1

Preliminaries and tools

The material we present in this book relies heavily on basic harmonic analysis tools on the real line $\mathbb{R}$ and on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. There are many excellent textbooks on the subject, e.g. Katznelson [84], Stein–Weiss [135], Stein [133], Folland [61], Wolff [154], and Muscalu–Schlag [116]. In this preliminary chapter, we state without proof the results we need in order to develop the wellposedness theory of dispersive partial differential equations (PDEs).

We first recall the Lebesgue spaces of measurable functions (for $K = \mathbb{T}$ or $\mathbb{R}$):

$$L^p(K) = \left\{ f : K \to \mathbb{C} : \|f\|_{L^p}^p := \int_K |f|^p < \infty \right\}, \quad p \in [1, \infty),$$

$$L^\infty(K) = \{ f : K \to \mathbb{C} : \|f\|_{L^\infty} := \operatorname{esssup}|f| < \infty \},$$

and Hölder's inequality

$$\left| \int_K fg \right| \leq \|f\|_{L^p(K)} \|g\|_{L^q(K)}, \;\; 1 \leq p, q \leq \infty, \;\; \frac{1}{p} + \frac{1}{q} = 1.$$

Also recall

$$\ell^p = \left\{ a : \mathbb{Z} \to \mathbb{C} : \|a\|_{\ell^p}^p = \sum_{k \in \mathbb{Z}} |a_k|^p < \infty \right\}, \quad p \in [1, \infty), \quad \text{and}$$

$$\ell^\infty = \left\{ a : \mathbb{Z} \to \mathbb{C} : \|a\|_{\ell^\infty} = \sup_{k \in \mathbb{Z}} |a_k| < \infty \right\}.$$

For linear operators between L^p spaces, we have the Riesz–Thorin interpolation theorem, see Folland [61]:

Theorem 1.1 *Let T be a linear operator mapping $L^{p_0} + L^{p_1}$ to $L^{q_0} + L^{q_1}$. Fix*

$\theta \in (0,1)$ *and define*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then T maps L^{p_θ} to L^{q_θ}, and

$$\|T\|_{L^{p_\theta}\to L^{q_\theta}} \le \|T\|^{1-\theta}_{L^{p_0}\to L^{q_0}} \|T\|^{\theta}_{L^{p_1}\to L^{q_1}}.$$

An easy consequence of the above theorem is Young's inequality for the convolution of two functions (see Exercise 1.1)

$$\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}, \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad 1 \le p, q, r \le \infty. \tag{1.1}$$

The corresponding statement holds also for the ℓ^p spaces.

Another useful convolution inequality is the Hardy–Littlewood–Sobolev theorem; see Stein [133]:

Theorem 1.2 *For any $1 < p < r < \infty$*

$$\left\| |\cdot|^{-\alpha} * f \right\|_{L^r(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}, \quad \alpha = 1 + \frac{1}{r} - \frac{1}{p}.$$

A useful extension of Riesz–Thorin theorem is the complex interpolation theorem of Stein [132]:

Theorem 1.3 *Let $\{T_z\}$ be a family of linear operators analytic in the strip $\{z \in \mathbb{C} : 0 < Re(z) < 1\}$ and continuous on the closure. Namely, for any test functions f, g, the inner product $\langle f, T_z g\rangle$ is analytic on the strip and continuous on the closure. Assume that there exists $b < \pi$ so that for any simple functions f, g, and for any z in the strip*

$$|\langle f, T_z g\rangle| \le C_{f,g} e^{b|\Im(z)|}.$$

Also assume that

$$\|T_{0+iy}\|_{L^{p_0}\to L^{q_0}} \le M_0(y), \quad \|T_{1+iy}\|_{L^{p_1}\to L^{q_1}} \le M_1(y)$$

and that M_j, $j = 1, 2$, grow at most exponentially as $y \to \pm\infty$. Then for all $0 \le \theta \le 1$

$$\|T_\theta\|_{L^{p_\theta}\to L^{q_\theta}} \le C,$$

where C depends on M_j and θ.

For a Banach space X, the dual space X' is the space of all bounded linear maps from X to $\mathbb{C}$. From now on, we will use p' for the dual exponent $\frac{p}{p-1}$ of p, and similarly for q' and r'. Recall that the dual space of $L^p(K)$ is $L^{p'}(K)$ for $1 \le p < \infty$. Also recall that the adjoint $T^* : Y' \to X'$ of a bounded linear map T between two Banach spaces X and Y is defined by (see Folland [61] for more details)

$$[T^*\ell](x) = \ell(T(x)), \quad \ell \in Y', \quad x \in X.$$

The next lemma provides a standard method to establish the boundedness of linear operators between L^p spaces; see Stein [133, p:280]

Lemma 1.4 *(TT^* method) Let T be a linear operator defined on a dense subset of L^2, and with a formal adjoint T^*. Then the following are equivalent:*
(i) $\|T\|_{L^2\to L^p} \leq A$,
(ii) $\|T^*\|_{L^{p'}\to L^2} \leq A$,
(iii) $\|TT^*\|_{L^{p'}\to L^p} \leq A^2$.

To study low regularity solutions of PDEs, we need to define the solution in a distributional sense. We introduce the following test function spaces

$$C^\infty(K) = \{f : K \to \mathbb{C} : f \text{ is infinitely differentiable}\}, \quad K = \mathbb{R} \text{ or } \mathbb{T},$$

$$C_0^\infty(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : f \text{ is compactly supported}\},$$

$$\mathcal{S}(\mathbb{R}) = \left\{f \in C^\infty(\mathbb{R}) : P_{m,n}(f) := \left\|\langle x\rangle^m f^{(n)}(x)\right\|_{L^\infty} < \infty, m, n \geq 0\right\}.$$

Recall that these spaces are locally convex topological vector spaces. The topology on C^∞ is given by uniform convergence of each derivative on compact sets. Similarly, we say f_j converges to f in C_0^∞ if there is a compact set F containing the support of each f_j and f, and that f_j and all of its derivatives converges to f and its derivatives uniformly on F. We say f_j converges to f in $\mathcal{S}(\mathbb{R})$ if $P_{m,n}(f - f_j)$ converges to 0 for each $m, n \geq 0$.

Note that the dual space S' of a topological vector space S is defined analogously as the space of linear continuous maps from S to $\mathbb{C}$. We define the space of distributions on $\mathbb{R}$, $\mathcal{D}(\mathbb{R})$, as the dual of $C_0^\infty(\mathbb{R})$, and the space of periodic distributions, $\mathcal{D}(\mathbb{T})$, as the dual of $C^\infty(\mathbb{T})$. We also define the space of tempered distributions, $\mathcal{S}'(\mathbb{R})$, as the dual of $\mathcal{S}(\mathbb{R})$. We refer the reader to Folland [61] for the basic properties of distributions. We denote the action of a distribution u on a test function ϕ by $u(\phi)$. If u is an L^p function, then we have (see Exercise 1.2)

$$u(\phi) = \int_{\mathbb{R}} u(x)\phi(x)\, dx.$$

We define the Fourier transform for functions in $L^1(\mathbb{R})$ as

$$\mathcal{F} f(\xi) = \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-i\xi x}dx, \quad \xi \in \mathbb{R}.$$

Recall that $\widehat{f}$ is a continuous bounded function on $\mathbb{R}$ that decays to zero at infinity. In the case when $\widehat{f} \in L^1(\mathbb{R})$, one has the inversion formula

$$f(x) = \mathcal{F}^{-1}\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \widehat{f}(\xi) d\xi, \quad x \in \mathbb{R}.$$

Similarly, for $f \in L^1(\mathbb{T})$, we define the Fourier series as

$$\mathcal{F} f(k) = \widehat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx, \quad k \in \mathbb{Z}.$$

In the case $\widehat{f} \in \ell^1$, we have

$$f(x) = \mathcal{F}^{-1}\widehat{f}(x) = \sum_{k\in\mathbb{Z}} e^{ikx} \widehat{f}(k).$$

We have the Poisson summation formula ; see e.g. Folland [61]

$$\sum_{k=-\infty}^{\infty} f(x + 2\pi k) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{ixk}, \quad f \in \mathcal{S}(\mathbb{R}). \tag{1.2}$$

For $f, g \in L^1(\mathbb{R})$, we have the Fourier multiplication formula

$$\int_{\mathbb{R}} f(x)\widehat{g}(x)\, dx = \int_{\mathbb{R}} \widehat{f}(x) g(x)\, dx. \tag{1.3}$$

This leads to Parseval's identity

$$\langle f, g\rangle = \int_{\mathbb{R}} \overline{f(x)} g(x)\, dx = \int_{\mathbb{R}} \overline{\widehat{f}(\xi)} \widehat{g}(\xi)\, d\xi = \left\langle \widehat{f}, \widehat{g} \right\rangle, \quad f, g \in L^2(\mathbb{R}),$$

and, in particular, we have Plancherel's theorem

$$\|f\|_{L^2(\mathbb{R})} = \left\|\widehat{f}\right\|_{L^2(\mathbb{R})}.$$

Similar formulas hold in the case of Fourier series. Interpolating Plancherel's theorem with the inequality

$$\left\|\widehat{f}\right\|_{L^\infty} \le \|f\|_{L^1}$$

yields the Hausdorff–Young theorem

$$\left\|\widehat{f}\right\|_{L^{p'}} \le \|f\|_{L^p}, \quad 1 \le p \le 2, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

By noting that $\mathcal{F}$ is a continuous bijection from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$, one can extend the definition of the Fourier transform to the space $\mathcal{S}'(\mathbb{R})$ by the formula

$$\mathcal{F} u(\phi) := u(\mathcal{F}\phi).$$

Similarly, one can define the Fourier series of periodic distributions as

$$\mathcal{F}u(k) = \frac{1}{2\pi} u(e^{-ik\cdot}).$$

Most of the basic properties of $\mathcal{F}$ can be extended to the distributional definitions, see [61]. In particular, both distributional definitions agree with the usual definition for $u \in L^1(K)$.

For $s > 0$, we often use the operators $D^s = (-\Delta)^{\frac{s}{2}}$ given on the Fourier side as

$$\widehat{D^s f}(\xi) = |\xi|^s \widehat{f}(\xi).$$

Similarly, we have the operators J^s given on the Fourier side as

$$\widehat{J^s f}(\xi) = \langle \xi \rangle^s \widehat{f}(\xi).$$

We define the L^2 based Sobolev spaces, for $s \in \mathbb{R}$

$$H^s(\mathbb{R}) = \{ f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{H^s(\mathbb{R})} := \|J^s f\|_{L^2} < \infty \},$$

where $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$. Similarly, for $s \in \mathbb{R}$, we have

$$H^s(\mathbb{T}) = \{ f \in \mathcal{D}(\mathbb{T}) : \|f\|_{H^s(\mathbb{T})} := \|J^s f\|_{L^2} < \infty \}.$$

The homogenous Sobolev spaces $\dot{H}^s$ are defined analogously with D^s instead of J^s.

Recall that C_0^∞ functions are dense in $H^s(K)$ for any s and in $L^p(K)$ for $1 \leq p < \infty$. Note that for $\alpha > 0$, $D^\alpha : H^s \to H^{s-\alpha}$, and for any α, $J^\alpha : H^s \to H^{s-\alpha}$. Also note that

$$\partial_x f = HDf = DHf,$$

where H is the Hilbert transform

$$\widehat{Hf}(\xi) = i\text{sign}(\xi)\widehat{f}(\xi),$$

which is bounded in $H^s(\mathbb{R})$ for any s, and in L^p for $1 < p < \infty$.

We collect some basic properties of Sobolev spaces in the following lemmas (see Exercises 1.3 and 1.4):

Lemma 1.5 *(Sobolev embedding) For $K = \mathbb{R}$ or $\mathbb{T}$, we have*

$$\|f\|_{L^p(K)} \lesssim \|f\|_{\dot{H}^s(K)}, \quad s = \frac{1}{2} - \frac{1}{p}, \ 2 < p < \infty,$$

$$\|f\|_{L^\infty(K)} \lesssim \|f\|_{H^s(K)}, \quad s > \frac{1}{2}.$$

Lemma 1.6 *(Algebra property) For $K = \mathbb{R}$ or $\mathbb{T}$, and $s > \frac{1}{2}$, we have*

$$\|fg\|_{H^s(K)} \lesssim \|f\|_{H^s(K)}\|g\|_{H^s(K)}.$$

We also need basic definitions and theorems from the Littlewood–Paley theory. We start with Littlewood–Paley projections

$$\widehat{P_k f}(\xi) = \varphi(2^{-k}\xi)\widehat{f}(\xi),$$

where φ is a smooth cut-off function supported in $\{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ with the property

$$\sum_k \varphi(2^{-k}\xi) = 1, \quad \text{for each } \xi \neq 0.$$

We use the same definition for functions on the torus. Similarly, $P_{\geq k}$ is given as

$$\widehat{P_{\geq k} f}(\xi) = \widehat{f}(\xi)\sum_{j=k}^{\infty} \varphi(2^{-j}\xi),$$

and $P_{<k} = Id - P_{\geq k}$. It is important to note the uniform bounds

$$|P_k f(x)|, |P_{<k} f(x)| \leq C\, Mf(x). \tag{1.4}$$

Here M is the Hardy–Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy,$$

where $B(x,r)$ is the ball of radius r centered at x, and $|B(x,r)|$ is the volume of $B(x,r)$. Both inequalities follow by majorizing the inverse Fourier transform of the cutoff function by a sum of characteristic functions of balls. The Hardy–Littlewood theorem states that for any $1 < p \leq \infty$

$$\|Mf\|_{L^p} \leq C_p\|f\|_{L^p},$$

both on $\mathbb{R}$ and $\mathbb{T}$. This implies uniform L^p boundedness of the Littlewood–Paley projections via (1.4).

We also have the following characterization of the Sobolev spaces in terms of Littlewood–Paley projections (see Exercise 1.5)

$$\|f\|_{H^s}^2 \approx \|P_{<1} f\|_{L^2}^2 + \sum_{k=1}^{\infty} 2^{2ks}\|P_k f\|_{L^2}^2. \tag{1.5}$$

Also recall that for $f \in \mathcal{S}(\mathbb{R})$, we have

$$P_{<1} f + \sum_{k=1}^{n} P_k f \to f \text{ in } \mathcal{S}(\mathbb{R}), \tag{1.6}$$

as $n \to \infty$.

We finish this section by stating some additional results on Sobolev spaces. The proofs of these results are more involved, and Taylor [146] is a good reference for these and related inequalities.

We first state the Gagliardo–Nirenberg inequality [117]:

Theorem 1.7 *Let $f : \mathbb{R} \to \mathbb{C}$. Fix $1 \leq p, q, r \leq \infty$ and a natural number m. Suppose also that a real number α and a natural number j are such that*

$$\frac{1}{p} = j + \left(\frac{1}{r} - m\right)\alpha + \frac{1-\alpha}{q}$$

and

$$\frac{j}{m} \leq \alpha \leq 1.$$

Then

$$\|D^j f\|_{L^p} \lesssim \|D^m f\|_{L^r}^{\alpha} \|f\|_{L^q}^{1-\alpha}.$$

We will mainly use the following corollary:

Corollary 1.8 *For any $j, k \in \mathbb{N}$, we have*

$$\left\|\partial_x^j f\, \partial_x^k f\right\|_{L^2} \lesssim \|f\|_{H^{k+j}} \|f\|_{L^\infty}.$$

Proof We can assume that $j, k \geq 1$. Let $m = j + k$. Let

$$p_1 = \frac{2m}{j}, \quad p_2 = \frac{2m}{k}, \quad \alpha_1 = \frac{j}{m}, \quad \alpha_2 = \frac{k}{m}.$$

Note that in this case $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$ and $\alpha_1 + \alpha_2 = 1$. By Hölder's and Gagliardo–Nirenberg's inequalities, and the L^p boundedness of the Hilbert transform, we have

$$\begin{aligned}\left\|\partial_x^j f\, \partial_x^k f\right\|_{L^2} &\leq \left\|\partial_x^j f\right\|_{L^{p_1}} \left\|\partial_x^k f\right\|_{L^{p_2}} \\ &\lesssim \left\|D^j f\right\|_{L^{p_1}} \left\|D^k f\right\|_{L^{p_2}} \lesssim \|D^m f\|_{L^2}^{\alpha_1+\alpha_2} \|f\|_{L^\infty}^{2-(\alpha_1+\alpha_2)} \\ &\leq \|f\|_{H^m} \|f\|_{L^\infty}.\end{aligned}$$

□

The following is a standard commutator estimate that we use in Chapter 3 to establish the wellposedness theory for some dispersive PDEs. For the proof, see Kato–Ponce [83], Kenig–Ponce–Vega [88], and Taylor [146, page 106].

Lemma 1.9 *For $s \in (0, 1)$, we have*

$$\|J^s(fg) - fJ^s g\|_{L^2} \lesssim \|f\|_{H^s} \|g\|_{L^\infty}.$$

For $s > 1$, we have

$$\left\|J^s(fg) - fJ^s g\right\|_{L^2} \lesssim \|f\|_{H^s}\|g\|_{L^\infty} + \|f_x\|_{L^\infty}\|g\|_{H^{s-1}}.$$

We state the following version of fractional Leibniz rule; see Taylor [146].

Lemma 1.10 *For $s \geq 0$*

$$\frac{1}{2} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}, \quad 2 < p_1, q_2 \leq \infty,$$

we have

$$\|fg\|_{H^s} \lesssim \|f\|_{L^{p_1}}\|J^s g\|_{L^{q_1}} + \|J^s f\|_{L^{p_2}}\|g\|_{L^{q_2}}.$$

Finally, we have the following L^r version of this lemma; see Taylor [146].

Lemma 1.11 *For $s \geq 0$, $1 < r < \infty$, we have*

$$\|J^s(fg)\|_{L^r} \lesssim \|f\|_{L^{p_1}}\|J^s g\|_{L^{q_1}} + \|J^s f\|_{L^{p_2}}\|g\|_{L^{q_2}},$$

where $p_j, q_j \in (1, \infty)$, and

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.$$

Exercises

1.1 Prove that

$$\|f * g\|_{L^p} \leq \|f\|_{L^p}\|g\|_{L^1}, \quad \text{and}$$

$$\|f * g\|_{L^\infty} \leq \|f\|_{L^p}\|g\|_{L^{p'}}.$$

Derive Young's inequality (1.1) using these bounds and Theorem 1.1.

1.2 Fix $p \in [1, \infty]$. Prove that any $f \in L^p(\mathbb{R})$ is a tempered distribution with the action given by

$$f(\phi) = \int_{\mathbb{R}} f(x)\phi(x)\,dx, \quad \phi \in \mathcal{S}(\mathbb{R}).$$

1.3 (a) Consider the tempered distribution $|x|^{-\alpha}$, $0 < \alpha < 1$ on $\mathbb{R}$. Prove that

$$\mathcal{F}\left(|\cdot|^{-\alpha}\right)(\xi) = c_\alpha|\xi|^{\alpha-1},$$

where c_α is a constant depending only on α.

(b) Use the Hardy–Littlewood–Sobolev theorem, part (a), and duality to prove the Sobolev embedding theorem on $\mathbb{R}$

$$\|f\|_{L^p(\mathbb{R})} \leq C\|D^s f\|_{L^2(\mathbb{R})}$$

with $\frac{1}{2} = \frac{1}{p} + s$ and $2 < p < \infty$.

(c) Use part (b) to prove the following version of the Gagliardo–Nirenberg inequality on $\mathbb{R}$, for $p \in [2, \infty]$ and $\theta = \frac{1}{2} - \frac{1}{p}$

$$\|f\|_{L^p} \le C\|f_x\|_{L^2}^{\theta}\|f\|_{L^2}^{1-\theta}. \tag{1.7}$$

(d) Give an alternative proof of (1.7) by interpolating the bounds for $p = 2$ and $p = \infty$. For the latter bound, express f^2 using the fundamental theorem of calculus.

1.4 Prove the algebra of Sobolev spaces stated in Lemma 1.6.

1.5 Prove (1.5) using Plancherel's theorem.

1.6 (Gronwall's inequality) Assume that for a.e. $t \in [0, T]$, we have

$$f(t) \le A + \int_0^t g(\tau)f(\tau)d\tau$$

for some $A \ge 0$ and some nonnegative functions f and g such that $fg \in L^1([0, T])$. Prove that

$$f(t) \le A \exp\left(\int_0^t g(\tau)d\tau\right), \quad t \in [0, T].$$

1.7 Consider the linear Schrödinger equation on $\mathbb{R}$

$$\begin{cases} iu_t + u_{xx} = 0, x \in \mathbb{R}, t \in \mathbb{R}, \\ u(0, \cdot) = g(\cdot) \in H^s(\mathbb{R}). \end{cases} \tag{1.8}$$

(a) For each $t \in \mathbb{R}$, define $u(t, \cdot)$ as a tempered distribution by the formula

$$u(t, \cdot) = \mathcal{F}^{-1}\left[e^{-it\xi^2}\mathcal{F}g(\xi)\right](\cdot).$$

Prove that $u \in C_t^0H_x^s(\mathbb{R} \times \mathbb{R})$ and that $\|u(t, \cdot)\|_{H^s(\mathbb{R})}$ is constant.

(b) Prove that if g_n converges to g in H^s, then, for each t, $u_n(t, \cdot)$ converges to $u(t, \cdot)$ in H^s.

(c) Prove that u solves (1.8) in the sense of distributions, i.e.

$$u(- i\phi_t + \phi_{xx}) = 0,$$

for all $\phi \in \mathcal{S}(\mathbb{R}^2)$.

(d) Prove that if $v \in C_t^0H_x^s(\mathbb{R} \times \mathbb{R})$, $v(0, x) = g(x)$, and v solves (1.8) in the sense of distributions, then $v = u$.

1.8 (a) Let $\phi \in \mathcal{S}(\mathbb{R})$ and $z \in \mathbb{C}\backslash\{0\}$ with a nonnegative real part. Then

$$\int_{\mathbb{R}} e^{-z|x|^2}\widehat{\phi}(x)dx = \frac{1}{\sqrt{2z}}\int_{\mathbb{R}} e^{-\frac{|x|^2}{4z}}\phi(x)dx.$$

(b) Assuming that $g \in \mathcal{S}(\mathbb{R})$, express the solution of (1.8) as a convolution of the tempered distribution $\frac{1}{\sqrt{4\pi it}} e^{i\frac{|x|^2}{4t}}$ with g.
(c) Similarly, prove that the solution of the linear Schrödinger equation on $\mathbb{R}^n$

$$\begin{cases} iu_t + \Delta u = 0, x \in \mathbb{R}^n, t \in \mathbb{R}, \\ u(0,\cdot) = g(\cdot) \in \mathcal{S}(\mathbb{R}^n) \end{cases} \tag{1.9}$$

is given by

$$e^{it\Delta} g = \frac{1}{(4\pi it)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} g(y)\, dy.$$

(d) Conclude that the following dispersive estimate holds

$$\left\|e^{it\Delta} g\right\|_{L^\infty_x(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}} \|g\|_{L^1(\mathbb{R}^n)}.$$

1.9 Let $f \in C_0^\infty(\mathbb{R})$, $\phi \in C^\infty$, and $\phi'(x) \neq 0$ for any x in the support of f. Then

$$I(\lambda) = \int_{\mathbb{R}} e^{i\lambda\phi(x)} f(x) dx = O(\lambda^{-k}), \text{ as } \lambda \to \infty$$

for any $k \in \mathbb{Z}^+$.

1.10 (Van der Corput lemma) Let $\phi : \mathbb{R} \to \mathbb{R}$ be C^∞.
(a) Assume that for some $k \in \mathbb{Z}^+$, we have $|\phi^{(k)}(x)| \geq 1$ for any $x \in [a, b]$, with $\phi'(x)$ monotonic when $k = 1$. Then

$$\left|\int_a^b e^{i\lambda\phi(x)} dx\right| \leq c_k \lambda^{-\frac{1}{k}},$$

where the constant c_k is independent of a and b.
Hint: The case $k = 1$ follows from an integration by parts noting that ϕ'' does not change sign. For the general case, use induction, noting that $\phi^{(k-1)}$ can vanish at most at a single point, and decompose $[a, b]$ into two disjoint sets appropriately.
(b) Under the hypothesis of part (a), prove that

$$\left|\int_a^b e^{i\lambda\phi(x)} f(x) dx\right| \leq c_k \lambda^{-\frac{1}{k}} \left(\|f\|_{L^\infty} + \|f'\|_{L^1}\right).$$

1.11 (Bernstein's inequality) Prove that if P is a trigonometric polynomial of degree N, then

$$\left\|P'\right\|_{L^p} \lesssim N \|P\|_{L^p}, \quad 1 \leq p \leq \infty.$$

Hint: Express P' as

$$P' = P * K_N,$$

where K_N is a suitable convolution kernel with the property $\|K_N\|_{L^1} \lesssim N$, see Katznelson [84].

1.12 Using the Poisson summation formula, (1.2) prove that the convolution kernel φ_k of P_k on the torus satisfies the bound

$$|\varphi_k(x)| \leq C \frac{2^k}{(1+2^k|x|)^2}, \quad k \in \mathbb{N}, |x| \leq \frac{1}{2}.$$