ANALYTICAL GEOMETRY OF THREE DIMENSIONS

CHAPTER I

CARTESIAN COORDINATE-SYSTEM

1·1. Cartesian coordinates.

In a plane the position of a point P is determined by two coordinates, x and y, referred to two straight lines OX, OY, the coordinate-axes; viz. if $NP \parallel OX$ and $MP \parallel OY$, so that we have a parallelogram $OMPN$, then $x = OM = NP$, $y = ON = MP$.

To fix the position of a point in space we take three planes. These have a point O in common and intersect in pairs in three lines $X'OX$, $Y'OY$, $Z'OZ$. O is called the origin, the three lines the coordinate-axes, and the three planes the coordinate-planes.

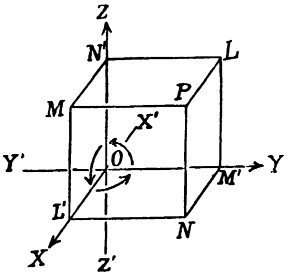

Fig. 1

Let P be any point. Through P draw PL parallel to XOX' cutting the plane YOZ in L, and similarly PM and PN parallel to the other axes. Let the plane MPN cut OX in L', and similarly obtain M' and N'. We obtain then a parallelepiped whose faces are parallel to the coordinate-planes, and edges parallel to the coordinate-axes, and OP is a diagonal. The figure is determined by the lengths of OL', OM', ON'.

In the usual way we attach signs to lengths measured along the axes, defining by a convention that distances measured in one direction are positive, distances measured in the opposite direction being negative. With these conventions we then define the *coordinates* of the point P as the three lengths

$$OL' = x, \quad OM' = y, \quad ON' = z.$$

To every point P there corresponds uniquely a set of three numbers $[x, y, z]$, and conversely to every set of three numbers, positive or negative, there corresponds a unique point.

1·11. Convention of signs.

Let the positive directions along the axes of x and y be defined arbitrarily, say OX and OY; then in the plane XOY we may pass from OX to OY by a rotation through an angle XOY less than two right angles. Viewed from one side of the plane this rotation is clockwise, and from the other side it appears counter-clockwise. We define that side of the plane from which the rotation appears to be counter-clockwise as the *positive side of the plane*. Then the positive direction of the axis of z is defined to be that which lies on the positive side of the plane XOY. This relation then holds for each of the axes, viz. the positive direction of the axis of x is on the positive side of the plane YOZ, and the positive direction of the axis of y is on the positive side of the plane ZOX. This is called a *right-handed* system of cartesian coordinates.

1·2. When the planes are mutually at right angles we call it a *rectangular* system, otherwise it is *oblique*. We shall confine our attention for the present to rectangular coordinates.

OP is called the *radius-vector* of P, denoted by r. Since PN is perpendicular to the plane XOY, ON is the orthogonal projection, or simply the projection, of OP on the plane XOY. Again, since the plane $PML'N \perp OX$, $PL' \perp OX$ and OL' is the projection of OP on the line OX.

Let the angles which OP makes with the positive directions of the axes be α, β, γ, then

$$x = r\cos\alpha, \quad y = r\cos\beta, \quad z = r\cos\gamma.$$

The position of P is determined by the angles α, β, γ and the radius-vector r, for these then determine x, y, z. The angles by themselves determine only the direction of the line OP. We call them the *direction-angles* of the line OP. As the cosines of these angles occur repeatedly it is convenient to call them the *direction-cosines*, and frequently we denote them by single letters l, m, n. There is a redundance in fixing the position of a point by the radius-vector and its direction-angles, for these are four numbers, and three, x, y, z, are sufficient to fix the position. We shall find that the three direction-cosines are connected by an identical relation.

1·21. The direction-angles are not uniquely defined since each is indeterminate to an added multiple of 2π, but the direction-cosines are unique. In fact, r *being always positive*, the direction-cosines l, m, n are uniquely defined as

$$l=x/r, \quad m=y/r, \quad n=z/r.$$

1·22. *To determine the radius-vector in terms of the rectangular coordinates.*

By the theorem of Pythagoras

$$OP^2=ON^2+NP^2=OL'^2+L'N^2+NP^2.$$

Hence $$r^2=x^2+y^2+z^2.$$

1·23. *Identity connecting the direction-cosines.*

Putting $x=r\cos\alpha$, $y=r\cos\beta$, $z=r\cos\gamma$, we find, on dividing by r^2,

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

It is often convenient to speak of a line whose direction-cosines are proportional to three given numbers l, m, n. The actual values of the direction-cosines are obtained by dividing each by $\sqrt{(l^2+m^2+n^2)}$. For suppose the actual values to be kl, km, kn; then

$$k^2l^2+k^2m^2+k^2n^2=1,$$

hence $$k=(l^2+m^2+n^2)^{-\frac{1}{2}}.$$

Ex. Find the direction-cosines of the line joining the origin to the point $(-1, 2, 2)$.

Here $r^2=1+4+4=9$, hence $r=3$.

Then $l=-\frac{1}{3}$, $m=\frac{2}{3}$, $n=\frac{2}{3}$.

1·3. Change of origin.

Let a new coordinate-system be constructed with origin $O'\equiv[X, Y, Z]$ and coordinate-planes parallel to the old ones. Let the coordinates of a point P referred to the two systems be $[x, y, z]$ and $[x', y', z']$. Draw through P a line parallel to the axis of x cutting the planes yOz and $y'O'z'$ in L and L', and let the plane $y'O'z'$ cut Ox in K. Then since the parallel planes yOz, $y'O'z'$ intercept

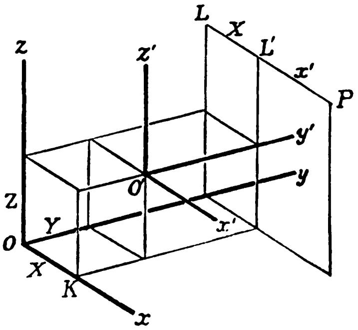

Fig. 2

equal segments on parallel lines, $LL' = OK$. But $LP = x$, $L'P = x'$, $OK = X$, hence

similarly
$$\left.\begin{aligned} x &= x' + X \\ y &= y' + Y \\ z &= z' + Z \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x' &= x - X \\ y' &= y - Y \\ z' &= z - Z \end{aligned}\right\}.$$

x', y', z' are the coordinates of P relative to $O' \equiv [X, Y, Z]$.

This may be described in vector language thus. The step from O to P can be broken up into the two steps O to O' and O' to P. Further the step from O to O' can be broken up into three steps of lengths X, Y, Z parallel to the axes, and similarly for the steps from O to P and O' to P. Parallel steps are then simply added.

1·4. Distance between two points.

Let $P_1 \equiv [x_1, y_1, z_1]$, $P_2 \equiv [x_2, y_2, z_2]$. Then the coordinates of P_2 referred to parallel axes through P_1 are

$$x_2 - x_1, \quad y_2 - y_1, \quad z_2 - z_1,$$

and P_1P_2 is the relative radius-vector. Hence

$$(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2.$$

1·5. Angle between two lines.

Let the two lines pass through O and have direction-angles $[\alpha_1, \beta_1, \gamma_1]$ and $[\alpha_2, \beta_2, \gamma_2]$. Take two points $P_1 \equiv [x_1, y_1, z_1]$ and $P_2 \equiv [x_2, y_2, z_2]$, one on each; let $OP_1 = r_1$, $OP_2 = r_2$ and let the angle $P_1OP_2 = \theta$. Then

$$(P_1P_2)^2 = OP_1^2 + OP_2^2 - 2OP_1 . OP_2 \cos\theta,$$

therefore $\quad \Sigma(x_2 - x_1)^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta,$

hence $\quad \Sigma x_2^2 + \Sigma x_1^2 - 2\Sigma x_1x_2 = \Sigma x_1^2 + \Sigma x_2^2 - 2r_1r_2\cos\theta.$

But $\quad x_1 = r_1\cos\alpha_1, \quad x_2 = r_2\cos\alpha_2$, etc.

Therefore $\quad \Sigma r_1r_2\cos\alpha_1\cos\alpha_2 = r_1r_2\cos\theta,$

i.e.
$$\begin{aligned} \cos\theta &= \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 \\ &= \Sigma\cos\alpha_1\cos\alpha_2 = \Sigma l_1 l_2. \end{aligned}$$

1·51. If the direction-cosines are only proportional to the numbers $[l_1, m_1, n_1]$ and $[l_2, m_2, n_2]$, then

$$\cos\theta = \frac{\Sigma l_1 l_2}{\sqrt{(\Sigma l_1^2 \Sigma l_2^2)}}.$$

1·52. A useful expression may be found also for $\sin^2\theta$.

We have
$$\sin^2\theta = 1-(\Sigma l_1 l_2)^2 = \Sigma l_1^2 \Sigma l_2^2 - (\Sigma l_1 l_2)^2$$
$$= \Sigma(m_1^2 n_2^2 + m_2^2 n_1^2) - 2\Sigma m_1 m_2 n_1 n_2$$
$$= \Sigma(m_1 n_2 - m_2 n_1)^2.$$

These results may be applied to any two lines. When two lines do not intersect we define the angle determined by them as the angle between two lines through O parallel to the given lines. All parallel lines are then considered as having the same direction-angles.

1·61. Condition for perpendicularity.

If two lines $[l_1, m_1, n_1]$ and $[l_2, m_2, n_2]$ are perpendicular or orthogonal, their angle $\theta = \frac{1}{2}\pi$, and $\cos\theta = 0$, hence
$$\left.\begin{aligned} l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \\ \text{or} \qquad \Sigma\cos\alpha_1 \cos\alpha_2 = 0 \end{aligned}\right\}.$$

Conversely, if this condition is satisfied, $\cos\theta = 0$ and $\theta = \frac{1}{2}\pi$.

The expression $\Sigma l_1 l_2$ is linear in each of the two sets of direction-cosines l_1, m_1, n_1 and l_2, m_2, n_2, and also symmetrical as regards these two sets. It is the *bilinear symmetrical* expression associated with the quadratic expression Σl^2.

1·62. Conditions for parallelism.

By definition, two lines are parallel when they have the same direction-angles. It follows then that $\sin\theta = 0$ and $\theta = 0$.

Conversely, if $\theta = 0$, $\sin\theta = 0$ and $\Sigma(m_1 n_2 - m_2 n_1)^2 = 0$. For real values of the direction-cosines this can be true only if
$$m_1 n_2 - m_2 n_1 = 0, \quad n_1 l_2 - n_2 l_1 = 0, \quad l_1 m_2 - l_2 m_1 = 0,$$
hence
$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}.$$

If l_1, m_1, n_1 and l_2, m_2, n_2 are the actual direction-cosines we have also
$$l_1^2 + m_1^2 + n_1^2 = 1 = l_2^2 + m_2^2 + n_2^2.$$
Putting each of the equal ratios equal to t and substituting
$$l_1 = tl_2, \quad m_1 = tm_2, \quad n_1 = tn_2$$
we get
$$t^2(l_2^2 + m_2^2 + n_2^2) = l_2^2 + m_2^2 + n_2^2.$$
Hence
$$t = \pm 1.$$

If $t=+1$, the direction-cosines are identical; if $t=-1$, they are equal but of opposite sign. The latter case is interpreted to mean that the lines are in opposite senses. In both cases they are parallel.

If l_1, m_1, n_1 and l_2, m_2, n_2 are only numbers proportional to the direction-cosines, the necessary and sufficient conditions for parallelism are

$$l_1 : m_1 : n_1 = l_2 : m_2 : n_2,$$

equivalent to *two* conditions only.

1·7. Position-ratio of a point with regard to two base-points.

The formulae for the coordinates (rectangular or oblique) of a point $P \equiv [x, y, z]$ dividing the join of two points $P_1 \equiv [x_1, y_1, z_1]$ and $P_2 \equiv [x_2, y_2, z_2]$ in a given ratio $l : m$ or $k : 1$ are exactly the same as in plane geometry. For if the planes through P_1, P_2, P parallel to the plane of yz cut Ox in L_1, L_2, L, then L cuts L_1L_2 in this same ratio, and $OL_1 = x_1$, $OL_2 = x_2$, $OL = x$. Hence

$$x = \frac{lx_2 + mx_1}{l+m} = \frac{kx_2 + x_1}{k+1},$$

with similar formulae for y and z. The ratio $l/m = k$ is called the *position-ratio* of P with regard to P_1 and P_2. The formulae are sometimes referred to as Joachimsthal's formulae.

1·71. Another set of section-formulae is useful. If

$$P_1P = t . P_1P_2,$$

we have

$$\left.\begin{aligned} x &= x_1 + t(x_2 - x_1) \\ y &= y_1 + t(y_2 - y_1) \\ z &= z_1 + t(z_2 - z_1) \end{aligned}\right\}.$$

1·72. If two particles of masses m_1, m_2 are placed at P_1 and P_2 their centre of mass divides P_1P_2 in the ratio $m_2 : m_1$, hence the coordinates of the centre of mass are

$$x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}, \text{ etc.}$$

This point is also called the *mean point* for the multiples m_1, m_2.

1·73. Similarly, if three masses m_1, m_2, m_3 are placed at three points P_1, P_2, P_3 the coordinates of the centre of mass are

$$x=\frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}, \text{ etc.}$$

By admitting negative masses any point in the plane can be represented in this way.

1·8. General cartesian coordinates.

In the general cartesian system the planes are not necessarily at right angles. The system will be determined by the angles between the coordinate-axes, viz. $\angle YOZ=\lambda$, $\angle ZOX=\mu$, $\angle XOY=\nu$.

The direction-angles are then no longer convenient, but in place of the direction-cosines we define the *direction-ratios* as follows
$$l=x/r,\quad m=y/r,\quad n=z/r.$$

1·81. To find the radius-vector r we have (Fig. 1)

$$r^2=OP^2=ON^2+NP^2+2ON\,.\,NP\cos NOZ.$$

Now the projection of ON on OZ is equal to the sum of the projections of OM' and $M'N$, hence

$$\begin{aligned}ON\cos NOZ&=OM'\cos YOZ+M'N\cos ZOX\\&=y\cos\lambda+x\cos\mu.\end{aligned}$$

Also
$$\begin{aligned}ON^2&=OL'^2+L'N^2+2OL'\,.\,L'N\cos XOY\\&=x^2+y^2+2xy\cos\nu.\end{aligned}$$

Hence

$$r^2 = x^2 + y^2 + z^2 + 2yz\cos\lambda + 2zx\cos\mu + 2xy\cos\nu.$$

1·82. Then substituting from (1·8) we have

$$l^2 + m^2 + n^2 + 2mn\cos\lambda + 2nl\cos\mu + 2lm\cos\nu = 1,$$

as the identity connecting the direction-ratios.

1·83. The square of the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is found by substituting in (1·81) $x_1 - x_2$ for x, etc.

1·84. The angle between two straight lines (l_1, m_1, n_1) and (l_2, m_2, n_2) is found as in 1·5,

$$\cos\theta=\frac{\Sigma l_1 l_2+\Sigma(m_1n_2+m_2n_1)\cos\lambda}{\sqrt{(\Sigma l_1^2+2\Sigma m_1n_1\cos\lambda)(\Sigma l_2^2+2\Sigma m_2n_2\cos\lambda)}}.$$

The numerator is the bilinear symmetrical expression associated with the quadratic expressions which occur in the denominator.

1·9. EXAMPLES.

1. Find the direction-angles of the lines joining the origin to the following points: (i) $[\sqrt{2}, 1, 1]$, (ii) $[-1, 2, 2]$, (iii) $[2, -3, 6]$.

Ans. (i) $[45°, 60°, 60°]$, (ii) $[\pi - \cos^{-1}\frac{1}{3}, \cos^{-1}\frac{2}{3}, \cos^{-1}\frac{2}{3}]$, (iii) $[\cos^{-1}\frac{2}{7}, \pi - \cos^{-1}\frac{3}{7}, \cos^{-1}\frac{6}{7}]$.

2. A line makes angles 60° and 45° with the positive axes of x and y respectively; what angle does it make with the positive axis of z?

Ans. 60° or 120°.

3. Show that the point $[3, -1, 2]$ is the centre of the sphere which passes through the four points $[2, 1, 4]$, $[5, 1, 1]$, $[4, -3, 0]$, $[1, -3, 3]$, and find its radius.

Ans. $r = 3$.

4. Find the centre and the radius of the sphere which passes through the four points $[-2, -2, 3]$, $[1, -5, 3]$, $[1, -2, 0]$, $[0, -6, -1]$.

Ans. $[-1, -4, 1]$, $r=3$.

5. A regular tetrahedron is placed with a vertex at the origin O, the altitude through O making equal angles with each of the three rectangular axes, and each of the edges through O lying in the same plane with the altitude and the corresponding coordinate-axis. Find the direction-cosines of the edges through O.

Ans. $[4, 1, 1]$, $[1, 4, 1]$, $[1, 1, 4]$ or $[0, 1, 1]$, $[1, 0, 1]$, $[1, 1, 0]$.

6. Find the actual direction-cosines of the line joining the origin to the point $[2u, 2v, u^2 + v^2 - 1]$.

Ans. Each divided by $u^2 + v^2 + 1$.

7. Show that the four points $[1, -1, -1]$, $[-1, 1, -1]$, $[-1, -1, 1]$, $[1, 1, 1]$ form the vertices of a regular tetrahedron and find the length of the edge.

Ans. $2\sqrt{2}$.

8. Show that $[-3, -3, -3]$, $[5, -1, -1]$, $[-1, 5, -1]$, $[-1, -1, 5]$ are the vertices of a regular tetrahedron whose centre is at the origin.

9. Prove that $[a, b, c]$, $[c, a, b]$, $[b, c, a]$, $[d, d, d]$ are the vertices of a regular tetrahedron with its centre at the origin when

$$a = t^2 + 3t + 1, \quad b = t^2 - t - 1, \quad c = -t^2 - t + 1, \quad d = -t^2 - t - 1,$$

t being any parameter.

10. Show that the four points [2, 9, 12], [1, 8, 8], [−2, 11, 8], [−1, 12, 12] are the vertices of a square.

11. Show that the six points [0, 1, −1], [0, −1, 1], [1, 0, −1], [1, −1, 0], [−1, 0, 1], [−1, 1, 0] form the vertices of a regular hexagon; and so also the points whose coordinates are $[a, b, c]$ and the permutations of these, where a, b, c are in arithmetical progression.

12. Show that the six points [−1, 2, 2], [2, −1, 2], [2, 2, −1], [1, −2, −2], [−2, 1, −2], [−2, −2, 1] form the vertices of a regular octahedron.

13. Show that the six points [1, 5, 6], [4, 2, 6], [4, 5, 3], [3, 1, 2], [0, 4, 2], [0, 1, 5] form the vertices of a regular octahedron.

14. OP, OQ are lines in the planes of zx, xy, bisecting the angles between the positive directions of the axes in these planes. Prove that the angle $POQ = 60°$. Hence show that six regular octahedra and eight regular tetrahedra will exactly fill up the space about a point.

15. Show that the 12 points $[0, \pm 1, \pm 1]$, taking all permutations, form the vertices of a polyhedron bounded by 6 squares and 8 equilateral triangles.

16. Show that the 24 points $[0, \pm a, \pm b]$, taking all permutations, form the vertices of a polyhedron bounded by 6 squares and 8 hexagons, and that the hexagons are regular if $a = 2b$.

17. Show that the 24 points $[\pm a, \pm b, \pm b]$, taking all permutations, form the vertices of a polyhedron bounded by 6 squares, 12 rectangles, and 8 equilateral triangles; and find the relation between a and b if the rectangles are squares. $(a > b.)$
Ans. $a^2 - 2ab - b^2 = 0$.

18. Show that the 48 points $[\pm 1, \pm(1+\sqrt{2}), \pm(1+2\sqrt{2})]$, where all permutations are taken, form the vertices of a polyhedron bounded by 6 regular octagons, 8 regular hexagons, and 12 squares.

19. Show that if $a^2-ab-b^2=0$ the 12 points $[0, \pm a, \pm b]$, $[\pm b, 0, \pm a]$, $[\pm a, \pm b, 0]$ are vertices of a regular icosahedron, and the 20 points $[0, \pm b, \pm(a+b)]$, $[\pm(a+b), 0, \pm b]$, $[\pm b, \pm(a+b), 0]$, $[\pm a, \pm a, \pm a]$ are vertices of a regular dodecahedron.

20. If the position of a point P is determined with reference to a rectangular coordinate-system by its radius-vector ρ, the angle $ZOP=\phi$, and the angle XOL, or θ, which the plane ZOP makes with ZOX, show that

$$x=\rho\sin\phi\cos\theta,\quad y=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi,$$

and

$$ds^2=d\rho^2+\rho^2 d\phi^2+\rho^2\sin^2\phi\, d\theta^2.$$

21. If A, B, C are three consecutive vertices of a parallelogram show that the coordinates of the fourth vertex are

$$x=x_A+x_C-x_B, \text{ etc.}$$