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Transcendence and Linear Relations of 1-Periods

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In memoriam Alan Baker

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Prologue

The study of transcendence properties of periods has a long history. It began in 1882 with the famous theorem of Lindemann on the transcendence of π, which showed that squaring the circle is not possible. This settled a problem more than 2000 years old from the time of the Greeks. At the same time, he showed that α and e^{α} cannot both be algebraic unless $\alpha = 0$. In particular, $\log \alpha$ is transcendental for algebraic $\alpha \neq 0, 1$. Lindemann actually proved more: his method gives us that if $\alpha_1, \ldots, \alpha_N$ are pairwise distinct algebraic numbers then $e^{\alpha_1}, \ldots, e^{\alpha_N}$ are linearly independent over $\overline{\mathbb{Q}}$. This was carried out in full detail, and with the approval of Lindemann, in 1885 by Weierstraß in [Wei85, p. 1067, footnote 2].

In his famous address at the ICM 1900, Hilbert went further. In the seventh of his 23 problems, he asked when α, β and $\gamma = \alpha^{\beta}$ can all three be algebraic numbers? There are some obvious cases where this is true, namely when $\alpha = 0$, $\alpha = 1$ or β is rational. But Hilbert went further and asked whether these were the only cases. He considered this problem as more difficult to prove than the Riemann Hypothesis.

To much surprise, Gelfond [Gel34] and Schneider [Sch34b], applying different methods, independently succeeded in 1934 in answering Hilbert's Problem. An almost equivalent formulation is that for $\alpha \neq 0$ the three numbers are algebraic if $\log(\alpha)$ and $\log(\alpha^{\beta})$ are linearly dependent over $\mathbb{Q}$.

Linear Forms of Logarithms

The work of Gelfond and Schneider initiated extensive work on so-called linear forms in logarithms. It was known that lower bounds for such linear forms would give solutions to several outstanding problems. One of them is the famous class number 1 problem; another is finding effectively the integral solutions of classes of diophantine equations. The main open problem in this context was to deal with linear forms in three logarithms of algebraic numbers with algebraic coefficients.

The methods that had been developed so far could not handle more than two logarithms and it was considered a very difficult problem to make progress on. This hurdle was overcome in the case of classical logarithms in 1966 by Baker. In his famous paper he was able to solve this problem in full generality not only for three but even for any finite number of logarithms. In particular, he showed that if $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers, then a linear form with algebraic coefficients in $\log \alpha_1, \ldots, \log \alpha_n$ vanishes if and only if the logarithms are linearly dependent over $\mathbb{Q}$. This result is exactly in the spirit of Gelfond and Schneider's solutions of the Hilbert Problem.

Elliptic and Abelian Integrals

The number $2\pi i$ is a period of the integral $\int dx/x$ of the rational differential form dx/x taken over a closed path in $\mathbb{C}$. Other period numbers appear in the theory of the Weierstraß elliptic function as elliptic integrals of the first kind, and, consequently, it was no surprise that Siegel [Sie32] took up the topic. He proved that not all periods of the Weierstraß elliptic functions with algebraic invariants g_2 and g_3 are algebraic; in particular it follows that in the complex multiplication case all non-zero periods are transcendental.

Schneider further developed the theory to deal also with elliptic integrals of the first and second kind and complete or incomplete periods. In a series of papers based on his solution of Hilbert's seventh problem [Sch34b, Sch34a], he went as far as the technical tools of the time allowed. He proved in [Sch37], for example, that if u is chosen such that the Weierstraß $\wp$-function takes an algebraic value, then 1, u and $\zeta(u)$ are linearly independent over $\overline{\mathbb{Q}}$. In particular, if ω is in the period lattice then 1, ω and $\eta(\omega)$ are linearly independent over $\overline{\mathbb{Q}}$. This was the first paper in which he proved a result about the Weiserstrass $\wp$-function and the exponential function. In particular he showed that π/ω is transcendental.

In the subsequent years Schneider extended his work, studying the transcendence properties of abelian functions and integrals and obtaining the first, albeit partial, results. The most striking example was the transcendence of the values of the B-function at rational arguments; see also [Sie49]. In his book [Sch57], Schneider asked, as open problems, for proofs of similar results for elliptic integrals of the third kind and for abelian integrals. As he noted, it was clear that the methods were exhausted and new methods would be necessary to solve the problems.

In a series of papers following his work on logarithms, Baker was also able to extend in a very limited way Schneider's results about transcendence of values of elliptic functions in the case of the Weierstraß $\wp-$ and ζ-functions which also contain transcendence results on elliptic integrals of the second kind; see, for example, [Bak69]. His results were extended to linear independence results on periods of integrals of the second kind by himself, Coates, Masser, Laurent and Bertrand. However, they were also strongly limited by the lack of a general tool for handling the case of an arbitrary number of elliptic or abelian logarithms. The problem was that Baker's approach did not work in general in the case of abelian varieties or more generally in the case of commutative algebraic groups.

This changed only when the Analytic Subgroup Theorem [Wüs89] became available in 1982. It allowed one to deal with 1-periods in general and linear relations between them. The case of *complete* periods in the general case where ω is an algebraic 1-form on a curve of arbitrary genus and γ a closed path on the corresponding Riemann surface was settled in 1986 by the second author in [Wüs87]: if a period is non-zero, it is transcendental. Both cases can arise. A simple example is a hyperelliptic curve whose Jacobian is isogenous to a product of two elliptic curves. Then 8 of the 16 standard periods are 0. The others are transcendental.

The Analytic Subgroup Theorem opened a fruitful interplay between transcendence theory and algebraic groups through the exponential map for Lie groups. More recently it turned out that the right frame is the theory of 1-motives introduced by Deligne in 1974; see [Del74]. It added to the algebraic groups extra data in the shape of a homomorphism of a free abelian group into the group, which lets one deal with so-called *incomplete* periods in an elegant and natural way. We develop this point of view in full detail in the present monograph.

Grothendieck's Period Conjecture

The transcendence properties of periods make it natural to ask questions about linear and algebraic relations between them. A conceptual interpretation of possible relations is provided by what came to be known as the *Period Conjecture*. Periods are given a cohomological interpretation and all relations between them should be induced by relations between motives.

This string of ideas was started by Grothendieck in [Gro66, p. 101]. He discussed the comparison of the de Rham cohomology of a smooth variety X over a number field K with its singular cohomology. The entries of the comparison matrix comparing $H^1_{\mathrm{dR}}(X)$ and $H^1_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Q})$ of a complete non-singular curve X are classical periods of the first and second kind. Grothendieck asks for instance if Schneider's Theorem generalises in some way to these periods.[1] Subsequently, he came to a conceptual conjecture and went further by predicting the transcendence degree of the field of periods of $H^n(X)$ for a smooth projective variety X (or more generally of a pure motive M) as the dimension of the motivic Galois group or alternatively the dimension of the Mumford–Tate group of the Hodge structure on M.

However, he did not publish the conjecture himself. We refer to the first-hand account of André in [And19] on the history of the conjecture. A complete formulation and discussion was finally given by André [And04, Chapter 23]. This includes a straightforward extension of the conjecture to the case of mixed motives. The formulation of the conjecture for 1-motives is discussed by Bertolin in [Ber02] and more recently [Ber20]. The only known result in this direction is a theorem of Chudnovsky, who showed in [Chu80] that for any elliptic curve defined over $\overline{\mathbb{Q}}$ at least two of the numbers ω_1, ω_2, $\eta(\omega_1)$ and $\eta(\omega_2)$ are algebraically independent provided that ω_1 and ω_2 generate the period lattice over $\mathbb{Q}$. In the case of complex multiplication this implies that ω and $\eta(\omega)$ are algebraically independent, which confirms the prediction in the CM-case. In general Grothendieck's Conjecture is out of reach. We shall present an overview of further developments below.

The Period Conjecture as Formulated by Kontsevich and Zagier

In a series of papers, Kontsevich and Zagier [KZ01, Kon99] also promoted the study of periods of non-smooth, non-projective varieties, or more generally of mixed motives. In [Kon99] Kontsevich formulated an alternative version of the Period Conjecture. As André pointed out, it has a very different flavour based on calculus rather than algebraic geometry. Relations between periods are induced by the transformation rule and Stokes's Theorem. This approach puts relative cohomology front and centre. Kontsevich views periods as the

[1] In footnote 10 of [Gro66] Grothendieck recalls the belief that the periods ω_1, ω_2 of a non-CM elliptic curve should be algebraically independent. 'This conjecture extends in an obvious way to the set of periods $(\omega_1, \omega_2, \eta_1, \eta_2)$ and can be rephrased also for curves of any genus, or rather for abelian varieties of dimension g, involving $4g$ periods.'

numbers in the image of the period pairing given by period integrals for relative cohomology

$$H^*_{\mathrm{dR}}(X,Y) \times H_*^{\mathrm{sing}}(X,Y;\mathbb{Q}) \to \mathbb{C}$$

for algebraic varieties X over $\overline{\mathbb{Q}}$ and subvarieties $Y \subset X$. By *Kontsevich's Period Conjecture* all $\overline{\mathbb{Q}}$-linear relations between such periods should be induced by bilinearity and functoriality of mixed motives. More explicitly, he introduces an algebra of formal periods $\widetilde{\mathcal{P}}$ (also called motivic periods by some authors) with explicit generators and relations. His conjecture predicts that the evaluation map $\widetilde{\mathcal{P}} \to \mathbb{C}$ (sending a formal period to the actual value of the integral) is injective. In the present monograph, we give an answer in the case of periods in degree 1, or, equivalently, periods of curves.

Grothendieck's Period Conjecture on algebraic relations between periods is essentially equivalent to Kontsevich's Period Conjecture on linear relations between periods; see, for example, [Ayo14a], [And17], [HMS17, Section 13.2.1] or [Hub20, Section 5.3] for the precise relation. It rests on a key insight of Nori, who realised that $\mathrm{Spec}(\widetilde{\mathcal{P}})$ of the algebra $\widetilde{\mathcal{P}}$ is a torsor under a motivic Galois group, in fact the torsor of tensor isomorphisms between the de Rham realisation and the singular realisation of the category of mixed Nori motives.

Dimensions of Period Spaces

For any given variety, the space of periods is finite dimensional. This makes it natural to ask about their dimension. A qualitative prediction already follows from the Period Conjecture. The precedent for the kind of formula that we have in mind is Baker's Theorem [Bak66] on logarithms. Such a formula can be made explicit: for $\beta_1, \ldots, \beta_n \in \overline{\mathbb{Q}}^*$ let $\langle \beta_1, \ldots, \beta_n \rangle$ be the multiplicative subgroup of $\overline{\mathbb{Q}}^*$ generated by these numbers. Then the dimension of the vector space generated by the principal determinations of $\log\beta_1, \ldots, \log\beta_n$ over $\overline{\mathbb{Q}}$ (but modulo multiples of π) is equal to the rank of the group generated by $\beta_1, \ldots, \beta_n$.

In addition to Baker's Theorem, a number of cases have been considered in the past: for example, the case of elliptic logarithms in [BW07, Section 6.2] or the extension of an elliptic curve by a torus of dimension n in [Wüs84b]. An interesting new case came up recently in connection with curvature lines and geodesics for billiards on a triaxial ellipsoid; see [Wüs21]. This leads to a period space generated by 1, $2\pi i$ and the periods $\omega_1, \omega_2, \eta(\omega_1), \eta(\omega_2), \lambda(u,\omega_1), \lambda(u,\omega_2)$ of the first, second and third kind. Its dimension over $\overline{\mathbb{Q}}$ is 8, 6 or 4 depending on the endomorphisms of the elliptic curve involved and on the nature of the differential of the third kind. This case serves as a model for a completely general result.

In the case of 1-motives this is the question about the dimension of the vector space generated over $\overline{\mathbb{Q}}$ by their periods. It turns out that to state and prove such a general formula for the dimension of the period space of a 1-motive is difficult. The difficulties arise from periods of the third kind and the formulas we shall give are quite involved.

Outlook

As we have already stated, the Period Conjecture itself seems currently far out of reach. Even the special case of values of the Riemann ζ-functions is widely open. The Period Conjecture implies that the $\zeta(2n+1)$ for $n \in \mathbb{N}$ are algebraically independent. On the one hand we have the theory of motives and on the other hand transcendence theory. The interaction between both is a wonderful topic. It has been exploited in order to deduce upper bounds for the spaces of periods. However, on the transcendence side, only comparatively weak lower bounds are available. Only for the case of 1-motives over $\overline{\mathbb{Q}}$ do we have a complete description of the transcendental aspects. This is what our book explains.

It is appropriate to give a short overview of other types of motives which were studied with respect to transcendence. This concerns motives over $\mathbb{Q}$, motives over function fields, both over $\mathbb{Q}$ as well as over finite fields. The more structure the base field has the more complete the transcendence situation becomes. In the following we go through some cases of motives for which transcendence has been studied and we also mention some problems about effectivity.

Mixed Tate Motives

The Riemann ζ-function has been, since the work of Euler, one of the central objects in number theory. Euler showed that its value at positive integers $2n$ is, up to a constant, of the form $(2\pi)^{2n}$. This implies that its values are transcendental over the rationals. This is Lindemann's Theorem. The only other known fact about irrationality or transcendence of integral values of the ζ-function is Ápery's discovery of irrationality of $\zeta(3)$; see [Apé79]. Only since about 2000 has there been more intensive study of these values, starting with Rivoal, Zudilin and others; see, for example, [Zud01], [BR01]. They considered the space over $\mathbb{Q}$ generated by odd ζ values up to a fixed integer n for small n and first showed that its dimension is at least 1. More recently they have been able to prove that the dimension tends to infinity with n at a rate of at least order $\log n$.

From the other side, upper bounds on spaces generated by ζ and multi-zeta values (the periods of mixed Tate motives over $\mathbb{Z}$) were provided by Deligne–Goncharov in [DG05]. The reason why these results could be established is that the motivic picture is completely understood; see also Brown's work [Bro12] on the structure of the motivic Galois group in this case.

Motives over Function Fields over $\mathbb{Q}$

Ayoub reformulated Kontsevich's Period Conjecture with fewer generators. Only polydisks are needed as domains of integration. Based on this description, he was able to formulate and prove a function field version of the conjecture in [Ayo15].

For a closed polydisk $\overline{\mathbb{D}}^n$ he considered the subspace

$$\mathcal{O}^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^n) \subset \mathcal{O}(\overline{\mathbb{D}}^n)[[T]][T^{-1}]$$

of Laurent series $F = \sum_{i>-\infty} f_i(z_1,\dots,z_n)T^i$ with coefficients in $\mathcal{O}(\overline{\mathbb{D}}^n)$ which are algebraic over $\mathbb{C}(T,z_1,\dots,z_n)$. The dimension n is allowed to vary and $\mathcal{O}^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty) = \bigcup_{n\in\mathbb{N}} \mathcal{O}^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^n)$. In analogy to Kontsevich's space of formal periods $\widetilde{\mathcal{P}}$, he defined $\mathcal{P}^{\dagger}$ as a quotient of $\mathcal{O}^{\dagger}_{\mathrm{alg}}(\overline{\mathbb{D}}^\infty)$ by certain relations and showed that there is an evaluation map $\mathcal{P}^{\dagger} \to \mathbb{C}((T))$. The main result (and geometric analogue of the Period Conjecture) is the injectivity of this evaluation map. There is also independent work of Nori (unpublished) in the same direction.

Motives over Function Fields over Finite Fields

All that is known about transcendence over $\mathbb{Q}$ is also known for function fields over finite fields: indeed often more is known. Let p be a prime, $q = p^n$ and $\mathbb{F}_q[x]$ the ring of polynomials in one variable over the finite field $\mathbb{F}_q$. In 1935 Carlitz introduced the so-called Carlitz ψ-functions in [Car35], defined as

$$\psi(t) = \sum_0^\infty \frac{(-1)^k}{F_k} t^{q^k},$$

where $F_k = [k][k-1]^q \cdots [1]^{q^{k-1}}$ and $[k] = x^{q^k} - x$. On the basis of these functions, Wade started studying transcendence theory over function fields over finite fields [Wad46]. Their minimal algebraic closure is complete with respect to the standard valuation so that the function $\psi(x)$ exists and is a replacement of the exponential function. Its inverse exists as a multi-valued function and is the analogue of the logarithm. Wade proved among other things the analogue of the theorem of Gelfond and Schneider.

In 1983 Jing Yu took up the topic and proved the analogue of Lindemann's Theorem in the realm of Drinfeld modules and started a very interesting transcendence theory for Drinfeld modules and t-motives. As a highlight of a sequence of papers including periods and quasi-periods of Drinfeld modules as well as special zeta values in characteristic p, he obtained an analogue of the Analytic Subgroup Theorem for Drinfeld modules and, more generally, Anderson's t-motives. Once the theory and the techniques had been established, a whole spectrum of applications followed, including linear independence of zeta values, by Yu, Chieh-Yu Chang, Papanikolas and Thakur; see [Yu97], [CY07], [CPTY10]. They were even able to determine the transcendence degree of fields generated by logarithms or zeta values in this setting. The survey paper of Chang [Cha17] gives a very nice and substantial report about the newest achievements in this theory.

Hypergeometric Period Relations, Periods of Higher Weight

It is well known that values of hypergeometric functions can be expressed as a quotient of two abelian integrals, in general of the second kind. This leads to a period relation between the two periods of the second kind with the hypergeometric function as coefficient. Algebraic values of the hypergeometric function provide linear relations between the two periods with algebraic coefficients. This cannot be true in general and leads to special points on certain Shimura varieties as explained very carefully in Chapter 5 of Tretkoff's beautiful monograph [Tre17]. In particular, new transcendence results are given for the Appell–Lauricella (hypergeometric) functions in $n \geq 2$ variables. They exceed the known results on the values of the classical hypergeometric function in one variable.

Hodge Level 1

An obvious problem is to extend the transcendence results to periods of higher weight. In general this seems to be a hopeless undertaking. However, there are cases when periods of higher weight can be related to 1-periods. Tretkoff gives some nice examples dealing with periods of Fermat hypersurfaces. This is the case for certain algebraic $K3$-surfaces and smooth complete intersections over $\mathbb{C}$ of Hodge level 1 as explained in [Del72]; see also [Wüs87].

We are not going to expand our monograph in this direction but mention some interesting research dealing with this kind of problem. The starting point was given in [Wüs87] and then taken up by Tretkoff.

In Chapter 6 of her monograph [Tre17], P. Tretkoff discusses among other things algebraic K3-surfaces X defined over a number field. She considers a holomorphic 2-form ω on X and shows that if the vector space generated by the periods $\int_\gamma \omega$ for $\gamma \in H_2(X, \mathbb{Z})$ has dimension 1, then X has complex multiplication, i.e. its Mumford–Tate group is abelian. This implies that if X has complex multiplication these periods are all transcendental unless $\gamma = 0$.

In Chapter 7 Tretkoff deals with arbitrary smooth projective varieties X defined over $\overline{\mathbb{Q}}$. One of the questions she raises is whether the Hodge filtration of $H^k(X, \mathbb{C})$ for $0 \leq k \leq \dim X$ has complex multiplication if it is defined over $\overline{\mathbb{Q}}$. She gives some nice examples dealing with periods on Fermat hypersurfaces.

All this is restricted to holomorphic differential forms and complete periods. In our monograph we deal with meromorphic differential forms and incomplete periods. It would be interesting to try to get more general cases involving incomplete periods.

Effectivity, Lower Bounds

The main applications of Baker's work on linear forms in logarithms were the lower bounds he derived. He showed that if Λ is a linear form in logarithms of algebraic numbers with algebraic coefficients, a lower bound for the absolute value of Λ can be obained. For a detailed account on this, see [BW07]. A similar theory exists also for the p-adic analogue, started by Coates and finally brought to the level of the archimedean case by Kunrui Yu. Similar results were also obtained for elliptic and abelian logarithms. One might ask whether this can be extended to 1-motives in a modified way. First steps in this direction were the work of Masser and Wüstholz [MW93] on isogeny estimates. It says that, given an isogeny between two abelian varieties, there exists an isogeny with degree bounded by the original data: height of the source isogeny, degree of the number field, dimension of the abelian variety. The result is completely effective and a crucial ingredient for the proof of the famous Tate Conjecture. It has also been used for the proof of the André–Oort Conjecture for the coarse moduli space of prinicipally polarised abelian varieties by Tsimerman [Tsi18]. It would be interesting to formulate an isogeny estimate type statement for 1-motives and to find applications in diophantine geometry.

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