CHAPTER 1

INTRODUCTION

This book is about the principles and methods of actuarial statistics. By 'actuarial statistics' we mean the statistics that actuaries compile for the purposes of their professional work. Our subject is thus a severely practical one, and the methods used are such as are sufficient for the practical purposes to be served. Elaborate theoretical development would be inappropriate for our purpose; utilitarianism is the keynote and approximation pervades the whole subject. The modern developments of mathematical statistics have made hardly any impact in this field, partly because of this inevitable element of approximation, partly because small samples would be inappropriate for most of the practical purposes to be served and partly because the observed statistics in this field are subject to considerable variation, both from one group to another and at different times for the same group. Random variations affect the statistics of course, but the problems arising do not usually present themselves in the form of repeated samples from the same universe.

Actuarial statistics not only have to be compiled on sound principles to provide a solid bedrock of observational fact, but they have to be interpreted and adjusted and then often used only as a guide in fixing the statistical basis required for the solution of the practical problems in connexion with which they were compiled. These problems are nearly always concerned with the future, and to cope with them an actuary has to use his experienced judgment in setting up suitable hypotheses with regard to the future on the basis of his knowledge of observed statistical facts. It would, however, give a false indication of the actuary's work and function and of the claims that he makes for the value of his work to suggest that the hypotheses that he regards as suitable for his practical problems are often in any real sense intended to be close or even unbiased forecasts or estimates of the future. Occasionally a best shot—an educated guess—at future developments may be required, but for the most part the actuary's problems are arranged in such a way that the discrepancies between actual future fact and the prior assumptions regarding the future provide sources of profit or loss to the organization concerned or are absorbed in some system of periodical readjustment in the light of actual experience such as a bonus system in life assurance or an adjustment of the contribution rates in a pension fund following a valuation of assets and liabilities. It will be appreciated that the statistical aspects of an actuary's problems do not usually operate in isolation; these problems are nearly always financial and nearly always involve administration. Thus investment, interest rates and administrative expense come into the picture as well and, indeed, they not infrequently predominate in importance over the strictly statistical aspects.

A constant appreciation of this background to the actuary's statistical work is essential if this work and the methods used are to be kept in proper focus. We are concerned with the observational aspect of actuarial science—the appeal to statistical fact as the basis of sound procedure—but the approach is not on the lines of 'pure observation' that is found in certain sciences in which every little fact is grist to the mill. The observational processes of the actuary are nearly always directed to particular purposes, and they are organized in relation to conceptual models that have been developed from past experience and found useful as modes of procedure for analogous problems that have arisen before. The typical conceptual model used by the actuary is, of course, the mortality table which exhibits for a hypothetical population all of the same age x their subsequent life and death history, i.e. it shows how many out of l_x living at age x may be 'expected' to survive to the successive ages $x+1$, $x+2$, $x+3$ and so on, or what comes to the same thing, how many may be expected to die between ages x and $x+1$, between ages $x+1$ and $x+2$, and so on. The mortality table is thus a frequency (or probability) distribution of ages at death. Similarly, a 'service table' such as is used in connexion with certain types of pension schemes, expresses a frequency distribution of ages at withdrawal, at incapacity retirement and at normal retirement as well as ages at death in service.

When a surveyor makes certain measurements—lengths, angles and so on—he tacitly assumes a Euclidean model in which straight lines, circular arcs and other simple geometrical shapes are substituted for the obviously rough shapes that would more accurately describe the real situation. From the combination of his measurements and his Euclidean model the surveyor makes a number of logical deductions dependent upon his knowledge of mensuration based on Euclidean geometry and trigonometry, and he reaches certain propositions about his particular model which he then, without giving a moment's thought to the complicated logical structure of his procedure, treats as true propositions about the real situation. The rationale of this process depends for its justification and practical success on a trained ability to ignore those features of the real situation which are irrelevant to the final practical purpose of his work. The surveyor judges how far he can go in treating a curve or an irregular line as a straight line, or an angle of 87° as a right angle and so on; he makes approximations and reaches approximate answers. The validity of his work depends essentially upon his ultimate purpose.

When a rational gambler repeatedly gambles on the toss of a coin he adopts the conceptual model of a probability distribution represented by the binomial expression $(\frac{1}{2}+\frac{1}{2})^n$, and he is content to operate by analogy between the practical situation and the propositions logically deduced from the binomial model. He does not trouble to test the coin and tossing system for complete symmetry or to test the probability assumption that $p=\frac{1}{2}$ by a trial run of tosses subject to a significance test on the statistical result. He is content to assume the approximate basis that $p=\frac{1}{2}$ for his practical purpose and to abide by the consequences unless as a result of his experience he begins to suspect some kind of bias.

When a card player shuffles the pack of cards before dealing for a hand of bridge he is content with a limited shuffle. As a rational player he adopts as his conceptual model for determining his calls and play the assumption that the order of the cards in the pack is at random, and his experience tells him that this is a good enough approximation to the practical situation for his purpose.

In solo whist shuffling is forbidden. After each hand (with certain exceptions) the cards are gathered together, two cuts are made and then the cards are dealt in threes to the four players with a final singleton each, the last card dealt being turned up as trumps. The assumption of randomness in the order of the cards in the pack and in the deal would be a fatal assumption to make in these circumstances, and a player who made such an assumption would undoubtedly lose heavily against experienced players. The peculiar fascination of solo whist does in fact depend on the non-random distribution of the cards in the four hands. To play the game successfully a certain amount of experience is necessary to enable the player to gauge the effect on the distribution of the cards, of the absence of shuffling and of the form of dealing the cards. There is no conceptual model that can be appealed to for use in making bids and in playing the hand. No doubt statistics could be compiled of the biased distributions arising, but to carry out an adequate investigation would be an immense undertaking. In practice, players develop a very keen sense of the run of the cards, and are sometimes able to make a coup by assuming a distribution of certain cards highly relevant to success or failure which would be extremely unlikely on the random principles associated with ordinary whist or bridge as modified of course by the inferences to be drawn from the bids made and the play so far.

In the solution of financial problems involving future payments a conceptual model based on the assumption of a uniform rate of compound interest is often adopted leading to the typical exponential form for the value of a unit payable t years hence, viz. $e^{-t\delta}$. The actual interest earned each year may, or may not, in fact approximate to the amounts of interest assumed in the model.

A physicist knows that observation has established that a lump of radioactive material in constant conditions emits α-particles at a constant rate subject to random fluctuations. Apparently, the emission of particular particles cannot be foreseen, the observational facts being expressed by saying that the particles in a lump of radioactive material diminish in number according to the exponential probability distribution function Ne^{-at}, where a is the force of emission which is estimated by counting the number of particles emitted in a given period of time. The exponential function represents the conceptual model and a represents the parameter which when estimated from the observations enables the model to be fully described in numerical terms.

In many statistical problems a convenient and sufficiently accurate model to adopt is the normal distribution for which the two parameters—the mean and the standard deviation—have to be estimated in some suitable way.

The scientific approach of the actuary to his problems is very similar in principle to that used in all the other examples that have been given. In some respects it combines particular features from all of them, while in others it is considerably more complicated. There is usually a conceptual model—the idea of a mortality table or a service table—but the model is not usually defined in a specific form such as in surveying or in coin-tossing and bridge, nor does it often take an explicit mathematical form involving a few unknown parameters such as for radioactive material or in many ordinary statistical problems. Instead, each numerical term in the mortality or service table requires to be individually estimated, partly on the basis of observed statistics and partly by experienced judgment.

The subject of this book is the compilation and adjustment of the observed facts rather than the elucidation of the element of experienced judgment. If the conditions of human life remained unchanged over a long period, mortality tables for groups of people of various kinds could be compiled by the simple process of observing the life history of samples of these groups and then used either as they stand or after applying some kind of graduation process to remove, or to reduce, the random variations. Conditions of human life have not, however, remained unchanged, and there is no sign that they are likely to remain unchanged in the foreseeable future, and so a mortality table compiled in this way would be quite out of date by the time that it had been completed. Quite apart from this continual flux the simple process of counting the number dying at each subsequent age out of a number of lives chosen at a given age x would take a very long time to carry out—at least $100-x$ years. Accordingly, instead of this process the early actuaries of the eighteenth and nineteenth centuries adopted the plan of observing groups of people of all ages over a short period of time and computing yearly mortality rates at each age from the observed numbers. After adjusting the observed rates to secure a smooth progression from age to age without altering their general level over short stretches of ages, they combined them together to form a mortality table by the process implied by the relations

$$p_x = 1 - q_x \quad \text{and} \quad l_{x+1} = p_x l_x,$$
$$l_{x+2} = p_{x+1} l_{x+1} = p_x p_{x+1} l_x,$$

and so on. Similar processes were adopted in building up service and other tables dependent on other contingencies of human life.

For some purposes and despite the doubt about the validity of the process, mortality and other tables are still built up in this way, while for other purposes attempts are made, on the basis of rates of mortality applicable to various earlier periods, to project rates that may be assumed to be appropriate for future periods and to build up models applicable to the future. A more simple purpose is often to compare the actual mortality of a group of lives in a given period with the number of deaths that would have occurred if the rates by some particular mortality table had in fact been closely experienced by the group during the period. Whatever may be the ultimate process to be used the essential starting point is to examine the mortality experience of one or more groups of lives. The typical measure of mortality is the ratio of the number of deaths occurring among a group of persons in a selected period of time; for example, E_x persons aged x are observed (exposed to risk) for a year, θ_x represents the number of such persons dying in the year and θ_x/E_x represents the rate of mortality.

This typical ratio illustrates the basic feature of all characteristic actuarial statistics. It may be expressed in general terms as the ratio of the number of times that a particular event occurs during a fixed period of time among a given group to the number of individuals in the group. This general expression of what we may call a time-rate is, in fact, too wide in its ambit for our purpose, because it includes many phenomena which would not normally come within the sphere of an actuary's work. It will be found, however, that the kind of examples that arise in other spheres rarely require more than a simple commonsense counting procedure which can

Cambridge University Press
978-1-316-60363-5 - Mortality and Other Investigations: Volume I
H. W. Haycocks and W. Perks
Excerpt
More information

be regarded as a special case of the more general processes used by actuaries. The more complicated processes of actuarial statistics have until fairly recent years usually been confined to the contingencies of human life, but the principles involved are of much wider application in theory, although the recognition of practical problems for this wider application has so far been relatively limited. However, the contingencies of human life, such as death, sickness, marriage, fertility, retirement and so on, which are the subject of actuarial treatment, nearly always exhibit a marked correlation in their incidence with age—and often with duration since the happening of some other contingency. Thus the incidence of death and of fertility both show considerable correlation with duration since marriage as well as with age. It is this correlation with age—and duration—that constitutes the special characteristic of the time-rates treated by actuaries. Age and duration themselves are both time characteristics, that is to say, they march forward together with the essential time element of the rates themselves. We might divide a group of lives into two subgroups according to, say, their sex, and then ascertain the mortality rates in a given year for the two sexes in the subgroups. If, on the other hand, we subdivided into ages and durations since entry into assurance a group of lives assured at a given point of time, we should not only have to fix successive ranges of ages and durations for the purpose of the subdivision but we should have to cope with the fact that as we observe the lives through the period of the investigation (e.g. a year) so would their ages and durations steadily progress. There is no doubt that as adult lives get older, so, on the average, does their health deteriorate and their mortality rates increase. Similarly, a group of lives, all of whom have passed a medical examination, steadily deteriorate, on the average, as the duration since the examination progresses. There is obviously a kind of progression towards destruction which is manifest throughout nature. The degree and rate of deterioration and the consequent level of mortality rates are affected by environmental features, and it is changes in these features and their effects that create the need for mortality investigations to be made at frequent intervals.

From a statistical point of view death is a simple event because it happens only once to an individual and must happen some time to every individual. It operates to remove the individual from the group of living persons—it is a 'decremental' phenomenon. Actuarial statistics are often concerned with events which operate as decrements, but they are by no means confined to decremental events. Some decremental events are reversible, e.g. a revival of a life policy is a negative lapse. Some events which are not decremental or reversible are repetitive. For example, in a sickness investigation the event is a day's sickness, and any individual may be the subject of one or more days of sickness in the period of the investigation. Fertility is even more complicated; multiple births take place—twins, triplets, etc.—but after a maternity a woman cannot normally again have a child until about 10 or 11 months have expired. This does not rigorously apply to a man—although it does sufficiently in practice—because he could have another child by another woman.

These various features of the events that are the subject of the various time-rates investigated by actuaries are important in interpreting the observed statistics and in adapting them for practical purposes. In particular, they require to be taken into account in considering the nature and level of the random element in the observed rates which we shall now briefly discuss. It is, of course, through the random element that the subject of actuarial statistics is connected with the general theory of mathematical statistics.

For mortality rates it is often useful to assume that the mortality rate for a given year of a group of persons aged x at the beginning of the year is a binomial variate generated by the binomial distribution $(p_x+q_x)^{E_x}$, where p_x and q_x are 'ideal' survival and mortality rates, so that the observed mortality rate is equal to q_x plus a random variable, of mean zero and variance $p_x q_x/E_x$. We need not assume that all the E_x lives are subject to identical probabilities of death like a set of symmetrical dice with $(q_x)^{-1}$ equally likely sides. All we need to do is to assume that E_x is a random sample from a hypothetical population (subject to a mortality rate of q_x) of sufficiently large numbers so that the extraction of E_x from the population makes no material difference to the mortality rate of the remainder. This avoidance of the assumption of homogeneity is not only more realistic than the dice model but leaves room for the development of an adequate theory of selection such as is required in life assurance and annuity mortality statistics, and in a reversed form in the mortality statistics of persons who have retired on incapacity pension. The validity of the assumption of binomiality of mortality rates, in the absence of certain disturbing features that are referred to later, is supported by the success attendant upon many examples of graduation of mortality rates.

The object of the process of graduation is to smooth the progression of the rates from age to age. That is the practical aim. The theoretical basis is the assumption that each rate contains a random error and that, shorn of these errors, the rates would show a smooth progression from age to age. Smoothness is a concept that has eluded a precise and generally accepted mathematical definition, but for actuarial purposes it usually means that the successive differences of the function concerned diminish and that third differences are small. Assuming that a particular graduation has succeeded in achieving smoothness the results can be tested for 'departure from the observed rates' (such a test may also be referred to as a 'test of fidelity' to the observed rates, although it would be more logical to refer to the closeness of the observed rates to the graduated rates which are, for the purpose of the test, being assumed to be 'true'). The process is to assume that the random errors in the observed rates are binomial and independent of each other. It is possible to devise various significance tests such that if the graduation survives the test, not only is the graduation judged acceptable but the assumption of binomiality is also supported.

The kind of disturbing features referred to above are the duplication of lives in an experience when an investigation of death claims under life policies is made without excluding the second, third, fourth, etc., policies on the same life. The existence of duplicates increases the variance of the random errors, and a correction of the significance test for this disturbance is always necessary when duplicates have not been eliminated. Various writers have discussed the subject. Redington and Michaelson (*Trans. 12th Int. Congr. Actu.*) provided a measure of the extent of the departure of the variances from the binomial variances including the effect of duplicates, and R. H. Daw (*J.I.A.* **72**, 174) has discussed the application of the measure in detail. Perks (*J.I.A.* **75**, 31) has suggested that a measure of the effect of duplicates alone could be obtained by taking a random sample of the deaths and ascertaining from them the proportion of lives in the investigation at each age with 1, 2, 3, etc., policies each. From these figures an estimate of the effect on the variances could easily be made (see, for example, *J.I.A.* **75**, 75, Beard and Perks).

A minor feature which affects the validity of the binomial assumption is the treatment of new entrants and withdrawals in computing the exposed-to-risk and ascertaining the deaths. If E_x is the computed exposed-to-risk, the number of lives contributing to E_x may be considerably greater than E_x because fractions of a year's exposure will have been accorded to the new entrants and withdrawals in the year. In the limit, if the number of lives contributing to E_x was very great, the average period of exposure in the year for each life being very small, the sampling distribution would tend to the Poisson limit. For example, if E_x were made up of nE_x lives all exposed for $1/n$ of a year each and subject to a rate of mortality of q_x/n, the binomial variance of the number of deaths $(nE_x)(q_x/n)(1-q_x/n)$ would be insignificantly different from the Poisson variance $E_x q_x$. Since $E_x p_x q_x \doteqdot E_x q_x$ when q_x is not very large, the prevalence of fractional exposures suggests that the Poisson variance (subject to adjustment for duplicates) might as well always be used in mortality investigations. Care needs to be exercised, however, at the extreme old ages where p_x becomes significantly smaller than unity so that the binomial and Poisson variances depart significantly from each other.

In other cases when the event is not a simple once-for-all event like death, the sampling variance will be correspondingly modified. An example of this is given in Chapter 6 in connexion with sickness rates where the variance is obtained from the frequency distribution of durations of sickness. In the case of sickness, however, the assumption of independence between the rates at successive ages is probably seriously inappropriate. This applies with particular force to the 'all-periods' sickness and with less force to the rates for the separate periods.