QUARTIC SURFACES

WITH SINGULAR POINTS

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PREFACE

THE purpose of the present treatise is to give a brief account of the leading properties, at present known, of quartic surfaces which possess nodes or nodal curves.

A surface which would naturally take a prominent position in such a book is the Kummer surface, together with its special forms, the tetrahedroid and the wave surface, but the admirable work written by the late R. W. H. T. Hudson, entitled *Kummer's Quartic Surface*, renders unnecessary the inclusion of this subject. Ruled quartic surfaces have also been omitted.

For the convenience of readers, a brief summary of all the leading results discussed in this book has been prefixed in the form of an Introduction.

I have to express my great obligation to Prof. H. F. Baker, Sc.D., F.R.S., who has given much encouragement and valuable criticism. Finally I feel greatly indebted to the staff of the University Press for the way in which the printing has been carried out.

C. M. JESSOP.

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Throughout, the vertices of the tetrahedron of reference are denoted by A_1, A_2, A_3, A_4: see p. 50.

pp. 38, 45. The ∞^1 quadrics $\psi + 2\lambda\phi + \lambda^2 w^2 = 0$ touch the surface $\phi^2 = w^2\psi$ along quadri-quartics. They are the quadrics mentioned on p. 59.

CORRIGENDA

p. 38, line 6, *for* $4w^2\psi$ *read* $w^2\psi$.
line 9, *for* close-points *read* pinch-points.

p. 40, last line but one, *for* be *read* be taken to be.
omit foot-note.

p. 76, foot-note, *insert* fourth edition.

INTRODUCTION

Ch. I. Quartic surfaces with isolated singular points.

This chapter, which is based on the results of Cayley* and Rohn, gives a method of classification of quartic surfaces which possess a definite number of isolated nodes and no nodal curves. The number of such nodes cannot exceed sixteen. Rohn has given a mode of classification for the surfaces having more than seven nodes, based on the properties of a type of seven-nodal plane sextic curves.

The equation of a quartic surface which has a node at the point $x = y = z = 0$, will be of the form

$$u_2w^2 + 2u_3w + u_4 = 0,$$

where $u_2 = 0$, $u_3 = 0$, $u_4 = 0$ are cones whose vertex is this point.

The tangent cone to the quartic whose vertex is the point is therefore

$$u_2u_4 - u_3^2 = 0.$$

The section of this cone by any plane gives a plane sextic curve having a *contact-conic* u_2, i.e. a conic which touches the sextic where it meets it. When the surface has eight nodes the tangent cone whose vertex is any one of them will have seven double edges which give seven nodes on the plane sextic.

Such sextics are divided into two classes, viz. those for which there is an infinite number of cubics through the seven nodes, and two other points of the curve, and those for which there is only one such cubic. When a quartic surface is such that it has eight nodes consisting of the common points of three quadrics, the tangent cone from any node to the surface gives rise to a plane sextic of the first kind: such a quartic surface is said to be

* *Recent researches*, etc., Proc. Lond. Math. Soc. (1869–71

syzygetic: the equation of the surface is represented by an equation of the form

$$(a \between A, B, C)^2 = 0,$$

where $A = 0$, $B = 0$, $C = 0$ represent quadrics whose intersections give the eight nodes.

The second, or general kind of sextic, arises from the general type of eight-nodal quartic surface which is said to be *asyzygetic*.

Similarly in the case of nine-nodal and ten-nodal quartic surfaces we have two kinds of plane sextics distinguished as above, giving rise to syzygetic and asyzygetic surfaces.

For ten-nodal surfaces there are two varieties of asyzygetic surfaces, one of which, the *symmetroid* (see Ch. IX), arises when the sextic curve consists of two cubic curves. The tangent cone from *each* of the ten nodes of this surface then consists of two cubic cones. There are also two varieties of ten-nodal syzygetic surfaces.

Seven points may be taken arbitrarily as nodes of a quartic surface, but if there is an eighth node it must either be the eighth point of intersection of the quadrics through the seven points, or, in the case of the general surface, lie upon a certain sextic surface, the *dianodal* surface, determined by the first seven nodes; hence it may not be taken arbitrarily.

When an eight-nodal surface has a ninth node the latter must lie on a curve of the eighteenth order, the *dianodal* curve.

Plane sextics with ten nodes and a contact-conic are divided into three classes according as they are the projections of the intersection of a quadric with (1) a cubic surface, (2) a quartic surface which also contains two generators of the same set of the quadric, (3) a quintic surface which also contains four generators of the same set of the quadric.

The first and second types of sextics are connected with eleven-nodal surfaces which are respectively asyzygetic and syzygetic; the third type gives a symmetroid with eleven nodes. A fourth surface arises when the sextic breaks up into two lines and a nodal quartic.

Twelve nodes on the quartic surface give rise to eleven nodes on the sextic, which must therefore break up into simpler curves; this process of decomposition goes on until we arrive at six straight lines, which case corresponds to the sixteen-nodal or Kummer surface.

There are four varieties of surfaces with twelve nodes of which one is a symmetroid: there are only *two* varieties of surfaces with thirteen nodes and only *one* with fourteen nodes, viz. that given by the equation

$$\sqrt{xx'} + \sqrt{yy'} + \sqrt{zz'} = 0.$$

An additional node arises for a surface having this equation, when there exists between the planes $x \dots z'$ the identity

$$Ax + By + Cz + A'x' + B'y' + C'z' \equiv 0,$$

with the condition

$$AA' = BB' = CC'.$$

If another such relation exists between the planes $x \dots z'$, there is a sixteenth node.

Ch. II. Desmic surfaces.

A surface of special interest which possesses nodes and no singular curve is the desmic surface. Three tetrahedra $\Delta_1, \Delta_2, \Delta_3$ are said to form a desmic system when an identity exists of the form

$$\alpha\Delta_1 + \beta\Delta_2 + \gamma\Delta_3 \equiv 0,$$

where Δ_i is the product of four factors linear in the coordinates.

It is easily deducible from this identity that the tetrahedra are so related that every face of Δ_3 passes through the intersection of faces of Δ_1 and Δ_2; hence we have sixteen lines through each of which one face of each tetrahedron passes. It is deducible as a consequence, that any pair of opposite edges of Δ_1 together with a pair of opposite edges of Δ_2 form a skew quadrilateral; and so for Δ_1 and Δ_3, Δ_2 and Δ_3.

It also follows that if the edges A_1A_2, A_1A_3, A_1A_4 of Δ_1 meet the respective edges of Δ_2 in LL', MM', NN'; then A_1, L, A_2, L' are four harmonic points; and so for A_1MA_3M', A_1NA_3N'. The relationship between the three tetrahedra is entirely symmetrical.

Hence we may construct a tetrahedron desmic to a given tetrahedron Δ, by drawing through any point A the three lines which meet the three pairs of opposite edges of Δ, then if the intersections of these three lines with the edges of Δ be LL', MM', NN' respectively, the fourth harmonics to A, L, L'; A, M, M'; A, N, N' will, with A, form a tetrahedron desmic to Δ.

The join of any vertex of Δ_1 and any vertex of Δ_2 passes through a vertex of Δ_3: there are therefore sixteen lines upon each of which one vertex of each tetrahedron lies. Hence any two desmic tetrahedra have four centres of perspective, viz. the vertices of the third tetrahedron.

If Δ_1 be taken as tetrahedron of reference the identity connecting Δ_1, Δ_2, Δ_3 is given by the equation

$$16xyzt$$
$$-(x+y+z+t)(x+y-z-t)(x-y+z-t)(\ \ x-y-z+t)$$
$$-(x+y+z-t)(x+y-z+t)(x-y+z+t)(-x+y+z+t)=0.$$

Closely connected with the system of tetrahedra Δ_i is a second desmic system of three tetrahedra D_i. They are afforded by the identity

$$(x^2-y^2)(z^2-t^2)+(x^2-t^2)(y^2-z^2)+(x^2-z^2)(t^2-y^2)=0.$$

The sixteen lines joining the vertices of the Δ_i are the sixteen intersections of the faces of the D_i.

A desmic surface is such that a pencil of such surfaces contains each of three such tetrahedra D_i in desmic position. The surface has as nodes the vertices of the corresponding tetrahedra Δ_i; hence the sixteen lines joining the vertices of the latter tetrahedra lie on the surface: along each of them the tangent plane to the surface is the *same*, i.e. the line is torsal; the tangent plane meets the surface also in a conic, and hence there are sixteen conics on the surface lying in these tangent planes.

There is a doubly-infinite number of quadrics through the vertices of any two tetrahedra Δ_i, the surface is therefore syzygetic; these quadrics meet the surface in three singly-infinite sets of quadri-quartics; one curve of each set passes through any point of the surface.

The coordinates of any point on the surface can be expressed in terms of two variables u, v as follows:

$$\rho x=\frac{\sigma_1(u)}{\sigma_1(v)},\qquad \rho y=\frac{\sigma_2(u)}{\sigma_2(v)},\qquad \rho z=\frac{\sigma_3(u)}{\sigma_3(v)},\qquad \rho t=\frac{\sigma(u)}{\sigma(v)};$$

since this leads to

$$(e_1-e_2)(x^2y^2+z^2t^2)+(e_3-e_1)(x^2z^2+y^2t^2)$$
$$+(e_2-e_3)(x^2t^2+y^2z^2)=0,$$

which is one form of equation belonging to the surface.

The three systems of twisted quartics are obtained by writing respectively

$$v = \text{constant}, \quad u - v = \text{constant}, \quad u + v = \text{constant}.$$

The generators of the preceding doubly-infinite set of quadrics form a cubic complex which depends merely on the twelve desmic points; all the lines through these points belong to the complex. Any line of this complex meets the surface in points whose arguments (u, v) are respectively

$$(\beta + \mu, \alpha), \quad (\beta - \mu, \alpha), \quad (\alpha + \mu, \beta), \quad (\alpha - \mu, \beta).$$

The tangents to the three quadri-quartics which pass through any point of the surface are bitangents of the surface, and their three other points of contact are collinear.

The curves $u = \text{constant}$, $v = \text{constant}$ form a conjugate system of curves on the surface: the system conjugate to $u + v = \text{constant}$ is $3v - u = \text{constant}$; the system conjugate to $u - v = \text{constant}$ is $3v + u = \text{constant}$; hence we derive the differential equation of conjugate tangents as

$$du du_1 + 3 dv dv_1 = 0.$$

The points of any plane section of the surface are divided into sets of sixteen points, lying upon three sets of four lines belonging to the cubic complex, where each line contains four of the sixteen points; denoting these twelve lines by $a_1 \dots a_4$, $b_1 \dots b_4$, $c_1 \dots c_4$, then if C is the curve enveloped by the lines of the cubic complex in the plane, the points of contact of the lines a lie on a tangent α of C, those of the lines b on a tangent β, and those of the lines c on a tangent γ; where α, β, γ are three concurrent lines.

If p, q, r are three lines of a cubic surface forming a triangle, then any three planes through p, q, r respectively meet the cubic surface in conics which lie on the same quadric; the locus of the vertices of such of these quadrics as are cones is a desmic surface.

Ch. III. Quartic surfaces with a double conic.

The equation of a quartic surface with a nodal conic has the form

$$\phi^2 = w^2 \psi.$$

This may be written

$$(\phi + \lambda w^2)^2 = w^2 (\psi + 2\lambda\phi + \lambda^2 w^2);$$

and hence can be brought to the form

$$\Phi^2 = w^2 V,$$

where $V = 0$ is a quadric cone, in five ways. Each tangent plane of the cones V_i meets the surface in a pair of conics. Among the conics arising from any particular cone V_1 there are eight pairs of lines; hence the surface contains sixteen lines. The relationship of these lines as regards intersection is the same as that of sixteen lines of the general cubic surface obtained by omitting any of its twenty-seven lines, p, together with the ten lines which intersect p.

The coordinates of any point on the surface can be expressed as cubic functions of two parameters by the equations

$$\rho x_i = f_i(\xi_1, \xi_2, \xi_3), \qquad (i = 1, 2, 3, 4);$$

so that every plane section of the surface is represented by a member of the family of curves $\sum_1^4 \alpha_i f_i = 0$; where $f_1 = 0, \ldots, f_4 = 0$ are plane cubic curves which have five common points; hence the surface is rational and is represented on a plane. Each of these five points, the *base-points* of the representation, is the *image* of a line of the surface. The other lines of the surface are represented in the plane by the conic through the base-points and by the ten lines joining pairs of base-points.

This method enables us to determine the varieties of curves of different orders which can exist on the surface, by use of the equation

$$N = 3n - \Sigma\alpha_i,$$

where N is the order of the curve on the surface, n that of its image in the plane, and α_i the number of times the curve on the surface meets one of the lines represented by the base-points. It is found that the sixteen lines previously mentioned are the only lines on the surface; the only conics on the surface, apart from the double conic, are those in the tangent planes of the cones V_i.

We obtain ∞^2 twisted cubics on the surface, and also ∞^4 quadri-quartics together with ∞^3 twisted quartics of the second species. It is seen that the quadrics

$$\psi + 2\lambda\phi + \lambda^2 w^2 = 0$$

touch the surface along quartics. The class of the surface is twelve.

The surface may also be obtained by aid of any two given quadrics Q and H and any given point O, as follows: the surface is the locus of a point P such that the points O, P, K, P' are harmonic, P and P' conjugate for H, and K any point of Q; P' also lies on the surface.

The twenty-one constants of the surface are seen to arise from those of Q and H, and the coordinates of O. This point is the vertex of one of the five cones V_i; the vertices of the other four cones are the vertices of the tetrahedron which is self-polar for Q and H. The double conic is the intersection of H with its polar plane for O.

From the foregoing mode of origin of the surface O is said to be a *centre of self-inversion* of the surface with regard to the quadric H.

The surface may be related to the general cubic surface by a (1, 1) correspondence in two ways, the relationship being a perspective one in each case.

The surface is connected with the general quartic curve as follows: the tangent cone drawn to the surface from any point P of the double conic is of the fourth order, its section being the general quartic curve; the tangent planes from P to the five cones V_i, and the tangent planes to the surface at P, meet the plane of the quartic curve in lines bitangent to this curve.

The other sixteen bitangents arise from the planes passing through P and the sixteen lines of the surface. The cone whose vertex is P and base a conic of the surface meets the plane of the quartic curve in a conic which has four-point contact with the quartic.

The general quartic surface with a double conic is obtained by Segre as the projection from any point A of the intersection Γ of two quadratic manifolds or *varieties* $P=0$, $\Phi=0$, in four dimensions, upon any given hyperplane S_3. Among the varieties of the pencil $F+\lambda\Phi=0$ there are five *cones*, i.e. members of the pencil containing only four variables homogeneously; each cone possesses an infinite number of generating planes consisting of two sets, and each generating plane meets Γ in a conic. These generating planes are projected from A upon S as the tangent planes of a quadric cone. Hence arise the five cones of Kummer, and the conics lying in their tangent planes.

The double conic is obtained as the projection from A on S of the quadri-quartic which is the intersection with Γ of the tangent hyperplane at A of the variety which passes through A. When A lies on one of the five cones of the pencil $F+\lambda\Phi=0$, this quadri-quartic becomes two conics in planes whose line of intersection passes through A. Hence the conics are projected into intersecting *double lines* of the quartic surface. By this projective method the lines and conics of the quartic surface may be obtained, as also its properties generally.

Ch. IV. Quartic surfaces with a nodal conic and additional nodes.

A quartic surface with a nodal conic may also have isolated nodes, but their number cannot exceed four. Each such node is the vertex of a cone of Kummer, and for every node the number of these cones is reduced by unity. There are two kinds of surfaces with two nodes, in one case the line joining the nodes lies on the surface, and in the other case it does not. Nodes arise when the base-points of the representation of the surface on a plane have certain special positions; if either two base-points coincide, or if three are collinear, there is a node on the surface. If *either* a coincidence of two base-points *or* a collinearity of three base-points occurs twice, the quartic surface has two nodes and is of the first kind just mentioned; if there is one coincidence together with one collinearity, the quartic surface is of the second kind.

There are three nodes when two base-points coincide and also two of three collinear base-points coincide; finally, when the join of two coincident base-points meets the join of two other coincident base-points in the fifth base-point, there are four nodes.

Three coincident base-points give rise to a binode, *four* coincident base-points give rise to a binode of the second kind, i.e. when the line of intersection of the tangent planes lies in the surface, and *five* to a binode of the third species, i.e. when the line of intersection is a line of contact for one of the nodal planes.

When four base-points come into coincidence in an indeterminate manner we have a ruled surface; a special variety occurs when the fifth base-point coincides with them in a determinate manner.

The double conic may be *cuspidal*, i.e. when the two tangent planes to the surface at each point of it coincide; the class of this surface is six. The equation of the surface may in this case be reduced to the form

$$U^2 + x_1^3 x_2 = 0.$$

The surface has two *close-points* C, C' given by

$$x_1 = x_2 = U = 0.$$

If K be any point of CC' and π the polar plane of K for $U=0$, then if any line through K meets π in L, it will meet the surface in four points P, P'; Q, Q' such that the four points K, P, L, P' and K, Q, L, Q' are harmonic.

The double conic may consist of two lines; the necessary condition for this is that three cubics of the system representing plane sections should be

$$\alpha u = 0, \quad \alpha v = 0, \quad \beta u = 0,$$

where $\alpha = 0$, $\beta = 0$ are lines, and $u = 0$, $v = 0$ are conics. Either or both of the double lines may be cuspidal.

Segre's method (Ch. III) affords a means of complete classification of quartic surfaces with a double conic, by aid of the theory of elementary factors. We thus obtain seven types, each type leading to sub-types.

There exists in the case of certain of these sub-types a *cone of the second order* in the pencil (F, Φ), i.e. a cone whose equation contains only three variables, say x_1, x_2, x_3; if the line $x_1 = x_2 = x_3$, which may be termed the *edge* of this cone, lies upon Γ, the surface is ruled. If the point of projection, A, is so chosen that the tangent hyperplane for A, of the variety which passes through A, is also a tangent hyperplane of this cone of the second order, the double conic is cuspidal.

When the pencil (F, Φ) consists entirely of cones of the first order having a common generator, and a common tangent hyperplane along this generator, the surface is that of Steiner.

Segre's table, which distinguishes each surface that can arise, is given on pp. 82–85.

Ch. V. The cyclide.

When the double conic is the section of a sphere by the plane at infinity, we obtain the cyclide. The equation of the cyclide is therefore $S^2 + U = 0$, where $S = 0$ is a sphere and $U = 0$ is a quadric.

This equation may be written in the form

$$\{x^2+y^2+z^2-2\lambda\}^2+4\{(A_1+\lambda)x^2+(A_2+\lambda)y^2+(A_3+\lambda)z^2 + 2B_1x+2B_2y+2B_3z+C-\lambda^2\}=0.$$

The second member of the left side will give a cone when λ is a root of the quintic $F(\lambda)=0$, where $F(\lambda)$ is the discriminant of the second member. We thus obtain as in Ch. III five cones V_i; the tangent planes of each cone meet the surface in pairs of circles.

There are five sets of bitangent spheres of the surface; each sphere of any set cuts a fixed sphere orthogonally, and its centre lies on a fixed quadric. The centres of the five fixed spheres are the vertices of the cones V_i.

These five spheres $S_1 \ldots S_5$ are mutually orthogonal, and the centres of any four of them form a self-polar tetrahedron for the fifth sphere and its corresponding quadric Q.

The equations of a pair S_i, Q_i are respectively

$$x^2+y^2+z^2+\frac{2B_1x}{A_1+\lambda_i}+\frac{2B_2y}{A_2+\lambda_i}+\frac{2B_3z}{A_3+\lambda_i}+2\lambda_i=0;$$

$$\frac{x^2}{A_1+\lambda_i}+\frac{y^2}{A_2+\lambda_i}+\frac{z^2}{A_3+\lambda_i}+1=0;$$

where λ_i is one of the roots of $F(\lambda)=0$.

The five quadrics $Q_1 \ldots Q_5$ are confocal; the curve of intersection of a pair S_i, Q_i is a focal curve of the surface.

The centre of a sphere S_i is a centre of self-inversion for the surface.

Three of the quadrics Q_i are necessarily real together with their corresponding spheres: one is an ellipsoid, one a hyperboloid of one sheet and one a hyperboloid of two sheets.

The surface is also obtained as the locus of the limiting points defined by S_i and the tangent planes of Q_i. Taking Q_i as an ellipsoid, this shows the shape of the surface to be one of the following:

(i) two ovals, one within the other, when S_i, Q_i do not intersect;

(ii) two ovals, external to each other, *or* a tubular surface similar to the anchor-ring, when the focal curve (S_i, Q_i) consists of two portions;

(iii) one oval, when the focal curve (S_i, Q_i) consists of one portion.

When $(\lambda + A_i)^2$ is a factor of $F(\lambda)$, one of the cones V is a pair of planes. If two roots of $F(\lambda)$ are equal, one of the principal spheres is a point-sphere. In a real cyclide only one principal sphere can be a point-sphere. Real cyclides must possess at least two principal spheres which are not point-spheres.

If $S_1 = 0, \dots S_5 = 0$ are any five spheres, there is a quadratic identity between the quantities $S_1 \dots S_5$, viz. that given by the equation

$$\begin{vmatrix} 0 & S_1 & \dots\dots\dots\dots & S_5 \\ S_1 & -2r_1^2 & \pi_{12} \;\dots & \pi_{15} \\ S_2 & \pi_{12} & -2r_2^2 & \dots\dots \\ \vdots & \vdots & & \\ S_5 & \pi_{15} & \dots\dots\dots & -2r_5^2 \end{vmatrix} = 0,$$

where $r_1 \dots r_5$ are the radii of the spheres, and π_{ij} is the mutual *power* of the spheres $S_i = 0$, $S_j = 0$.

By solution of the equations

$$S_1 \equiv x^2 + y^2 + z^2 + 2f_1x + 2g_1y + 2h_1z + c_1, \text{ etc.},$$

it is seen that $x^2 + y^2 + z^2$, x, y, z, and unity, can be expressed as linear functions of $S_1 \dots S_5$; hence the equation of a cyclide appears as a quadratic function of $\frac{S_1}{r_1}, \dots \frac{S_5}{r_5}$, which are themselves connected by a quadratic identity. This gives rise to seven chief types of cyclide, by application of the theory of elementary factors; but only three of them give real cyclides, viz.

$$[11111], \quad [2111], \quad [311].$$

Each of these types and the corresponding sub-types, with the exception of the general cyclide, arise as the inverses of quadrics. The sub-type [(11) 111] can be expressed in terms of three variables. It is the envelope of spheres which pass through a fixed point and whose centres lie on a conic; contact with the envelope here occurs along a circle. It has also two systems of bitangent spheres, as in the general case. A variable sphere of one of these systems makes with two fixed spheres of the first system angles whose sum or whose difference is constant. The inverse of this cyclide is a cone.

The cyclide [(11) (11) 1] is known as *Dupin's cyclide*. There are two systems of spheres which touch the cyclide along circles; the spheres of each system cut one of the principal spheres at a constant angle. The spheres of either system are obtained as those which touch any two fixed spheres of the other system and have their centres on a given plane.

Denoting $\frac{S_i}{r_i}$ by x_i, the equation of the general cyclide appears as $\sum_1^5 a_i x_i^2 = 0$, with the condition $\sum_1^5 x_i^2 = 0$.

The system of cyclides $\sum_1^5 \frac{x_i^2}{a_i + \lambda} = 0$ is confocal with the first cyclide. Three confocals pass through any point and cut orthogonally.

The system of quadrics $V = 0$, where $V \equiv U + kS - k^2$, which touch the cyclide $S^2 + 4U = 0$ along sphero-conics are such that *two* of them pass through any point, *three* touch any line, *four* touch any plane. The four points of contact of the surfaces V which touch any given plane π are the centres of self-inversion for the section of the cyclide by π.

The locus of points of contact of common tangent planes of the cyclide and any given quadric V is a line of curvature on the cyclide.

The Cartesian equation of the system of confocals is

$$(A_1 + \lambda)(A_2 + \lambda)(A_3 + \lambda) S^2 + 4F(\lambda) Q = 0,$$

where S, Q have the same form as the S_i, Q_i when λ is substituted for λ_i.

The confocals to the given cyclide $S^2 + 4U = 0$, where

$$S \equiv x^2 + y^2 + z^2 - 2\lambda,$$

may be obtained as follows: when $S + 2L = 0$ is a point-sphere and $U + L^2 = 0$ is a cone, the locus of the centres of these point-spheres is a cyclide confocal with $S^2 + 4U = 0$.

Ch. VI. Surfaces with a double line: Plücker's surface.

The quartic surface with a double line is cut by any plane through the double line in a conic also. In eight cases this conic breaks up into a pair of lines, giving sixteen lines on the surface. There is no other line on the surface with the exception of the double line.

There are sixty-two planes not passing through the double line each of which meets the surface in a pair of conics, one of whose intersections lies on the double line. By aid of one of these conics c^2 the surface may be represented on a plane; for through any point x of the surface one line can be drawn to meet c^2 and also the double line, so that with each point of the surface one such line is associated. This line is determined as the intersection of two planes, each of whose coefficients contains linearly and homogeneously three parameters ξ_1, ξ_2, ξ_3. A third equation, arising from the equation of the surface, is that of a plane whose coefficients are quadratic in the ξ_i, and the intersection of these three planes is a point on the surface; hence we obtain a (1, 1) correspondence between the points x of the surface and the points ξ of a plane.

There are nine *base-points* in the plane, eight of which we represent by $B_1 \ldots B_8$; they correspond to the points of eight non-intersecting lines of the surface, together with a point A which corresponds to any point of the conic coplanar with c^2. These nine points cannot constitute the complete intersection of two cubic curves.

To any plane section of the surface there corresponds, in the plane of ξ, a quartic curve having a node at A and passing through the points B_i. The cubic through the nine base-points corresponds to the double line.

The plane image of any curve of order M on the surface is a curve of order m, where

$$M = 4m - 2\beta - \Sigma\alpha,$$

β being the number of times the curve on the surface meets the conic corresponding to A, and $\Sigma\alpha$ the total number of passages of the image through the points B_i.

By applying Rohn's method to the surface, using any point on the double line as that from which a tangent cone is drawn, it is easy to see the modifications which arise when isolated nodes exist.

The section of the tangent cone, whose vertex is any point of the double line, is a sextic curve, meeting the double line in a quadruple point; with each additional node of the surface this curve acquires an additional node: when there are seven nodes the sextic becomes a nodal cubic, meeting the double line in one point together with three lines through this point. When the surface has eight nodes, the sextic curve becomes a conic together with four lines concurring at a point of the double line.

In the case of seven nodes there are three torsal lines meeting the double line, and each containing two nodes; also there are four tropes meeting in the seventh node, and each containing four nodes. If there are eight nodes we have Plücker's surface which has also eight tropes. The nodes form two tetrahedra, each of which is inscribed in the other. The nodes lie in pairs on four torsal lines meeting the double line. Through any two nodes not on the same torsal line there pass two tropes. The tropes can be arranged in four pairs so that the line of intersection of a pair meets the double line in a pinch-point.

Plane sections of Plücker's surface are represented by quartic curves having a common node and touching, at fixed points, four concurrent lines.

Ch. VII. Quartic surfaces containing an infinite number of conics: Steiner's surface: the quartic monoid.

The nature of the quartic surfaces which contain an infinite number of conics was investigated by Kummer. He showed the existence of the following classes: surfaces with a double conic or a double line; ruled quartic surfaces; the surface $\Phi^2 = \alpha\beta\gamma\delta$, where $\Phi = 0$ is a quadric and $\alpha, \beta, \gamma, \delta$ coaxal planes; Steiner's surface.

To these surfaces discussed by Kummer must be added the surface whose equation is

$$\{xw + f(y, z, w)\}^2 = (z, w \between a)^4.$$

The surface $\Phi^2 = \alpha\beta\gamma\delta$ has two tacnodes at the intersection of the common axis of the planes with Φ; it is birationally transformable into a cubic cone. The conics of the surface can be arranged in sets of four lying on the same quadric; the quadric cone whose vertex is on the axis of the planes $\alpha \ldots \delta$, and whose base is any conic of the surface, meets the surface in four conics.

Steiner's surface is of the third class and has four tropes; the coordinates of any point of the surface are expressible as homogeneous quadratic functions of three variables; conversely any surface, the coordinates of whose points are so expressible, is a Steiner surface. The surface has a triple point, three double lines meeting in the triple point, and a node on each double line. A characteristic property of the surface is that its section by any tangent plane breaks up into two conics. Every algebraic curve on the surface is of even order.

The surface being determined by the equations

$$\rho x_i = f_i(\eta_1, \eta_2, \eta_3) \qquad (i = 1, 2, 3, 4),$$

we are enabled to map the surface on the plane of the η_i.

Any conic of the surface is represented on the η-plane by a straight line, the pair of lines representing two conics in the same tangent plane of the surface are represented by the equations

$$\eta_1 + m\eta_2 + n\eta_3 = 0, \quad \eta_1 + \frac{1}{m}\eta_2 + \frac{1}{n}\eta_3 = 0.$$

The surface contains ∞^5 quartic curves of the second species, which are represented by the general conic in the plane of η; also ∞^4 quadri-quartics having a node on one of the double lines; they lie on quadrics passing through two double lines, and are represented by conics $\Sigma a_{ik}\eta_i\eta_k = 0$, in which two of the quantities a_{11}, a_{22}, a_{33} are equal.

The conics apolar to the four conics $f_i = 0$ form the pencil

$$u_\alpha^2 + \lambda u_\beta^2 = 0;$$

the conics of this pencil are inscribed in the same quadrilateral, and form the images on the plane of η of the asymptotic lines of the surface.

A form of the preceding property of the surface, that its coordinates are expressible as homogeneous quadratic functions of two variables, is the following: in the general quadric transformation

$$\rho x_i = f_i(\alpha_1, \alpha_2, \alpha_3, \alpha_4),$$

the locus of x is a Steiner surface when the locus of α is a plane. From this we derive the fact that Steiner's surface, and the cubic polar of a plane with reference to a general cubic surface, are reciprocal.

Another mode of origin of the surface, given by Sturm, is that if a pencil of surfaces of the second class is projectively related to the points of a line in such a way that the line meets one *conic* c^2 of the system in a point corresponding to c^2, and another *conic* c'^2 in a point corresponding to c'^2, then the envelope of the tangent cones drawn from the points of the line to the corresponding surfaces is a Steiner surface.

Weierstrass and Schröter have shown that a Steiner surface arises as a locus connected with a known theorem for the quadric. The theorem is that if through any given point A of a quadric any three mutually perpendicular lines are drawn, meeting the quadric again in L, M, N, then the plane LMN meets the normal at A in a fixed point.

This theorem may be generalized as follows: if A be joined to the vertices of *any* triangle self-polar for a given conic c^2 in a given plane α, and the joining lines meet the quadric again in L, M, N, then the plane LMN meets the line AR in a fixed point S, where R is the pole for c^2 of the trace on α of the tangent plane to the quadric at A.

If now c^2 is a member of the ∞^2 conics

$$\eta_1 U + \eta_2 V + \eta_3 W = 0,$$

where $U=0$, $V=0$, $W=0$ are given conics, we have a point S determined for each set of values of $\eta_1 : \eta_2 : \eta_3$. On giving these ratios all values the locus of S is a Steiner surface; for it can be shown that if the coordinates of S are $y_1 \ldots y_4$, we have

$$y_1 : y_2 : y_3 : y_4 = f_1(\eta) : f_2(\eta) : f_3(\eta) : f_4(\eta),$$

where the f_i are quadratic functions of the η_i.

Properties of Steiner's surface may be deduced by aid of the transformation

$$x_i y_i = \rho \qquad (i = 1, 2, 3, 4),$$

applied to any plane $\Sigma a_i x_i = 0$, giving the cubic surface

$$\Sigma \frac{a_i}{y_i} = 0,$$

which is the reciprocal of Steiner's surface.

Steiner's surface is one example of a type of surfaces known as *monoids*, viz. surfaces of the nth order which have an $(n-1)$-fold point. The equation of the quartic monoid may be written

$$wu_3 + u_4 = 0,$$

where $u_3 = 0$, $u_4 = 0$ are cones having their vertices at the triple point. The surface contains twelve lines, the intersections of $u_3 = 0$ and $u_4 = 0$. The surface is projectively related to any plane, e.g. the plane $w = 0$, in a $(1, 1)$ manner, except that *every* point of each of these twelve lines is represented by *one* point only, viz. where the line meets the plane $w = 0$.

The surface contains conics in planes through two of the twelve lines, and twisted cubics on quadric cones passing through five of the twelve lines. The ∞^1 quadric cones passing through any four of the twelve lines meet the surface in quartic curves having a node at the triple point; the ∞^1 cubic cones passing through any eight of the twelve lines meet the surface in quartic curves without double points. If the lines corresponding to a curve of each type together make up the twelve lines, these two curves lie on one quadric. All these quartic curves are quadri-quartics.

Quartic curves of the second species arise as the intersection with the surface of cubic cones having six of the twelve lines as simple lines and one of them as double line; there are 5544 such quartic curves on the surface. The surface will have a line not passing through the triple point provided that three of the twelve lines are coplanar.

The cases of the quartic monoid of special interest are those in which there are six nodes; here the twelve lines coincide in pairs six times.

There are two cases of such surfaces; in the first case the six nodes may have any positions, this surface is a special case of the symmetroid; for the symmetroid being the result of eliminating the x_i from the equations

$$\alpha_1 \frac{\partial S_1}{\partial x_i} + \alpha_2 \frac{\partial S_2}{\partial x_i} + \alpha_3 \frac{\partial S_3}{\partial x_i} + \alpha_4 \frac{\partial S_4}{\partial x_i} = 0, \qquad (i = 1, 2, 3, 4),$$

where the α_i are regarded as point-coordinates, the surface considered is the special case in which one of the quadrics $S_i = 0$ is a plane taken doubly. The tangent cone to the surface whose vertex is one of the six nodes breaks up into two cubic cones. In the other case the six nodes lie on a conic whose plane is a trope of the surface. Each kind of surface has the same number of constants, viz. twenty-one.

Ch. VIII. Rational quartic surfaces.

The quartic surfaces with a triple point or with a double curve have been seen to be rational, i.e. the coordinates of the points of such a surface are expressible as rational functions of two parameters. Nöther has shown that there are only three rational quartic surfaces apart from them. The first of these surfaces has a *tacnode,* i.e. is such that every plane through the node meets the surface in a quartic curve having two consecutive double points at the node. The coordinates of any point x of the surface are projectively related to the points y of a *double plane* by the equations

$$\rho x_1 = y_1, \quad \rho x_2 = y_2, \quad \rho x_3 = y_3, \quad \rho x_4 = \frac{-\chi_2(y) \pm \sqrt{\Omega(y)}}{y_1},$$

where $\chi_2(y) = 0$ is a conic and $\Omega(y) = 0$ is a general quartic curve.

Clebsch showed that the points y can be expressed as rational functions of new variables z_i in such a way as to render $\sqrt{\Omega(y)}$ a rational function of the z_i, viz. by equations of the form

$$\sigma y_i = f_i(z), \qquad (i = 1, 2, 3),$$

where the curves $f_i(z) = 0$ are cubics having seven points in common. The plane sections of the surface have then as their images, in the field of the z_i, *sextic curves* having the seven points as nodes and also four other common points; the eleven points lie on the same cubic.

If the quartic surface has the equation

$$x_4^2 f_2 + 2x_4 f_3 + f_4 = 0,$$

we obtain

$$x_1 : x_2 : x_3 : x_4 = y_1 : y_2 : y_3 : \frac{-f_3(y) \pm \sqrt{\Omega(y)}}{f_2(y)},$$

where $\Omega(y) = 0$ is a sextic curve. It is shown that $\sqrt{\Omega(y)}$ is capable of rationalization only in the following two cases, viz. (1) when $\Omega(y) = 0$ is a sextic with a quadruple point; (2) when $\Omega(y) = 0$ is a sextic with two consecutive triple points.

The transformation to the simple plane is effected by the consideration that to plane sections through the double point there must correspond, in the simple plane, curves of order n of the same genus as these sections, viz. two, and intersecting each other in two variable points. This gives the equations

$$n^2 - 2 = \alpha_1 + 4\alpha_2 + \dots + r^2\alpha_r,$$

$$\frac{n(n+3)}{2} - 1 = \alpha_1 + 3\alpha_2 + \dots + \frac{r(r+1)}{2}\alpha_r,$$

where α_1 is the number of points the curves in the z-plane have in common, α_2 the number of double points they have in common, and so on. By aid of Cremona transformations repeatedly applied, it is seen that these curves of order n are capable of being replaced by one of the following types: (1) the curves $c_4(a^2b_1 \dots b_{10})$, (2) the curves $c_6(a_1^2 \dots a_8^2 b_1 b_2)$; i.e. quartic curves with one common node and ten common points, or sextic curves with eight common nodes and two common points. In each case the fixed points lie on one cubic.

The substitutions $\rho y = c_4(z)$, $\rho y = c_6(z)$ will then rationalize $\sqrt{\Omega(y)}$ in the two cases respectively mentioned, and hence lead to two rational quartic surfaces.

Ch. IX. Determinant surfaces.

The quartic surface whose equation is $\Delta = 0$, where Δ is a determinant of four rows whose elements are linear functions of the coordinates, depends upon thirty-three constants, one less than the general quartic surface. Taking as its equation

$$\begin{vmatrix} p_x & q_x & r_x & s_x \\ p_x' & q_x' & \dots\dots & \\ p_x'' & \dots\dots & & \\ p_x''' & \dots\dots & & \end{vmatrix} = 0,$$

it is seen that the surface contains two sets of sextic curves, viz. the curves

$$\left\| \begin{matrix} p_x & \dots\dots & & s_x \\ p_x' & \dots\dots & & \\ p_x'' & \dots\dots & & \\ p_x''' & \dots\dots & & \\ a & b & c & d \end{matrix} \right\| = 0, \qquad \left\| \begin{matrix} p_x & q_x & r_x & s_x & A \\ p_x' & \dots\dots & & & B \\ p_x'' & \dots\dots & & & C \\ p_x''' & \dots\dots & & & D \end{matrix} \right\| = 0.$$

Denoting these two kinds of curves by c_6 and k_6, it is found that any two curves of the same kind meet in four points, any two curves of different kinds in fourteen points. Any two curves of different kinds lie on a cubic surface.

The surface can be birationally transformed into itself by aid of the three sets of equations

$$\begin{aligned} \lambda_1 p_x + \lambda_2 q_x + \lambda_3 r_x + \lambda_4 s_x &= 0, & \alpha_1 p_y + \alpha_2 p_y' + \alpha_3 p_y'' + \alpha_4 p_y''' &= 0, \\ \lambda_1 p_x' + \dots\dots &= 0, & \alpha_1 q_y + \dots\dots &= 0, \\ \lambda_1 p_x'' + \dots\dots &= 0, & \alpha_1 r_y + \dots\dots &= 0, \\ \lambda_1 p_x''' + \dots\dots &= 0; & \alpha_1 s_y + \dots\dots &= 0; \end{aligned}$$

$$\begin{aligned}\lambda_1 P_1 + \lambda_2 Q_1 + \lambda_3 R_1 + \lambda_4 S_1 &= 0,\\ \lambda_1 P_2 + \ldots\ldots\ldots\ldots\ldots &= 0,\\ \lambda_1 P_3 + \ldots\ldots\ldots\ldots\ldots &= 0,\\ \lambda_1 P_4 + \ldots\ldots\ldots\ldots\ldots &= 0;\end{aligned}$$

where $$P_i = \alpha_1 p_i + \alpha_2 p_i' + \alpha_3 p_i'' + \alpha_4 p_i'''.$$

If we regard the λ_i and the α_i as point-coordinates we pass, by aid of these equations, from a point x of Δ to a point λ of a surface Σ, thence to a point α of a surface Σ' and finally to a point y of Δ.

From the preceding equations we deduce that if x is any point of a curve c_6, the point x determines a trisecant of c_6 whose fourth intersection with Δ is the point y, which corresponds to x.

These trisecants, as x describes c_6, form a ruled surface of the eighth order, whose intersection with Δ is c_6 taken triply together with a curve of the fourteenth order, the locus of the points y on Δ corresponding to the points x of c_6.

When the determinant Δ is symmetrical, i.e. if

$$p' \equiv q, \quad p'' \equiv r, \quad p''' \equiv s, \text{ etc.,}$$

the surfaces Σ and Σ' coincide; and the quantities P_i, Q_i, etc. are in this case the partial derivatives of a quantity which is quadratic in the α_i; if, changing the notation, we represent this quantity by S_i, the last set of equations take the form

$$\sum_{i=1}^{i=4} x_i \frac{\partial S_j}{\partial y_i} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

on replacing λ and α by x and y respectively.

Thus the surface Σ is the Jacobian J, of four quadrics. The surface $\Delta = 0$, where Δ is a symmetrical determinant, is known as the *symmetroid*; if in the first set of preceding equations we replace x, λ, y and α by α, x, β and y respectively, and express that $q \equiv p'$, etc., these equations assume the form

$$\sum_{i=1}^{i=4} \alpha_i \frac{\partial S_i}{\partial x_j} = 0, \quad \sum_{i=1}^{i=4} \beta_i \frac{\partial S_i}{\partial y_j} = 0, \quad (j = 1, 2, 3, 4)\ldots\ldots(2).$$

The surface Δ, the locus of the points α, is obtained by eliminating the x_i, or the y_i, from these equations. The surface J is seen to be the locus of vertices of cones of the system

$$\sum_1^4 \alpha_i S_i = 0.$$