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Douglas Quadling , Julian Gilbey
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Chapter 1
Velocity and acceleration

This chapter introduces kinematics, which is about the connections between displacement, velocity and acceleration. When you have completed it, you should

- know the terms 'displacement', 'velocity', 'acceleration' and 'deceleration' for motion in a straight line
- be familiar with displacement–time and velocity–time graphs
- be able to express speeds in different systems of units
- know formulae for constant velocity and constant acceleration
- be able to solve problems on motion with constant velocity and constant acceleration, including problems involving several such stages
- understand what is meant by the terms 'average speed' and 'average velocity'.

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1.1 Motion with constant velocity

A Roman legion marched out of the city of Alexandria along a straight road, with a velocity of 100 paces per minute due east. Where was the legion 90 minutes later?

Notice the word **velocity**, rather than speed. This is because you are told not only how fast the legion marched, but also in which direction. Velocity is speed in a particular direction.

Two cars travelling in opposite directions on a north–south motorway may have the same speed of 90 kilometres per hour, but they have different velocities. One has a velocity of 90 k.p.h. north, the other a velocity of 90 k.p.h. south.

The abbreviation k.p.h. is used here for 'kilometres per hour' because this is the form often used on car speedometers. A scientist would use the abbreviation km h^{-1}*, and this is the form that will normally be used in this book.*

The answer to the question in the first paragraph is, of course, that the legion was 9000 paces (9 Roman miles) east of Alexandria. The legion made a **displacement** of 9000 paces east. Displacement is distance in a particular direction.

This calculation, involving the multiplication $100 \times 90 = 9000$, is a special case of a general rule.

An object moving with constant velocity u units in a particular direction for a time t units makes a displacement s units in that direction, where $s = ut$.

The word 'units' is used three times in this statement, and it has a different sense each time. For the Roman legion the units are paces per minute, minutes and paces respectively. You can use any suitable units for velocity, time and displacement provided that they are consistent.

The equation $s = ut$ can be rearranged into the forms $u = \frac{s}{t}$ or $t = \frac{s}{u}$. You decide which form to use according to which quantities you know and which you want to find.

EXAMPLE 1.1.1

An airliner flies from Cairo to Harare, a displacement of 5340 kilometres south, at a speed of 800 k.p.h. How long does the flight last?

You know that $s = 5340$ and $u = 800$, and want to find t. So use

$$t = \frac{s}{u} = \frac{5340}{800} = 6.675.$$

For the units to be consistent, the unit of time must be hours. The flight lasts 6.675 hours, or 6 hours and $40\frac{1}{2}$ minutes.

This is not a sensible way of giving the answer. In a real flight the aircraft will travel more slowly while climbing and descending. It is also unlikely to travel in a straight line, and the figure of 800 k.p.h. for the speed looks like a convenient approximation. The solution is based on a **mathematical model**, in which such complications are

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ignored so that the data can be put into a simple mathematical equation. But when you have finished using the model, you should then take account of the approximations and give a less precise answer, such as 'about 7 hours'.

The units almost always used in mechanics are metres (m) for displacement, seconds (s) for time and metres per second (written as $m\,s^{-1}$) for velocity. These are called **SI units** (SI stands for Système Internationale), and scientists all over the world have agreed to use them.

EXAMPLE 1.1.2

Express a speed of 144 k.p.h. in $m\,s^{-1}$.

If you travel 144 kilometres in an hour at a constant speed, you go $\frac{144}{60 \times 60}$ kilometres in each second, which is $\frac{1}{25}$ of a kilometre in each second.

A kilometre is 1000 metres, so you go $\frac{1}{25}$ of 1000 metres in a second.

Thus a speed of 144 k.p.h. is $40\,m\,s^{-1}$.

You can extend this result to give a general rule: to convert any speed in k.p.h. to $m\,s^{-1}$, you multiply by $\frac{40}{144}$, which is $\frac{5}{18}$.
Note that in this section you have met two pairs of quantities which are related in the same way to each other:

velocity is speed in a certain direction
displacement is distance in a certain direction.

Quantities that are directionless are known as **scalar quantities**: speed and distance are two examples. Quantities that have a direction are known as **vector quantities**: velocity and displacement are two examples. You will meet several different types of quantities during the mechanics course. For each one, it is important to ask yourself whether it is a scalar or vector quantity. It would be incorrect to say that something has a velocity of $30\,km\,h^{-1}$ without giving a direction, and it would also be wrong to say that it had a speed of $30\,km\,h^{-1}$ due south.

1.2 Graphs for constant velocity

You do not always have to use equations to describe mathematical models. Another method is to use graphs. There are two kinds of graph which are often useful in kinematics.

The first kind is a **displacement–time graph**, as shown in Fig. 1.1. The coordinates of any point on the graph are (t,s), where s is the displacement of the moving object after a time t (both in appropriate units). Notice that $s = 0$ when $t = 0$, so the graph passes through the origin. If the velocity is constant, then $\frac{s}{t} = u$, and the gradient of the line joining (t,s) to the origin has the constant value u. So the graph is a straight line with gradient u.

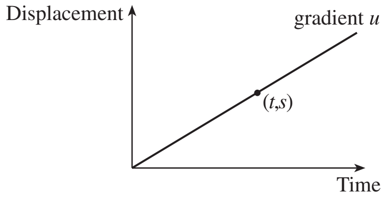

Fig. 1.1

For an object moving along a straight line with constant velocity u, the displacement–time graph is a straight line with gradient u.

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The second kind of graph is a **velocity–time graph** (see Fig. 1.2). The coordinates of any point on this graph are (t,v), where v is the velocity of the moving object at time t. If the velocity has a constant value u, then the graph has equation $v = u$, and it is a straight line parallel to the time-axis.

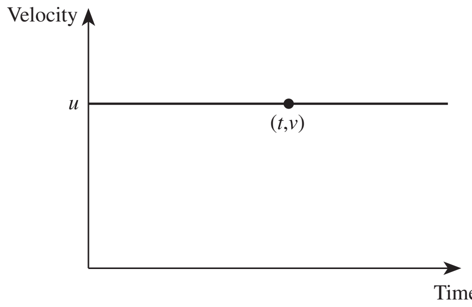

Fig. 1.2

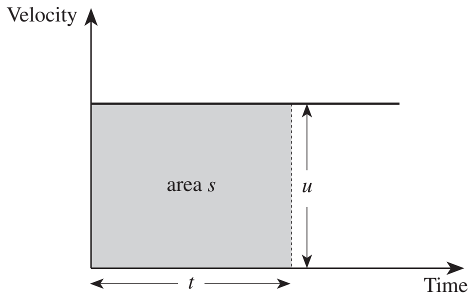

Fig. 1.3

How is displacement shown on the velocity–time graph? Fig. 1.3 answers this question for motion with constant velocity u. The coordinates of any point on the graph are (t,u), and you know that $s = ut$. This product is the area of the shaded rectangle in the figure, which has width t and height u.

For an object moving along a straight line with constant velocity, the displacement from the start up to any time t is represented by the area of the region under the velocity–time graph for values of the time from 0 to t.

Exercise 1A

1 How long will an athlete take to run 1500 metres at $7.5\,\text{m s}^{-1}$?

2 A train maintains a constant velocity of $60\,\text{m s}^{-1}$ due south for 20 minutes. What is its displacement in that time? Give the distance in kilometres.

3 How long will it take for a cruise liner to sail a distance of 530 nautical miles at a speed of 25 knots? (A knot is a speed of 1 nautical mile per hour.)

4 Some Antarctic explorers walking towards the South Pole expect to average 1.8 kilometres per hour. What is their expected displacement in a day in which they walk for 14 hours?

5 Here is an extract from the diary of Samuel Pepys for 4 June 1666, written in London.

‘We find the Duke at St James’s, whither he is lately gone to lodge. So walking through the Parke we saw hundreds of people listening to hear the guns.’

These guns were at the battle of the English fleet against the Dutch off the Kent coast, a distance of between 110 and 120 km away. The speed of sound in air is $344\,\text{m s}^{-1}$. How long did it take the sound of the gunfire to reach London?

6 Light travels at a speed of $3.00 \times 10^8\,\text{m s}^{-1}$. Light from the star Sirius takes 8.65 years to reach the Earth. What is the distance of Sirius from the Earth in kilometres?

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7 The speed limit on a motorway is 120 km per hour. What is this in SI units?

8 The straightest railway line in the world runs across the Nullarbor Plain in southern Australia, a distance of 500 kilometres. A train takes $12\frac{1}{2}$ hours to cover the distance.

Model the journey by drawing

a a velocity–time graph, **b** a displacement–time graph.

Label your graphs to show the numbers 500 and $12\frac{1}{2}$ and to indicate the units used.

Suggest some ways in which your models may not match the actual journey.

9 An aircraft flies due east at 800 km per hour from Kingston to Antigua, a displacement of about 1600 km. Model the flight by drawing

a a displacement–time graph, **b** a velocity–time graph.

Label your graphs to show the numbers 800 and 1600 and to indicate the units used. Can you suggest ways in which your models could be improved to describe the actual flight more accurately?

1.3 Acceleration

A vehicle at rest cannot suddenly start to move with constant velocity. There has to be a period when the velocity increases. The rate at which the velocity increases is called the **acceleration**.

In the simplest case the velocity increases at a constant rate. For example, suppose that a train accelerates from 0 to 144 k.p.h. in 100 seconds at a constant rate. You know from Example 1.1.2 that 144 k.p.h. is $40\,\mathrm{m\,s^{-1}}$, so the speed is increasing by $0.4\,\mathrm{m\,s^{-1}}$ in each second.

The SI unit of acceleration is ‘$\mathrm{m\,s^{-1}}$ per second’, or $(\mathrm{m\,s^{-1}})\,\mathrm{s^{-1}}$; this is always simplified to $\mathrm{m\,s^{-2}}$ and read as ‘metres per second squared’. Thus in the example above the train has a constant acceleration of $\frac{40}{100}\,\mathrm{m\,s^{-2}}$, which is $0.4\,\mathrm{m\,s^{-2}}$.

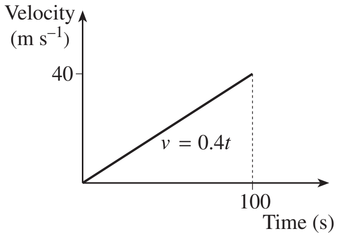

Fig. 1.4

Consider the period of acceleration. After t seconds the train will have reached a speed of $0.4t\,\mathrm{m\,s^{-1}}$. So the velocity–time graph has equation $v = 0.4t$. This is a straight line segment with gradient 0.4, joining (0,0) to (100,40). It is shown in Fig. 1.4.

This is a special case of a general rule.

The velocity–time graph for an object moving with constant acceleration a is a straight line segment with gradient a.

Now suppose that at a later time the train has to stop at a signal. The brakes are applied, and the train is brought to rest in 50 seconds. If the velocity drops at a constant rate, this is $\frac{40}{50}\,\mathrm{m\,s^{-2}}$, or $0.8\,\mathrm{m\,s^{-2}}$. The word for this is **deceleration** (some people use **retardation**).

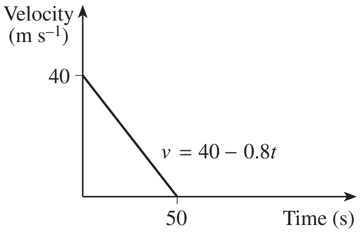

Fig. 1.5

Fig. 1.5 shows the velocity–time graph for the braking train. If time is measured from the instant when the brakes are applied, the graph has equation $v = 40 - 0.8t$.

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There are two new points to notice about this graph. First, it doesn't pass through the origin, since at time $t = 0$ the train has a velocity of $40\,\mathrm{m\,s^{-1}}$. The velocity when $t = 0$ is called the **initial velocity**.

Secondly, the graph has negative gradient, because the velocity is decreasing. This means that the acceleration is negative. You can either say that the acceleration is $-0.8\,\mathrm{m\,s^{-2}}$, or that the deceleration is $0.8\,\mathrm{m\,s^{-2}}$.

The displacement is still given by the area of the region between the velocity–time graph and the t-axis, even though the velocity is not constant. In Fig. 1.4 this region is a triangle with base 100 and height 40, so the area is $\frac{1}{2} \times 100 \times 40 = 2000$. This means that the train covers a distance of 2000 m, or 2 km, while gaining speed.

In Fig. 1.5 the region is again a triangle, with base 50 and height 40, so the train comes to a standstill in 1000 m, or 1 km.

A justification that the displacement is given by the area will be found in Section 11.3.

1.4 Equations for constant acceleration

You will often have to do calculations like those in the last section. It is worth having algebraic formulae to solve problems about objects moving with constant acceleration.

Fig. 1.6 shows a velocity–time graph which could apply to any problem of this type. The initial velocity is u, and the velocity at time t is denoted by v. If the acceleration has the constant value a, then between time 0 and time t the velocity increases by at. It follows that, after time t,

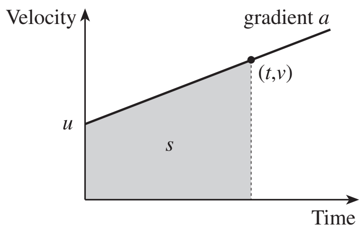

Fig. 1.6

$$v = u + at.$$

Remember that in this equation u and a are constants, but t and v can vary. In fact, this equation is just like $y = mx + c$ (or, for a closer comparison, $y = c + mx$). The acceleration a is the gradient, like m, and the initial velocity u is the intercept, like c. So $v = u + at$ is just the equation of the velocity–time graph.

There is, though, one important difference. This equation only applies so long as the constant acceleration lasts, so the graph is just part of the line.

There are no units in the equation $v = u + at$. You can use it with any units you like, provided that they are consistent.

To find a formula for the displacement, you need to find the area of the shaded region under the graph between $(0,u)$ and (t,v) in Fig. 1.6. You can work this out in either of two ways, illustrated in Figs. 1.7 and 1.8.

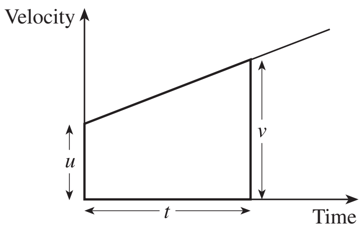

Fig. 1.7

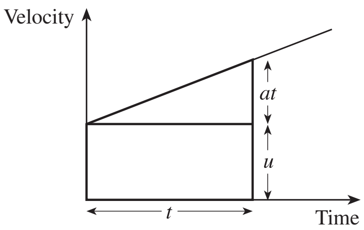

Fig. 1.8

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In Fig. 1.7 the region is shown as a trapezium, with parallel vertical sides of length u and v, and width t. The formula for the area of a trapezium gives

$$s = \tfrac{1}{2}(u+v)t.$$

Fig. 1.8 shows the region split into a rectangle, whose area is ut, and a triangle with base t and height at, whose area is $\tfrac{1}{2} \times t \times at$. These combine to give the formula

$$s = ut + \tfrac{1}{2}at^2.$$

EXAMPLE 1.4.1

A racing car enters the final straight travelling at $35\,\text{m s}^{-1}$, and covers the 600 m to the finishing line in 12 s. Assuming constant acceleration, find the car's speed as it crosses the finishing line.

Measuring the displacement from the start of the final straight, and using SI units, you know that $u = 35$. You are told that when $t = 12$, $s = 600$, and you want to know v at that time. So use the formula connecting u, t, s and v.

Substituting in the formula $s = \tfrac{1}{2}(u+v)t$,

$$600 = \tfrac{1}{2}(35+v)\times 12.$$

This gives $35 + v = \dfrac{600 \times 2}{12} = 100$, so $v = 65$.

Assuming constant acceleration, the car crosses the finishing line at $65\,\text{m s}^{-1}$.

EXAMPLE 1.4.2

A cyclist reaches the top of a slope with a speed of $1.5\,\text{m s}^{-1}$, and accelerates at $2\,\text{m s}^{-2}$. The slope is 22 m long. How long does she take to reach the bottom of the slope, and how fast is she moving then?

You are given that $u = 1.5$ and $a = 2$, and want to find t when $s = 22$. The formula which connects these four quantities is $s = ut + \tfrac{1}{2}at^2$, so displacement and time are connected by the equation

$$s = 1.5t + t^2.$$

When $s = 22$, t satisfies the quadratic equation $t^2 + 1.5t - 22 = 0$. Solving this by the quadratic formula (see P1 Section 4.4), $t = \dfrac{-1.5 \pm \sqrt{1.5^2 - 4 \times 1 \times (-22)}}{2}$, giving $t = -5.5$ or 4. In this model t must be positive, so $t = 4$. The cyclist takes 4 seconds to reach the bottom of the slope.

To find how fast she is then moving, you have to calculate v when $t = 4$. Since you now know u, a, t and s, you can use either of the formulae involving v.

The algebra is simpler using $v = u + at$, which gives

$$v = 1.5 + 2 \times 4 = 9.5.$$

The cyclist's speed at the bottom of the slope is $9.5\,\text{m s}^{-1}$.

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Exercise 1B

1 A police car accelerates from 15 m s^{-1} to 35 m s^{-1} in 5 seconds. The acceleration is constant. Illustrate this with a velocity–time graph. Use the equation $v = u + at$ to calculate the acceleration. Find also the distance travelled by the car in that time.

2 A marathon competitor running at 5 m s^{-1} puts on a sprint when she is 100 metres from the finish, and covers this distance in 16 seconds. Assuming that her acceleration is constant, use the equation $s = \frac{1}{2}(u+v)t$ to find how fast she is running as she crosses the finishing line.

3 A train travelling at 20 m s^{-1} starts to accelerate with constant acceleration. It covers the next kilometre in 25 seconds. Use the equation $s = ut + \frac{1}{2}at^2$ to calculate the acceleration. Find also how fast the train is moving at the end of this time. Illustrate the motion of the train with a velocity–time graph.

How long does the train take to cover the first half kilometre?

4 A long-jumper takes a run of 30 metres to accelerate to a speed of 10 m s^{-1} from a standing start. Find the time he takes to reach this speed, and hence calculate his acceleration. Illustrate his run-up with a velocity–time graph.

5 Starting from rest, an aircraft accelerates to its take-off speed of 60 m s^{-1} in a distance of 900 metres. Assuming constant acceleration, find how long the take-off run lasts. Hence calculate the acceleration.

6 A train is travelling at 80 m s^{-1} when the driver applies the brakes, producing a deceleration of 2 m s^{-2} for 30 seconds. How fast is the train then travelling, and how far does it travel while the brakes are on?

7 A balloon at a height of 300 m is descending at 10 m s^{-1} and decelerating at a rate of 0.4 m s^{-2}. How long will it take for the balloon to stop descending, and what will its height be then?

1.5 More equations for constant acceleration

All the three formulae in Section 1.4 involve four of the five quantities u, a, t, v and s. The first leaves out s, the second a and the third v. It is also useful to have formulae which leave out t and u, and you can find these by combining the formulae you already know.

To find a formula which omits t, rearrange the formula $v = u + at$ to give $at = v - u$, so $t = \frac{v-u}{a}$. If you now substitute this in $s = \frac{1}{2}(u+v)t$, you get

$$s = \frac{1}{2}(u+v) \times \frac{v-u}{a},$$

which is $2as = (u+v)(v-u)$. The right side of this is $(v+u)(v-u) = v^2 - u^2$, so that finally $2as = v^2 - u^2$, or

$$v^2 = u^2 + 2as.$$

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The fifth formula, which omits u, is less useful than the others. Turn the formula $v = u + at$ round to get $u = v - at$. Then, substituting this in $s = ut + \frac{1}{2}at^2$, you get $s = (v - at)t + \frac{1}{2}at^2$, which simplifies to

$$s = vt - \tfrac{1}{2}at^2.$$

For an object moving with constant acceleration a and initial velocity u, the following equations connect the displacement s and the velocity v after a time t.

$$v = u + at \qquad s = ut + \tfrac{1}{2}at^2 \qquad v^2 = u^2 + 2as$$
$$s = \tfrac{1}{2}(u + v)t \qquad s = vt - \tfrac{1}{2}at^2$$

You should learn these formulae, because you will use them frequently throughout this mechanics course.

EXAMPLE 1.5.1

The barrel of a shotgun is 0.9 m long, and the shot emerges from the muzzle with a speed of 240 m s^{-1}. Find the acceleration of the shot in the barrel, and the length of time the shot is in the barrel after firing.

In practice the constant acceleration model is likely to be only an approximation, but it will give some idea of the quantities involved.

The shot is initially at rest, so $u = 0$. You are given that $v = 240$ when $s = 0.9$, and you want to find the acceleration, so use $v^2 = u^2 + 2as$.

$$240^2 = 0^2 + 2 \times a \times 0.9.$$

This gives $a = \dfrac{240^2}{2 \times 0.9} = 32\,000.$

You can now use any of the other formulae to find the time. The simplest is probably $v = u + at$, which gives $240 = 0 + 32\,000t$, so $t = 0.0075$.

Taking account of the approximations in the model and the data, you can say that the acceleration of the shot is about 30 000 m s^{-2}, and that the shot is in the barrel for a little less than one-hundredth of a second.

EXAMPLE 1.5.2

The driver of a car travelling at 96 k.p.h. in mist suddenly sees a stationary bus 100 metres ahead. With the brakes full on, the car can decelerate at 4 m s^{-2} in the prevailing road conditions. Can the driver stop in time?

You know from Example 1.1.2 that 96 k.p.h. is $96 \times \frac{5}{18}$ m s^{-1}, or $\frac{80}{3}$ m s^{-1}. This suggests writing $u = \frac{80}{3}$ and $a = -4$ in the formula $v^2 = u^2 + 2as$ to find v when $s = 100$. Notice that a is negative because the car is decelerating.

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When you do this, you get $v^2 = \left(\frac{80}{3}\right)^2 - 2 \times 4 \times 100 = -\frac{800}{9}$. This is clearly a ridiculous answer, since a square cannot be negative.

The reason for the absurdity is that the equation only holds so long as the constant acceleration model applies. In fact the car stops before s reaches the value 100, and after that it simply stays still.

To avoid this, it is better to begin by substituting only the constants in the equation, leaving v and s as variables. The equation is then

$$v^2 = \frac{6400}{9} - 8s.$$

This model holds so long as $v^2 \geq 0$. The equation gives $v = 0$ when $s = \frac{6400}{9 \times 8} = \frac{800}{9}$, which is less than 100. So the driver can stop in time.

This example could be criticised because it assumes that the driver puts the brakes on as soon as he sees the bus. In practice there would be some 'thinking time', perhaps 0.3 seconds, while the driver reacts. At $\frac{80}{3}\,\mathrm{m\,s^{-1}}$, the car would travel 8 metres in this time, so you should add 8 metres to the distance calculated in the example. You can see that the driver will still avoid an accident, but only just.

Exercise 1C

1 Interpret each of the following in terms of the motion of a particle along a line, and select the appropriate constant acceleration formula to find the answer. The quantities u, v, s and t are all positive or zero, but a may be positive or negative.

a $u = 9,\ a = 4,\ s = 5$, find v

b $u = 10,\ v = 14,\ a = 3$, find s

c $u = 17,\ v = 11,\ s = 56$, find a

d $u = 14,\ a = -2,\ t = 5$, find s

e $v = 20,\ a = 1,\ t = 6$, find s

f $u = 10,\ s = 65,\ t = 5$, find a

g $u = 18,\ v = 12,\ s = 210$, find t

h $u = 9,\ a = 4,\ s = 35$, find t

i $u = 20,\ s = 110,\ t = 5$, find v

j $s = 93\ v = 42,\ t = \frac{3}{2}$, find a

k $u = 24,\ v = 10,\ a = -0.7$, find t

l $s = 35,\ v = 12,\ a = 2$, find u

m $v = 27,\ s = 40,\ a = -4\frac{1}{2}$, find t

n $a = 7,\ s = 100,\ v - u = 20$, find u

2 A train goes into a tunnel at $20\,\mathrm{m\,s^{-1}}$ and emerges from it at $55\,\mathrm{m\,s^{-1}}$. The tunnel is 1500 m long. Assuming constant acceleration, find how long the train is in the tunnel for, and the acceleration of the train.

3 A motor-scooter moves from rest with acceleration $0.1\,\mathrm{m\,s^{-2}}$. Find an expression for its speed, $v\,\mathrm{m\,s^{-1}}$, after it has gone s metres. Illustrate your answer by sketching an (s,v) graph.

4 A cyclist riding at $5\,\mathrm{m\,s^{-1}}$ starts to accelerate, and 200 metres later she is riding at $7\,\mathrm{m\,s^{-1}}$. Find her acceleration, assumed constant.

5 A train travelling at $55\,\mathrm{m\,s^{-1}}$ has to reduce speed to $35\,\mathrm{m\,s^{-1}}$ to pass through a junction. If the deceleration is not to exceed $0.6\,\mathrm{m\,s^{-2}}$, how far ahead of the junction should the train begin to slow down?

