b}'$ respectively, is the moment about either line of the unit vector localised in the other. Thus, being the moment about the second line of the unit vector $\mathbf{b}$ localised in the first, it is given by

$$\begin{aligned}M &= \mathbf{b}' \cdot (\mathbf{a} - \mathbf{a}') \times \mathbf{b} \\ &= [\mathbf{a} - \mathbf{a}', \mathbf{b}, \mathbf{b}'].\end{aligned}$$

The *condition of intersection* of two straight lines is therefore

$$[\mathbf{a} - \mathbf{a}', \mathbf{b}, \mathbf{b}'] = 0.$$

* *Elem. Vect. Anal.*, Art. 44. † *Ibid.*, Art. 45. ‡ *Ibid.*, Art. 46.

This is also obvious from the fact that the two given lines are then coplanar with the line joining the points **a**, **a**′, so that the vectors **b**, **b**′, **a** − **a**′ are coplanar.

Differentiation of Vectors

Let the vector **r** be a function of the scalar variable s, and let $\delta\mathbf{r}$ be the increment in the vector corresponding to the increment δs in the scalar. In general the direction of $\delta\mathbf{r}$ is different from that of **r**. The limiting value of the vector $\delta\mathbf{r}/\delta s$, as δs tends to zero, is called the *derivative* of **r** with respect to s and is written

$$\frac{d\mathbf{r}}{ds} = \underset{\delta s \to 0}{\mathrm{Lt}} \frac{\delta \mathbf{r}}{\delta s}.$$

When the scalar variable s is the arc-length of the curve traced out by the point whose position vector is **r**, the derivative is frequently denoted by **r**′. Its direction is that of the tangent to the curve at the point considered (Fig. 1, Art. 1).

The derivative is usually itself a function of the scalar variable. Its derivative is called the *second derivative* of **r** with respect to s, and is written

$$\frac{d}{ds}\left(\frac{d\mathbf{r}}{ds}\right) = \frac{d^2\mathbf{r}}{ds^2} = \mathbf{r}'',$$

and so on for derivatives of higher order. If

$$\mathbf{r} = (x, y, z),$$

then clearly
$$\mathbf{r}' = (x', y', z')$$
and
$$\mathbf{r}'' = (x'', y'', z'').$$

If s is a function of another scalar variable t, then, as usual,

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt}.$$

The ordinary rules of differentiation hold for sums and products of vectors*. Thus

$$\frac{d}{dt}(\mathbf{r} + \mathbf{s} + \ldots) = \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{s}}{dt} + \ldots,$$

$$\frac{d}{dt}(\mathbf{r} \cdot \mathbf{s}) = \frac{d\mathbf{r}}{dt} \cdot \mathbf{s} + \mathbf{r} \cdot \frac{d\mathbf{s}}{dt},$$

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{s}) = \frac{d\mathbf{r}}{dt} \times \mathbf{s} + \mathbf{r} \times \frac{d\mathbf{s}}{dt}.$$

* *Elem. Vect. Anal.*, Art. 56.

If r is the modulus of $\mathbf{r}$, then $\mathbf{r}^2 = r^2$. Hence on differentiating this formula we have

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = r\frac{dr}{dt},$$

which is an important result. In particular if $\mathbf{n}$ is a vector of constant length, but variable direction, we have

$$\mathbf{n} \cdot \frac{d\mathbf{n}}{dt} = 0.$$

Thus *a vector of constant length is perpendicular to its derivative.* This property is one of frequent application.

To differentiate a product of several vectors, differentiate each in turn, and take the sum of the products so obtained. For instance

$$\frac{d}{dt}[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \left[\frac{d\mathbf{a}}{dt}, \mathbf{b}, \mathbf{c}\right] + \left[\mathbf{a}, \frac{d\mathbf{b}}{dt}, \mathbf{c}\right] + \left[\mathbf{a}, \mathbf{b}, \frac{d\mathbf{c}}{dt}\right].$$

Suppose next that $\mathbf{r}$ is a function of *several independent variables* $u, v, w, \ldots$. Let the first variable increase from u to $u + \delta u$, while the others remain unaltered, and let $\delta\mathbf{r}$ be the corresponding increment in the vector. Then the limiting value of $\delta\mathbf{r}/\delta u$, as δu tends to zero, is called the partial derivative of $\mathbf{r}$ with respect to u, and is written $\frac{\partial \mathbf{r}}{\partial u}$. Similarly for partial derivatives with respect to the other variables.

These derivatives, being themselves functions of the same set of variables, may be again differentiated partially, yielding second order partial derivatives. We denote the derivatives of $\frac{\partial \mathbf{r}}{\partial u}$ with respect to u and v respectively by

$$\frac{\partial^2 \mathbf{r}}{\partial u^2} \text{ and } \frac{\partial^2 \mathbf{r}}{\partial u \partial v},$$

and, as in the scalar calculus,

$$\frac{\partial^2 \mathbf{r}}{\partial u \partial v} = \frac{\partial^2 \mathbf{r}}{\partial v \partial u}.$$

Also, in the notation of differentials, the *total differential* of $\mathbf{r}$ is given by the formula

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} du + \frac{\partial \mathbf{r}}{\partial v} dv + \ldots.$$

And, if $\mathbf{n}$ is a vector of constant length, $\mathbf{n}^2 = \text{const.}$, and therefore

$$\mathbf{n} \cdot d\mathbf{n} = 0.$$

Thus a vector of constant length is perpendicular to its differential

In the *geometry of surfaces*, the various quantities are usually functions of two independent variables (or parameters) u, v. Partial derivatives with respect to these are frequently indicated by the use of suffixes 1 and 2 respectively. Thus

$$\mathbf{r}_1 = \frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{r}_2 = \frac{\partial \mathbf{r}}{\partial v},$$

$$\mathbf{r}_{11} = \frac{\partial^2 \mathbf{r}}{\partial u^2}, \quad \mathbf{r}_{12} = \frac{\partial^2 \mathbf{r}}{\partial u \partial v}, \quad \mathbf{r}_{22} = \frac{\partial^2 \mathbf{r}}{\partial v^2},$$

and so on. The total differential of $\mathbf{r}$ is thus

$$d\mathbf{r} = \mathbf{r}_1\, du + \mathbf{r}_2\, dv.$$

SHORT COURSE

In the following pages the Articles marked with an asterisk may be omitted at the first reading.

The student who wishes to take first a short course of what is most essential in the development of the subject should read the following Articles:

1—5, 8, 13—17, 22—43, 46—53, 54 (first part),
56—57, 67—75, 84—86, 91—101.

The reader who is anxious to begin the study of Differential Invariants (Chap. XII) may do so at any stage after Chap. VI.

CHAPTER I

CURVES WITH TORSION

1. Tangent. A *curve* is the locus of a point whose position vector $\mathbf{r}$ relative to a fixed origin may be expressed as a function of a single variable parameter. Then its Cartesian coordinates x, y, z are also functions of the same parameter. When the curve is not a plane curve it is said to be *skew*, tortuous or twisted. We shall confine our attention to those portions of the curve which are free from singularities of all kinds.

It is usually convenient to choose as the scalar parameter the length s of the arc of the curve measured from a fixed point A on it. Then for points on one side of A the value of s will be positive; for points on the other side, negative. The positive direction along the curve at any point is taken as that corresponding to algebraical increase of s. Thus the position vector $\mathbf{r}$ of a point on the curve is a function of s, regular within the range considered. Its successive

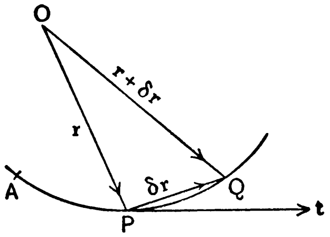

Fig. 1.

derivatives with respect to s will be denoted by $\mathbf{r}', \mathbf{r}'', \mathbf{r}'''$, and so on. Let P, Q be the points on the curve whose position vectors are $\mathbf{r}$, $\mathbf{r} + \delta\mathbf{r}$ corresponding to the values $s, s + \delta s$ of the parameter; then $\delta\mathbf{r}$ is the vector PQ. The quotient $\delta\mathbf{r}/\delta s$ is a vector in the same direction as $\delta\mathbf{r}$; and in the limit, as δs tends to zero, this direction becomes that of the tangent at P. Moreover the ratio of the lengths of the chord PQ and the arc PQ tends to unity as Q moves up to