CHAPTER I

QUARTIC SURFACES WITH ISOLATED SINGULAR POINTS

1. The singular points possessed by a quartic surface may consist either of a certain number of isolated nodes or may form double curves.

In the present chapter we discuss the quartic surfaces which have an assigned number of nodes, beginning with those which have four nodes, and give a definite method of classification for all the cases in which the number of nodes exceeds seven.

The number of isolated nodes of a quartic surface cannot exceed sixteen; for the class of a surface of order n which has δ double points is $n(n-1)^2-2\delta$, since this is the number of points of intersection of the surface and its first polars for two points A and B, diminished by the number 2δ of these intersections arising from each double point (a simple point on the polars of both A and B). Hence if n is four, δ cannot exceed sixteen.

2. Quartic surfaces with four to seven nodes.

Since the equation of the general quartic surface contains thirty-four constants, the surface with four given nodes should contain $34-16=18$ constants; if then $A=0$, $B=0$, $C=0$, $D=0$, $E=0$, $F=0$ are six linearly independent quadrics through the four nodes, the equation

$$(a \between A, B, C, D, E, F)^2=0,$$

containing apparently twenty constants, is a quartic surface having the given nodes.

The number of constants is really eighteen, since there are two quadratic relations between the six quadrics, as may be seen by taking the four given points as vertices of the tetrahedron of reference, in which case the quadrics may be taken to be

$$x_1x_2,\quad x_1x_3,\quad x_1x_4,\quad x_3x_4,\quad x_2x_4,\quad x_2x_3,$$

between which there exist the identities

$$x_1x_2 \,.\, x_3x_4=x_1x_3 \,.\, x_2x_4=x_1x_4 \,.\, x_2x_3.$$

For five nodes, taking $A \dots E$ as quadrics passing through the given nodes, the equation

$$(a \between A, B, C, D, E)^2 = 0,$$

containing fourteen constants, represents the general quartic having these given nodes.

The general quartic with six nodes is represented by the equation

$$(a \between A, B, C, D)^2 + \rho J = 0,$$

where A, B, C, D are quadrics through the six nodes, and J is the Jacobian of the four quadrics. For this equation contains ten constants and J has the given points $D_1 \dots D_6$ as nodes, moreover J cannot be expressed as a quadratic function of A, B, C, D.

The following properties of J may be used to establish these results. The surface $J=0$ is the locus of vertices of cones of the system

$$A + \lambda B + \mu C + \nu D = 0;$$

now each point of the line joining any two double points, e.g. $D_1 D_2$, is the vertex of such a cone, hence J contains the join of any two double points; also since $D_1 D_2 \dots D_1 D_6$ lie on J it follows that D_1 is a node of J; similarly for $D_2 \dots D_6$. Again there are ten pairs of planes passing through the points $D_1 \dots D_6$, and each point of the line of intersection of such a pair of planes satisfies the condition of being the vertex of a cone of the system. Hence such a line lies upon J, which thus contains $15 + 10 = 25$ lines. Again, since any quadric of the system is linearly expressible in terms of any four members of the system, it is so expressible in terms of any four of the previous pairs of planes; hence if J were expressible as a quadratic function of A, B, C and D, we should necessarily have a relation of the form

$$J \equiv (a \between \alpha\alpha', \beta\beta', \gamma\gamma', \delta\delta')^2,$$

in which we may take the planes α, β, γ to contain the line $D_1 D_2$, while δ, δ' do not contain it, e.g.,

$$\alpha \equiv (D_1, D_2, D_3), \quad \alpha' \equiv (D_4, D_5, D_6), \quad \text{etc.,}$$

while $$\delta \equiv (D_1, D_3, D_5), \quad \delta' \equiv (D_2, D_4, D_6).$$

Hence since J contains $D_1 D_2$ such a relation is impossible.

The general quartic with seven nodes is represented by the equation

$$(a \between A, B, C)^2 + \rho \Sigma = 0,$$

where A, B, C are quadrics through the given nodes and Σ is any quartic surface having the seven nodes*.

* This quartic surface may also be expressed in terms of the quartic surfaces which have one of the given points as a triple point and the other six as double points; if $T_1 \dots T_7$ are these surfaces, the required general quartic surface is $\Sigma a_i T_i = 0$, $i = 1, \dots 7$.

3. Quartic surfaces with more than seven nodes.

The equation of a quartic surface having a node at the point $x = y = z = 0$ will be of the form

$$u_2 w^2 + 2u_3 w + u_4 = 0,$$

where $u_2 = 0$, $u_3 = 0$, $u_4 = 0$ are cones whose vertex is the node.

The equation of the tangent cone drawn to the surface from this node is

$$u_2 u_4 - u_3^2 = 0.$$

The section of this cone by any plane, e.g. the plane $w = 0$, is a sextic curve with a "contact-conic," i.e. a conic which touches it wherever it meets it.

If the surface has any other node, the tangent cone will have a double line passing through this new node and giving rise to a node on this sextic; we obtain the different varieties of quartic surfaces possessing nodes by consideration of all special cases of sextic curves with a contact-conic*.

It is to be noted that the existence of a contact-conic $u_2 = 0$ of a sextic implies also a contact-quartic $u_4 = 0$; if a sextic has another contact-conic $v_2 = 0$, and hence another contact-quartic $v_4 = 0$, an identity exists of the form

$$u_2 u_4 - u_3^2 \equiv v_2 v_4 - v_3^2.$$

Now by multiplying the equation of the surface by u_2 we derive

$$(u_2 w + u_3)^2 + u_2 u_4 - u_3^2 = 0,$$

hence in the present case

$$(u_2 w + u_3)^2 + v_2 v_4 - v_3^2 = 0 \quad \dots\dots\dots\dots\dots\dots(1).$$

Denoting by c_8 the intersection of the quartic surface with the cone $v_2 = 0$, it is clear that $v_2 = 0$ meets the surface (1) in the curve c_8 and in the four lines $u_2 = v_2 = 0$; but v_2 meets (1) where it meets the two nodal cubic surfaces

$$u_2 w + u_3 - v_3 = 0,$$

$$u_2 w + u_3 + v_3 = 0,$$

hence in general c_8 must break up into two quartic curves, either of which is the partial intersection of v_2 with a cubic surface which contains also two generators of v_2. These curves are therefore quadri-quartics†. Hence the surface contains an infinite number

* This method is due to Rohn, see *Die Flächen vierter Ordnung hinsichtlich ihrer Knotenpunkte und ihrer Gestaltung*, Leipzig, 1886.

† We denote by *quadri-quartic* the type of twisted quartic through which an infinite number of quadrics pass.

of quadri-quartic curves which are projected from the node into quartic curves which touch the sextic $u_2u_4 - u_3^2 = 0$ at each point of intersection*.

Hence if the curve $u_2u_4 - u_3^2 = 0$ has more than one contact-conic it has an infinite number of contact-conics.

4. Nodal sextics†.

For the purpose of classification of nodal quartic surfaces we discuss various properties of sextic curves with a contact-conic. In the first place it may be seen that sextic curves with *six nodes* lying on a conic c_2 can have their equation expressed as above. For if $c_3 = 0$ is any cubic through the six points, any other cubic through them is of the form $c_3 + c_2L = 0$; and any sextic through the complete intersection of c_2 and c_3 being

$$c_2c_4 + c_3c_3' = 0,$$

if the six points are nodes on this sextic c_4 and c_3' must be of the form $c_3M + c_2N$, $c_3 + c_2R$ respectively.

Hence the required sextic takes the form

$$c_3^2 + c_2c_3A + c_2^2B = 0,$$

i.e. the form $$K_3^2 - c_2^2V = 0,$$

and hence has a contact-conic.

The corresponding quartic surface is $w^2V + 2wK_3 + c_2^2 = 0$; this has the plane $w = 0$ as a singular tangent plane or *trope*, which touches the surface along a conic.

Sextics with seven nodes.

There are two different kinds of seven-nodal sextics, viz. that for which it is possible to find a pair of points P, P' on the curve, such that through the seven nodes $D_1 \dots D_7$ and P, P' there pass an infinite number of cubics, and the one for which it is not possible; considering the former kind, then if one such pair of points exists there is an infinite number of such pairs; for taking c_3 and c_3' as two such cubics, then c_6, the given sextic, since it passes through the complete intersection of c_3 and c_3', has an equation of the form

$$c_3\Gamma_3 + c_3'\Gamma_3' = 0.$$

* For such a point of intersection P is the projection of an actual intersection Q of the quadri-quartic and the curve of contact of the tangent cone, and the tangents to these curves at Q lie in the tangent plane of the surface.

† See Rohn, *l.c.*

Now c_3 meets c_6 only in $D_1 \dots D_7$, P, P' and two further points Q, Q', hence Γ_3' passes through $D_1 \dots D_7$ and also through Q and Q'; so that two and therefore an infinite number of cubics pass through $D_1 \dots D_7$, Q and Q'. By varying the cubic through the nine points $D_1 \dots D_7$, P, P' we form an involution of points Q, Q' on c_6. If Q coincides with P, Q' will coincide with P'; therefore every cubic through the seven nodes which touches c_6 once will touch it twice.

Since Γ_3 is seen to pass through $D_1 \dots D_7$ and since only three linearly independent cubics pass through seven points, there is a linear connection between c_3, c_3', Γ_3 and Γ_3', hence the sextic which has the property considered is represented by an equation of the form

$$(a \between \phi, \psi, \chi)^2 = 0,$$

where ϕ, ψ, χ are any three cubics through the given nodes.

This class of sextic always has a contact-conic; for if the sextic is $c_6 \equiv c_3^2 - c_3'c_3''$, let the chord joining the intersections P_1, P_2 of c_3' and c_3'', apart from the nodes, be $f = 0$, and $f' = 0$, $f'' = 0$ similar chords for c_3'', c_3 and c_3, c_3'; then fc_3, $f'c_3'$, $f''c_3''$ all pass through the thirteen points $D_1 \dots D_7$, P_1, $P_2, \dots P_2''$, and hence through three other fixed points*. Hence we have a linear relation of the form

$$Afc_3 + Bf'c_3' + Cf''c_3'' = 0,$$

where A, B, C are definite constants.

Now if $\quad c_4 \equiv -\tfrac{1}{2}Afc_3 - Bf'c_3' \equiv \tfrac{1}{2}Afc_3 + Cf''c_3''$,

we have $\quad c_4^2 - \tfrac{1}{4}A^2f^2c_3^2 + BCf'f''c_3'c_3'' \equiv 0$;

that is $\quad 4c_4^2 - A^2f^2c_6 + c_3'c_3''(4BCf'f'' - A^2f^2) \equiv 0.$

Hence the conic $4BCf'f'' - A^2f^2 = 0$ touches c_6, viz. at six of its intersections with c_4, the other two being the points (f, f'), (f, f'').

This conic is touched by f' and f'', hence *the tangents of the contact-conic are the chords of contact of c_6 and its bitangent cubics.*

We observe that in this case there is a doubly infinite number of quartic curves c_4 which pass through the seven nodes and the six points of contact of c_6 and its contact-conic.

* Since all quartics through thirteen points which do not all lie on a curve of lower degree pass through three other fixed points and hence belong to a pencil.

Sextics with eight or with nine nodes.

If $f=0$ is *any* sextic with eight nodes $D_1 \dots D_8$ and $\phi=0$, $\psi=0$ any two cubics through them, the general sextic with the eight given nodes is

$$f+\lambda\phi^2+\mu\phi\psi+\nu\psi^2=0.$$

If this curve has a ninth node it either degenerates into two cubics through the nine points (which are then the complete intersection of two cubics) or the ninth node lies on the curve $J(f, \phi, \psi)=0$. This is of the ninth degree and will be denoted by c_9; it has each of the eight nodes as a triple point*.

The curve $f=0$ and the eight nodes completely determine c_9; if we take any point P of intersection of f and c_9, and suppose ϕ to pass through P, then any sextic with the eight given nodes is of the form $f+\rho\phi\phi'=0$ where ϕ' does not pass through P.

Since P lies on c_9 it follows from the equation of that curve that the linear polars of P for f, ϕ and ϕ' concur; but the first two are the tangents at P to f and ϕ, and the third cannot pass through P, hence f and ϕ touch at P, and ϕ touches every sextic with the eight given nodes which pass through P. Now f and c_9 meet in $9\times6-8\times6=6$ points apart from the nodes, hence every sextic *with eight nodes is touched by six cubics through these nodes.*

If $f=0$ is any sextic with nine nodes and $\phi=0$ the cubic through them, $f+\rho\phi^2=0$ is the equation of the general sextic with the given nine nodes. If there is a tenth node it will be included among the points determined by the equations

$$\left\| \begin{matrix} f_1 & f_2 & f_3 \\ \phi_1 & \phi_2 & \phi_3 \end{matrix} \right\| = 0.$$

The number of solutions given by these equations is thirty-nine, but each of the given nine nodes occurs as a triple solution. Hence the pencil of sextics $f+\rho\phi^2=0$ contains twelve curves which have a tenth node (see Art. 9).

The foregoing result as to contact-cubics is modified as follows: through any eight nodes of a sextic with nine nodes there pass four tangent cubics; through any eight nodes of a sextic with ten nodes there pass two tangent cubics.

5. Sextics with ten nodes.

The following result for ten-nodal sextics is important for our purpose: *every plane sextic with ten nodes and a contact-conic is the projection of a twisted sextic on a quadric*: for choosing any centre

* As may be seen by taking any one of them as $x=0$, $y=0$, $z=0$.

of projection O and any quadric whose section by the polar plane of O for the quadric projects into the given contact-conic, the sextic cone whose base is the given sextic meets the quadric in a curve c_{12} which has twenty-six actual double points, since each node of the plane sextic gives rise to two nodes on c_{12}, and each point of contact of the contact-conic and the sextic is the projection of a point at which two branches of c_{12} touch each other. Moreover c_{12} has thirty apparent double points*, hence the projection of c_{12} from any point has $30+26=56$ nodes, and this is one more than can be possessed by a curve of order 12 which does not break up into simpler curves. Hence c_{12} must break up into two sextic curves.

There are *three* varieties of twisted sextics on a quadric: (1) its intersection with a cubic surface, (2) its partial intersection with a quartic surface which also contains two generators of the quadric of the same species, (3) its partial intersection with a quintic surface which also contains four generators of the quadric of the same species.

The following result, which may be easily proved†, is of frequent application: through every point P of space there pass $n(n-1)$ double secants of the complete curve of intersection of a quadric with any surface of order n; these double secants form the intersection of a cone of order n with a cone of order $n-1$, the former cone passes through the $2n$ intersections of the polar plane of P and this curve.

Let us now consider the plane ten-nodal sextic which is the projection of the first of these three varieties. This has six apparent double points and, since its plane projection has ten

* Salmon, *Geom. of three dimensions* (fifth ed. 1912), vol. I. p. 356.

† If $V=0$ is the surface and $U=0$ the quadric, it is easy to see that the section of the curve of intersection by the polar plane of P for U is given by the equations

$$\Delta U=0,\quad \left(V-\frac{U}{U'}\frac{\Delta^2 V}{\underline{|2}}+\dots\right)^2+\frac{U}{U'}(\Delta V-\dots)^2=0,$$

where
$$\Delta U=\sum_1^4 x_i'\frac{\partial}{\partial x_i}U,$$

and x_i' are the coordinates of P. Relatively to its plane the equation of this curve is of the form

$$v_n{}^2+c_2 v^2_{n-1}=0;$$

this curve contains $n(n-1)$ nodes which arise solely from apparent double points of the curve $U=0$, $V=0$; also $v_n=0$ is seen to pass through the common intersections of $V=0$, $U=0$, $\Delta U=0$.

nodes, it must have four actual double points; by the last result six of the nodes of the plane sextic lie on a conic; it is therefore represented by an equation of the form

$$K_3^2 - c_2^2 V = 0. \qquad \text{(Art. 4.)}$$

The second species of twisted sextic lies on a quadric and a quartic surface, their intersection being completed by two generators of the quadric. This curve has seven apparent double points*; and therefore, to complete the number of nodes of the plane quartic, must have three actual double points. Each generator of the given species meets the curve four times. There is an infinite number of quartic surfaces passing through the sextic and any two generators of the quadric. For any quartic surface through five points of each generator and any seventeen points of the sextic will meet the sextic in $8+17=25$ points, and therefore contain it altogether: it will also contain the two generators. Let us denote the twisted sextic by c_6, its plane projection by c_6', and take any generator p and its consecutive generator as the pair of generators just mentioned; then the *cubic* cone which contains the seven double secants of c_6 will touch c_6' twice†; hence, varying p, we obtain *an infinite number of cubics through seven nodes of c_6' and bitangent to it.*

In the third type of twisted sextic c_6 is the partial intersection of a quadric and a quintic, the residual intersection being formed by four generators of the quadric of the same species. Each generator of this species meets the sextic five times. It may be shown as before that there is an infinite number of quintic surfaces passing through the given sextic and any four generators of the given species. The curve c_6 has ten apparent double points.

We may select the four generators as follows: let p and p' be those generators which are projected from the centre of projection O into the tangents of the contact-conic of c_6' drawn from some node D of c_6'; we then take as our four generators p, p' and the generators consecutive to them. The line OD thus meets c_6 twice, and serves as join of apparent intersections for c_6, p and for c_6, p'. The compound curve of intersection of order 10 has twenty apparent double points, of which nine are projected into D, viz. one point arising from c_6, two from $(c_6, p)(c_6, p+dp)$, two from $(c_6, p')(c_6, p'+dp')$ and four from p and p'.

Hence the two cones of orders 4 and 5 through the double secants must

* After deduction of five apparent double points arising from the two lines.

† Since p gives rise to two apparent double points of the compound curve.

each have a common triple edge; we therefore obtain the following results: if D is any one of the ten nodes there exists a quartic curve which has a triple point in D and passes through the nine other nodes and the points of contact of the tangents drawn from D to the contact-conic; also there exists a quintic curve which has a triple point in D, passes through the nine other nodes and touches the contact-conic where it is touched by its tangents drawn from D. This holds for each node.

6. Quartic surfaces with eight nodes.

Returning to the sextic curve $u_2u_4 - u_3^2 = 0$, derived from the surface $u_2w^2 + 2u_3w + u_4 = 0$, any quadric through the node is

$$2t_1 w + t_2 = 0;$$

if the quartic surface has any other node which also lies upon this quadric, since this node also lies on the surface

$$u_2w + u_3 = 0,$$

it is clear that the curve $u_2t_2 - 2t_1u_3 = 0$ will pass through the resulting node on $u_2u_4 - u_3^2 = 0$ or c_6.

This quartic curve passes through the points of contact of c_6 with its contact-conic u_2, and also through the nodes of c_6 which result from nodes on the quartic surface. If therefore the surface has eight nodes we have seven nodes on c_6: to each quartic through these seven nodes and the points of contact $B_1 \dots B_6$ of c_6 and u_2, there corresponds one quadric through the eight nodes, and vice-versâ.

Now it was stated (Art. 4) that plane sextics with seven nodes form two classes; in the more general case there is a singly infinite number of quartic curves through the nodes and $B_1 \dots B_6$, and we obtain corresponding to this case a singly infinite number of quadrics through the eight nodes. For the more special case where there is a doubly infinite number of quartic curves through the thirteen points we have a doubly infinite number of quadrics through the eight nodes, which therefore form *eight associated points*. Such a surface is represented by an equation of the form

$$(a \between A, B, C)^2 = 0.$$

It follows that any quadric through the eight nodes meets the quartic surface in two quadri-quartic curves which are projected from any node into two of the ∞^2 cubics which pass through the seven nodes of c_6.

These two classes of quartic surfaces will be termed *asyzygetic* and *syzygetic* respectively*.

The equation of the general seven-nodal surface being $F \equiv (a \between A, B, C)^2 + \rho\Sigma = 0$ (Art. 2), where A, B, C are quadrics through the seven nodes, if there is an eighth node we obtain, to determine it, the equations

$$A_i \frac{\partial F}{\partial A} + B_i \frac{\partial F}{\partial B} + C_i \frac{\partial F}{\partial C} + \rho\Sigma_i = 0, \quad (i = 1, \dots 4);$$

hence the eighth node lies on the surface

$$J(A, B, C, \Sigma) \equiv \begin{vmatrix} A_1 & B_1 & C_1 & \Sigma_1 \\ A_2 & B_2 & C_2 & \Sigma_2 \\ A_3 & B_3 & C_3 & \Sigma_3 \\ A_4 & B_4 & C_4 & \Sigma_4 \end{vmatrix} = 0.$$

The eighth node may therefore not be taken arbitrarily, as in the case of the first seven nodes.

If A, B are two quadrics through the eight nodes and T any eight-nodal asyzygetic surface, the general asyzygetic surface is represented by the equation

$$\alpha A^2 + \beta B^2 + 2\gamma AB + 2\rho T = 0.$$

The surface J is called the *dianodal surface*†, and is the locus of a point whose polar planes for A, B, C and Σ are concurrent, and therefore also concurrent for every quartic surface with the given seven nodes; thus if P is any point of the dianodal surface, all the quartics through P have a common tangent line thereat, which touches the quadri-quartic through P and the seven nodes, as is seen by taking as the quartic a doubled quadric through P and the seven nodes.

The dianodal surface.

The dianodal surface contains the line joining any two nodes D_1, D_2; for if P be any point on this line then, since we may take the surfaces A, B, Σ which appear in the equation of the seven-nodal quartic to pass through P, they will necessarily contain the line D_1D_2, hence the tangent planes at P to A, B, Σ all pass through D_1D_2 and therefore the point P satisfies the equation of

* The general syzygetic surface is the envelope of the quadrics $\lambda^2 D + \lambda E + F = 0$, where D, E, F are quadrics through the eight nodes.

† Cayley.