

LINEAR STATE/SIGNAL SYSTEMS

The authors explain in this work a new approach to observing and controlling linear systems whose inputs and outputs are not fixed in advance. They cover a class of linear time-invariant state/signal systems that is general enough to include most of the standard classes of linear time-invariant dynamical systems, but simple enough to make it easy to understand the fundamental principles. They begin by explaining the basic theory of finite-dimensional and bounded systems in a way suitable for graduate courses in systems theory and control. They then proceed to more advanced infinite-dimensional settings, opening up new ways for researchers to study distributed parameter systems, including linear port-Hamiltonian systems and boundary triplets. They include the general nonpassive part of the theory in continuous and discrete time, and provide a short introduction to the passive situation. Numerous examples from circuit theory are used to illustrate the theory.

Encyclopedia of Mathematics and Its Applications

This series is devoted to significant topics or themes that have wide application in mathematics or mathematical science, and for which a detailed development of the abstract theory is less important than a thorough and concrete exploration of the implications and applications.

Books in the **Encyclopedia of Mathematics and Its Applications** cover their subjects comprehensively. Less important results may be summarized as exercises at the end of the chapters. For technicalities, readers can be referred to the bibliography that is expected to be comprehensive. As a result, volumes are encyclopedic references or manageable guides to major subjects.

 ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing, visit

www.cambridge.org/mathematics.

- 133 P. Kornerup and D. W. Matula *Finite Precision Number Systems and Arithmetic*
 134 Y. Crama and P. L. Hammer (eds.) *Boolean Models and Methods in Mathematics, Computer Science, and Engineering*
 135 V. Berthé and M. Rigo (eds.) *Combinatorics, Automata and Number Theory*
 136 A. Kristály, V. D. Rădulescu and C. Varga *Variational Principles in Mathematical Physics, Geometry, and Economics*
 137 J. Berstel and C. Reutenauer *Noncommutative Rational Series with Applications*
 138 B. Courcelle and J. Engelfriet *Graph Structure and Monadic Second-Order Logic*
 139 M. Fiedler *Matrices and Graphs in Geometry*
 140 N. Vakil *Real Analysis through Modern Infinitesimals*
 141 R. B. Paris *Hadamard Expansions and Hyperasymptotic Evaluation*
 142 Y. Crama and P. L. Hammer *Boolean Functions*
 143 A. Arapostathis, V. S. Borkar and M. K. Ghosh *Ergodic Control of Diffusion Processes*
 144 N. Caspard, B. Leclerc and B. Monjardet *Finite Ordered Sets*
 145 D. Z. Arov and H. Dym *Bitangential Direct and Inverse Problems for Systems of Integral and Differential Equations*
 146 G. Dassios *Ellipsoidal Harmonics*
 147 L. W. Beineke and R. J. Wilson (eds.) with O. R. Oellermann *Topics in Structural Graph Theory*
 148 L. Berlyand, A. G. Kolpakov and A. Novikov *Introduction to the Network Approximation Method for Materials Modeling*
 149 M. Baake and U. Grimm *Aperiodic Order I: A Mathematical Invitation*
 150 J. Borwein et al. *Lattice Sums Then and Now*
 151 R. Schneider *Convex Bodies: The Brunn–Minkowski Theory (Second Edition)*
 152 G. Da Prato and J. Zabczyk *Stochastic Equations in Infinite Dimensions (Second Edition)*
 153 D. Hofmann, G. J. Seal and W. Tholen (eds.) *Monoidal Topology*
 154 M. Cabrera García and Á. Rodríguez Palacios *Non-associative Normed Algebras I: The Vidav–Palmer and Gelfand–Naimark Theorems*
 155 C. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several Variables (Second Edition)*
 156 L. W. Beineke and R. J. Wilson (eds.) with B. Toft *Topics in Chromatic Graph Theory*
 157 T. Mora *Solving Polynomial Equation Systems III: Algebraic Solving*
 158 T. Mora *Solving Polynomial Equation Systems IV: Buchberger Theory and Beyond*
 159 V. Berthé and M. Rigo (eds.) *Combinatorics, Words and Symbolic Dynamics*
 160 B. Rubin *Introduction to Radon Transforms: With Elements of Fractional Calculus and Harmonic Analysis*
 161 M. Ghergu and S. D. Taliaferro *Isolated Singularities in Partial Differential Inequalities*
 162 G. Molica Bisci, V. D. Rădulescu and R. Servadei *Variational Methods for Nonlocal Fractional Problems*
 163 S. Wagon *The Banach–Tarski Paradox (Second Edition)*
 164 K. Broughan *Equivalents of the Riemann Hypothesis I: Arithmetic Equivalents*
 165 K. Broughan *Equivalents of the Riemann Hypothesis II: Analytic Equivalents*
 166 M. Baake and U. Grimm (eds.) *Aperiodic Order II: Crystallography and Almost Periodicity*
 167 M. Cabrera García and Á. Rodríguez Palacios *Non-associative Normed Algebras II: Representation Theory and the Zel’manov Approach*
 168 A. Yu. Khrennikov, S. V. Kozyrev and W. A. Zúñiga-Galindo *Ultrametric Pseudodifferential Equations and Applications*
 169 S. R. Finch *Mathematical Constants II*
 170 J. Krajčiček *Proof Complexity*
 171 D. Bulacu, S. Caenepeel, F. Panaite and F. Van Oystaeyen *Quasi-Hopf Algebras*
 172 P. McMullen *Geometric Regular Polytopes*
 173 M. Aguiar and S. Mahajan *Bimonoids for Hyperplane Arrangements*
 174 M. Barski and J. Zabczyk *Mathematics of the Bond Market: A Lévy Processes Approach*
 175 T. R. Bielecki, J. Jakubowski and M. Niewęglowski *Structured Dependence between Stochastic Processes*
 176 A. A. Borovkov, V. V. Ulyanov and Mikhail Zhitlukhin *Asymptotic Analysis of Random Walks: Light-Tailed Distributions*
 177 Y.-K. Chan *Foundations of Constructive Probability Theory*
 178 L. W. Beineke, M. C. Golombic and R. J. Wilson (eds.) *Topics in Algorithmic Graph Theory*
 179 H.-L. Gau and P. Y. Wu *Numerical Ranges of Hilbert Space Operators*
 180 P. A. Martin *Time-Domain Scattering*
 181 M. D. de la Iglesia *Orthogonal Polynomials in the Spectral Analysis of Markov Processes*
 182 A. E. Brouwer and H. Van Maldeghem *Strongly Regular Graphs*

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Linear State/Signal Systems

DAMIR Z. AROV

South Ukrainian National Pedagogical University

OLOF J. STAFFANS

Åbo Akademi University, Finland



CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom
One Liberty Plaza, 20th Floor, New York, NY 10006, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India
103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org

Information on this title: www.cambridge.org/9781316519677

DOI: 10.1017/9781009024921

© Damir Z. Arov and Olof J. Staffans 2022

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2022

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data

Names: Arov, Damir Z., author. | Staffans, Olof J., 1947– author.
Title: Linear state/signal systems / Damir Z. Arov, South Ukrainian
National Pedagogical University, Olof J. Staffans, Åbo Akademi
University, Finland.

Description: Cambridge, United Kingdom ; New York, NY : Cambridge
University Press, 2022. | Series: Encyclopedia of mathematics and its
applications | Includes bibliographical references and index.

Identifiers: LCCN 2021058586 (print) | LCCN 2021058587 (ebook)
| ISBN 9781316519677 (hardback) | ISBN 9781009024921 (ebook)

Subjects: LCSH: Linear systems. | Operator theory. | Linear control
systems. | BISAC: MATHEMATICS / General

Classification: LCC QA402 .A755 2022 (print) | LCC QA402 (ebook)
| DDC 003/.74–dc23/eng/20220126

LC record available at <https://lccn.loc.gov/2021058586>

LC ebook record available at <https://lccn.loc.gov/2021058587>

ISBN 978-1-316-51967-7 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

Contents

<i>Preface</i>	<i>page</i> xxi
<i>List of Notations</i>	xxiii
1 Introduction and Overview	1
1.1 Linear Time-Invariant Dynamical Systems	1
1.1.1 State Systems	2
1.1.2 Systems That Interact with the Outside World	3
1.1.3 Input/State/Output Systems	3
1.1.4 Input/Output Systems	4
1.1.5 Classical (Sub)networks	5
1.1.6 Port-Hamiltonian Systems	5
1.1.7 Behavioral Systems	6
1.1.8 State/Signal Systems	7
1.1.9 State/Signal versus I/S/O Systems	8
1.1.10 Frequency Domain Systems	10
1.1.11 Boundary Triplets	11
1.1.12 State/Signal versus Behavioral Systems	11
1.1.13 How to Read This Book	12
1.1.14 H -Spaces	12
1.1.15 Where to Go from Here?	13
1.2 An Overview of State/Signal and Input/State/Output Systems	14
1.2.1 Input/State/Output Systems	14
1.2.2 Well-Posed I/S/O Systems	17
1.2.3 State/Signal Systems	18
1.2.4 I/S/O Representations	19
1.2.5 Similarity of I/S/O and State/Signal Systems	20
1.2.6 Input/Output Invariant Properties of I/S/O Systems	23
1.2.7 Properties of I/S/O Systems in the State/Signal Sense	25

1.2.8	Static Transformations of I/S/O and State/Signal Systems	26
1.2.9	Invariant Subspaces of I/S/O and State/Signal Systems	28
1.2.10	Interconnections of I/S/O and State/Signal Systems	29
1.2.11	External Characteristics of I/S/O and State/Signal Systems	31
1.2.12	Restrictions, Projections, and Compressions	31
1.2.13	I/S/O and State/Signal Systems in Discrete Time	33
1.2.14	The Resolvent Matrix of an I/S/O System	35
1.2.15	The Resolvent Set and the Characteristic Bundles of State/Signal Systems	37
1.2.16	Well-Posed I/S/O and State/Signal Systems in the Frequency Domain	39
1.2.17	General Resolvable Frequency Domain I/S/O and State/Signal Systems	40
1.2.18	Dual and Adjoint I/S/O and State/Signal Systems	42
1.2.19	Passive I/S/O and State/Signal Systems	44
1.2.20	Passive Finite-Dimensional Electrical n -Ports	45
1.2.21	Some Finite-Dimensional Passive 2-Ports	51
1.2.22	Some Distributed Parameter Passive Systems	59
1.3	Notes and Comments	68
2	State/Signal Systems: Trajectories, Transformations, and Interconnections	71
2.1	State/Signal Nodes and State/Signal Systems	71
2.1.1	The State/Signal System and Its Trajectories	71
2.1.2	Regular and Semiregular State/Signal Nodes	75
2.1.3	Kernel and Image Representations of Closed State/Signal Nodes	80
2.1.4	Bounded State/Signal Nodes and Systems	84
2.2	Some Basic Transformations of State/Signal Nodes	87
2.2.1	Similarity of Two State/Signal Nodes	87
2.2.2	Time Reflection of a State/Signal Node	89
2.2.3	Time Rescaling of a State/Signal Node	91
2.2.4	Exponentially Weighted State/Signal Nodes	92
2.3	Properties of Trajectories of State/Signal Systems	93
2.3.1	Classical, Generalized, and Mild Trajectories	93
2.3.2	Existence and Uniqueness of Trajectories	95

Contents

vii

	2.3.3 Connections between Classical, Generalized, and Mild Trajectories	100
2.4	Some Additional Transformations of State/Signal Nodes	104
	2.4.1 The (P, Q) -Image of a State/Signal Node	104
	2.4.2 Parts and Static Projections of a State/Signal Node	107
	2.4.3 Adding Inputs and Output to a State/Signal Node	111
2.5	Interconnections of State/Signal Nodes	118
	2.5.1 The Cross Product of Two State/Signal Nodes	118
	2.5.2 (P, Q) -Interconnections of State/Signal Nodes	119
	2.5.3 A Short Circuit Connection of State/Signal Nodes	120
	2.5.4 Examples of Interconnections of State/Signal Nodes	121
2.6	Examples of Infinite-Dimensional State/Signal Systems	123
3	State/Signal Systems: Dynamic and Frequency Domain Properties	132
3.1	Signal Behaviors and Their State/Signal Realizations	132
	3.1.1 Future Signal Behaviors	132
	3.1.2 External Equivalence of State/Signal Systems	133
3.2	Dynamic Properties of State/Signal Systems	133
	3.2.1 Controllability and Observability of State/Signal Systems	134
	3.2.2 Intertwinements of State/Signal Systems	140
	3.2.3 Compressions, Restrictions, and Projections of State/Signal Systems	142
	3.2.4 Examples of Minimal Compressions	148
	3.2.5 State/Signal Systems with the Continuation Property	153
3.3	State Systems	156
	3.3.1 A State System and Its Trajectories	157
	3.3.2 The Homogeneous Cauchy Problem	158
	3.3.3 Bounded State Systems and Uniformly Continuous Groups	159
	3.3.4 Well-Posed State Systems and Strongly Continuous Semigroups	163
	3.3.5 Transformations and Interconnections of State Nodes	169
	3.3.6 Invariant Subspaces of State Nodes	171
	3.3.7 Intertwinement of State Nodes	173
3.4	Frequency Domain Characteristics of State/Signal Nodes	174
	3.4.1 The Characteristic Node Bundle	174
	3.4.2 The Characteristic Control Bundle	178
	3.4.3 The Characteristic Observation Bundle	179
	3.4.4 The Characteristic Signal Bundle	180

3.4.5	The Characteristic Bundles of Transformed State/Signal Systems	182
3.4.6	The Resolvent of a Regular State Node	189
3.4.7	The Resolvent Set of a State/Signal Node	191
3.5	Invariance with Respect to Similarities	192
3.6	Dual and Adjoint State/Signal Nodes and Systems	194
3.6.1	The Dual of a State/Signal System	194
3.6.2	The Duals of Some Transformed State/Signal Nodes	202
3.6.3	The Characteristic Bundles of Dual State/Signal Nodes	206
3.6.4	The Adjoint State/Signal System	207
3.7	Notes to Chapters 2 and 3	213
4	Input/State/Output Representations	217
4.1	Input/State/Output Nodes and Systems	217
4.1.1	Regular I/S/O Nodes	217
4.1.2	General I/S/O Nodes and Systems	220
4.1.3	Kernal and Image Representations of Closed I/S/O Nodes	223
4.1.4	State, Input/State and State/Output Nodes and Systems	226
4.1.5	Input/State and State/Output Representations of State/Signal Systems	228
4.1.6	Free Inputs and Continuously Determined Outputs	229
4.1.7	Existence and Uniqueness of Trajectories	230
4.1.8	Bounded I/S/O Nodes and Systems	232
4.2	Input/State/Output Representations of State/Signal Nodes and Systems	234
4.2.1	The State/Signal Node Induced by an I/S/O Node	234
4.2.2	I/O Representations of the Signal Space	236
4.2.3	General I/S/O Representations of a State/Signal Node	239
4.2.4	Semiregular I/S/O Representations of a Semiregular State/Signal Node	241
4.2.5	Regular I/S/O Representations of a Regular State/Signal Node	242
4.2.6	Parametrization of I/S/O Representations	246
4.2.7	Bounded I/S/O Representations of Bounded State/Signal Nodes	248
4.2.8	Parametrization of Bounded I/S/O Representations	251
4.3	State Feedback and Output Injection Representations	256
4.3.1	State Feedback Representations	256
4.3.2	Output Injection Representations	259

4.4	Basic Transformations of Input/State/Output Nodes	261
4.4.1	Similarity of I/S/O Nodes	261
4.4.2	Time Reflection of an I/S/O Node	263
4.4.3	Time Rescaling of an I/S/O Node	264
4.4.4	Exponentially Weighted I/S/O Nodes	265
4.5	Properties of Trajectories of Input/State/Output Systems	267
4.5.1	Basic Properties of the Sets of Classical and Generalized Trajectories	267
4.5.2	Solvability and the Uniqueness Property	268
4.5.3	Connections between Classical, Generalized, and Mild Trajectories	270
4.6	Some Simple Input/State/Output Examples	273
5	Input/State/Output Systems: Dynamic and Frequency Domain Properties	276
5.1	Additional Transformations of Input/State/Output Nodes	276
5.1.1	Adding a Feedthrough Term to an I/S/O Node	276
5.1.2	Modifying Inputs and Outputs of an I/S/O Node	277
5.1.3	The (P, R, Q) -Image of an I/S/O Node	278
5.1.4	Parts and Static Projections of an I/S/O Node	281
5.1.5	Static Output Feedback	285
5.1.6	Adding Inputs and Outputs to an I/S/O Node	287
5.1.7	A Second Look at State Feedbacks and Output Injections	297
5.2	Interconnections of Input/State/Output Nodes	303
5.2.1	The Cross Product of Two I/S/O Nodes	303
5.2.2	(P, R, Q) -Interconnections of I/S/O Nodes	305
5.2.3	A Short Circuit Connection of I/S/O Nodes	306
5.2.4	T -Junctions, Sum Junctions, and Difference Junctions	307
5.2.5	Parallel and Difference Connections	311
5.2.6	Cascade Connections	313
5.2.7	Dynamic Feedback	316
5.2.8	Examples of I/S/O Interconnections	317
5.3	Realizations of Input/Output Behaviors	318
5.3.1	Future I/O Behaviors	318
5.3.2	External Equivalence of I/S/O Systems	319
5.4	Dynamic Properties of Input/State/Output Systems	320
5.4.1	Controllability and Observability of I/S/O Systems	320
5.4.2	Intertwinements of I/S/O Systems	325

5.4.3	Compressions, Restrictions, and Projections of I/S/O Systems	326
5.4.4	I/S/O Systems with the Continuation Property	330
5.5	Frequency Domain Characteristics of Input/State/Output Nodes	331
5.5.1	The Characteristic Node Bundle of an I/S/O Node	331
5.5.2	The I/S/O Resolvent Matrix of an I/S/O Node	334
5.5.3	Resolvability of Transformed I/S/O Nodes	338
5.5.4	Frequency Domain I/S/O-Admissible I/O Representations	341
5.6	The Correspondence between State/Signal and Input/State/Output Notions	343
5.6.1	I/O Invariant Notions	343
5.6.2	Properties of I/S/O Systems in the State/Signal Sense	349
5.7	Adjoint and Dual Input/State/Output Nodes and Systems	351
5.7.1	The Adjoint and the Dual of an I/S/O Node	351
5.7.2	Adjoint and Dual I/S/O Representations	354
5.7.3	I/S/O Lagrange Identities	355
5.7.4	Properties of Adjoint and Dual I/S/O Nodes and Systems	359
5.7.5	The Adjoints and Duals of Some Transformed I/S/O Nodes	360
5.7.6	The Adjoints and Duals of Some Interconnected I/S/O Nodes	364
5.8	Notes to Chapters 4 and 5	366
6	Bounded Input/State/Output Systems in Continuous and Discrete Time	370
6.1	Bounded State Operators and Nodes	370
6.1.1	The Spectral Radius of a Bounded State Operator	370
6.1.2	Invariant Subspaces of Bounded State Operators and Uniformly Continuous Groups	372
6.1.3	Parts and Projections of Bounded State Operators	373
6.1.4	Parts and Projections of Uniformly Continuous Groups	375
6.1.5	Intertwinements of Bounded State Operators and Uniformly Continuous Groups	377
6.1.6	Compressions of Bounded State Operators and Uniformly Continuous Groups	379
6.1.7	The General Structure of a Compression of a Bounded State Operator	385

Contents

xi

	6.1.8 The Adjoints of Bounded State Operators and Uniformly Continuous Groups	390
6.2	Static Properties of Bounded Input/State/Output Nodes	393
	6.2.1 Transformations of Bounded I/S/O Nodes	393
	6.2.2 Interconnections of Bounded I/S/O Nodes	403
	6.2.3 The I/S/O Resolvent Matrix of a Bounded I/S/O Node	406
6.3	Dynamic Properties of Bounded Input/State/Output Systems	407
	6.3.1 Strongly Invariant and Unobservably Invariant Subspaces	407
	6.3.2 External Equivalence of Bounded I/S/O Systems	415
	6.3.3 Intertwinements of Bounded I/S/O Systems	417
	6.3.4 Restrictions and Projections of Bounded I/S/O Systems	421
	6.3.5 Compressions of Bounded I/S/O Systems	423
	6.3.6 The General Structure of a Bounded I/S/O Compression	428
	6.3.7 Compressions into Minimal Bounded I/S/O Systems	435
6.4	The Adjoint and the Dual of a Bounded Input/State/Output Node	439
6.5	Discrete Time Input/State/Output Systems	444
	6.5.1 Introduction to Discrete Time I/S/O Systems	444
	6.5.2 Properties of Discrete Time I/S/O Systems	445
	6.5.3 Time Reflection of Discrete Time I/S/O Systems	448
	6.5.4 Power Weightings of Discrete Time I/S/O Systems	449
	6.5.5 Frequency Domain Shifts of Discrete Time I/S/O Systems	450
	6.5.6 Stable Discrete Time I/S/O Systems	451
	6.5.7 Connections between Continuous and Discrete Time I/S/O Properties	453
	6.5.8 Dynamic Notions for Bounded I/S/O Nodes	454
6.6	Bounded Input/State/Output Realizations	456
	6.6.1 Analyticity at Infinity of the I/S/O Resolvent Matrix	456
	6.6.2 Existence of a Bounded I/S/O Realization	457
7	Bounded State/Signal Systems in Continuous and Discrete Time	460
	7.1 Static Properties of Bounded State/Signal Nodes	460
	7.1.1 The I/S/O-Bounded Resolvent Set of a Bounded State/Signal Node	460
	7.1.2 Transformations of Bounded State/Signal Nodes	462
	7.1.3 Resolvability of Transformations of State/Signal Nodes	476
	7.2 Dynamic Properties of Bounded State/Signal Systems	481

7.2.1	Strongly Invariant and Unobservably Invariant Subspaces	481
7.2.2	External Equivalence of Bounded State/Signal Systems	489
7.2.3	Intertwinements of Bounded State/Signal Systems	490
7.2.4	Restrictions and Projections of Bounded State/Signal Systems	494
7.2.5	Compressions of Bounded State/Signal Systems	496
7.2.6	The General Structure of a Bounded State/Signal Compression	498
7.2.7	Compressions into Minimal Bounded State/Signal Systems	504
7.2.8	Bounded State/Signal Realizations	505
7.3	The Dual and the Adjoint of a Bounded State/Signal Node	506
7.4	Discrete Time State/Signal Systems	510
7.4.1	Introduction to Discrete Time State/Signal Systems	510
7.4.2	Properties of Discrete Time State/Signal Systems	511
7.4.3	Time Reflection of Discrete Time State/Signal Systems	513
7.4.4	Power Weightings of Discrete Time State/Signal Systems	513
7.4.5	Frequency Domain Shifts of Discrete Time State/Signal Systems	514
7.4.6	Stable Discrete Time State/Signal Systems	515
7.4.7	Connections between Continuous and Discrete Time State/Signal Properties	515
7.4.8	Dynamic Notions for Bounded State/Signal Nodes	516
7.5	Notes to Chapters 6 and 7	518
8	Semi-bounded Input/State/Output Systems	521
8.1	C_0 Semigroups and Well-Posed State Systems	521
8.1.1	On the Resolvents of Generators of C_0 Semigroups	521
8.1.2	The Inhomogeneous Cauchy Problem	524
8.1.3	Invariant Subspaces of C_0 Semigroups	530
8.1.4	Parts, Projections, and Restrictions of Single-Valued Resolvable Main Operators	530
8.1.5	Parts and Projections of C_0 Semigroups	533
8.1.6	Intertwinements of C_0 Semigroups	535
8.1.7	Compressions of C_0 Semigroups	536
8.1.8	The General Structure of a Compression of a C_0 Semigroup	539

<i>Contents</i>		xiii
8.1.9	The Adjoint of a C_0 Semigroup	542
8.2	Semi-bounded Input/State/Output Systems	544
8.2.1	Introduction to Semi-bounded I/S/O Systems	544
8.2.2	Transformations of Semi-bounded I/S/O Nodes	547
8.2.3	Interconnections of Semi-bounded I/S/O Nodes	551
8.2.4	The I/S/O Resolvent Matrix of a Semi-bounded I/S/O Node	552
8.3	Dynamic Properties of Semi-bounded Input/State/Output Systems	553
8.3.1	Strongly Invariant and Unobservably Invariant Subspaces	553
8.3.2	External Equivalence of Semi-bounded I/S/O Systems	559
8.3.3	Intertwinements of Semi-bounded I/S/O Systems	559
8.3.4	Restrictions and Projections of Semi-bounded I/S/O Systems	562
8.3.5	Compressions of Semi-bounded I/S/O Systems	563
8.3.6	The General Structure of a Semi-bounded I/S/O Compression	565
8.3.7	Compressions into Minimal Semi-bounded I/S/O Systems	570
8.4	The Adjoint of a Semi-bounded Input/State/Output Node	572
9	Semi-bounded State/Signal Systems	576
9.1	Static Properties of Semi-bounded State/Signal Nodes	576
9.1.1	Introduction to Semi-bounded State/Signal Nodes and Systems	576
9.1.2	The I/S/O Semi-bounded Resolvent Set of a Semi-bounded State/Signal Node	580
9.1.3	Transformations and Interconnections of Semi-bounded State/Signal Nodes	581
9.2	Dynamic Properties of Semi-bounded State/Signal Systems	581
9.2.1	Strongly Invariant and Unobservably Invariant Subspaces	581
9.2.2	External Equivalence of Semi-bounded State/Signal Systems	584
9.2.3	Intertwinements of Semi-bounded State/Signal Systems	585
9.2.4	Restrictions and Projections of Semi-bounded State/Signal Systems	587
9.2.5	Compressions of Semi-bounded State/Signal Systems	588

9.2.6	The General Structure of a Semi-bounded State/Signal Compression	589
9.2.7	Compressions into Minimal Semi-bounded State/Signal Systems	593
9.3	The Adjoint of a Semi-bounded State/Signal Node	594
9.4	Notes to Chapters 8 and 9	596
10	Resolvable Input/State/Output and State/Signal Nodes	599
10.1	Resolvable State Nodes	599
10.1.1	Linear Operator-Valued Pencils	599
10.1.2	The Resolvent of a State Node	601
10.1.3	The Interpolation Space of a Semiregular State Node	606
10.1.4	The Extrapolation Space of a Regular Resolvable State Node	607
10.1.5	The Duals of the Interpolation and Extrapolation Spaces	610
10.1.6	The Interpolation and Extrapolation Spaces of a Semigroup Generator	613
10.2	Resolvable Input/State/Output Nodes	614
10.2.1	Resolvability of an I/S/O Node	615
10.2.2	Kernel and Image Representations of the I/S/O Resolvent Matrix	618
10.2.3	The I/S/O Resolvent Identity	620
10.2.4	Representations of the System Operator	624
10.2.5	Semiregular and Regular Resolvable I/S/O Nodes	628
10.2.6	The Observation and Control Operators of a Regular Resolvable I/S/O Node	631
10.2.7	Some Examples of Regular Resolvable I/S/O Nodes	637
10.2.8	Resolvability of Transformed I/S/O Nodes	640
10.2.9	Resolvability of Interconnected I/S/O Nodes	647
10.2.10	The Resolvent Family of Bounded I/S/O Nodes	652
10.2.11	A Finite-Dimensional Nonregular Resolvable I/S/O Node	653
10.2.12	The Adjoint and the Dual of a Resolvable I/S/O Node	655
10.3	Resolvable State/Signal Nodes	658
10.3.1	On the Resolvent Set of a Closed State/Signal Node	658
10.3.2	Frequency Domain I/S/O-Admissible I/O Representations	662
10.3.3	Resolvability of Transformed State/Signal Nodes	669
10.3.4	The Resolvent Family of Bounded State/Signal Nodes	673

10.3.5	The Dual and the Adjoint of a Resolvable State/Signal System	674
10.4	Notes and Comments	676
11	Frequency Domain Input/State/Output Systems	679
11.1	Frequency Domain Input/State/Output Systems	679
11.1.1	Introduction to Frequency Domain I/S/O Systems	679
11.1.2	Frequency Domain Controllability and Observability	681
11.1.3	Frequency Domain Invariance	682
11.1.4	The Frequency Domain Behavior and External Equivalence	689
11.1.5	Frequency Domain Intertwinements	690
11.1.6	Frequency Domain Compressions, Restrictions, and Projections	696
11.1.7	Resolvable Frequency Domain Compressions, Restrictions, and Projections	698
11.1.8	The General Structure of a Resolvable Frequency Domain Compression	704
11.1.9	Compressions into Ω -Minimal I/S/O Systems	712
11.1.10	Results for Connected Frequency Domains	715
11.2	The Adjoint and the Dual of a Frequency Domain Input/State/Output System	725
11.2.1	Frequency Domain Lagrange Identities	726
11.2.2	Properties of Adjoint and Dual Frequency Domain I/S/O Systems	728
11.3	Frequency Domain Notions for Ω -Resolvable Input/State/Output Nodes	730
11.3.1	Dynamic Properties of the Resolvent Family of Bounded I/S/O Nodes	730
11.4	Resolvable Frequency Domain State Systems	733
11.4.1	Frequency Domain Invariance	734
11.4.2	Frequency Domain Intertwinements and Compressions	734
11.4.3	Results for Connected Frequency Domains	738
11.4.4	Frequency Domain Duality	740
11.5	Notes and Comments	741
12	Frequency Domain State/Signal Systems	743
12.1	Frequency Domain State/Signal Systems	743
12.1.1	Introduction to Frequency Domain State/Signal Systems	743

12.1.2	Separately and Jointly I/S/O Admissible Frequency Domains	745
12.1.3	Frequency Domain Controllability and Observability	747
12.1.4	Frequency Domain Invariance	748
12.1.5	The Frequency Domain Behavior and External Equivalence	752
12.1.6	Frequency Domain Intertwinements	755
12.1.7	Frequency Domain Compressions, Restrictions, and Projections	761
12.1.8	Resolvable Frequency Domain Compressions, Restrictions, and Projections	763
12.1.9	The General Structure of a Resolvable Frequency Domain Compression	769
12.1.10	Compressions into Ω -Minimal State/Signal Systems	773
12.2	Local Frequency Domain Notions	775
12.2.1	Local Frequency Domain Notions for Ω -Resolvable State/Signal Systems	776
12.2.2	Connected Frequency Domains	783
12.3	The Dual and the Adjoint of a Frequency Domain State/Signal System	793
12.3.1	Frequency Domain Lagrange Identities	794
12.3.2	Properties of Dual and Adjoint Frequency Domain State/Signal Systems	795
12.4	Frequency Domain Notions for Ω -Resolvable State/Signal Nodes	798
12.4.1	Dynamic Properties of the Resolvent Family of Bounded State/Signal Nodes	798
12.5	Notes and Comments	801
13	Internally Well-Posed Systems	802
13.1	Internally Well-Posed Input/State/Output Systems	802
13.1.1	Basic Definitions and Properties	802
13.1.2	Transformations and Interconnections	804
13.2	Frequency-Domain Internally Well-Posed Input/State/Output Systems	805
13.2.1	Frequency Domain Invariance	806
13.2.2	Frequency Domain Intertwinements	806
13.2.3	Frequency-Domain Restrictions, Projections, and Compressions	807
13.2.4	The General Structure of $\rho_{+\infty}(\Sigma)$ -Compressions	809

<i>Contents</i>		xvii
13.3	Internally Well-Posed State/Signal Systems	812
13.3.1	Basic Definitions and Properties	812
13.3.2	Frequency-Domain Compressions of Internally Well-Posed State/Signal Systems	813
13.4	Notes and Comments	814
14	Well-Posed Input/State/Output Systems	816
14.1	Basic Properties of Well-Posed Input/State/Output Systems	816
14.1.1	The Definition of a Well-Posed I/S/O System	816
14.1.2	Alternative Conditions for Well-Posedness	819
14.1.3	The Fundamental I/S/O Solution of a Well-Posed I/S/O System	825
14.2	The Growth Bound of a Well-Posed Input/State/Output System	833
14.2.1	The Growth Bound of a Well-Posed I/S/O System	833
14.2.2	Stable I/S/O Systems	838
14.3	Resolvability of Well-Posed Input/State/Output Systems	842
14.3.1	Well-Posed I/S/O Systems Are Resolvable	842
14.3.2	Growth Estimates for the I/S/O Resolvent Matrix	847
14.4	Realizations of Shift-Invariant Causal Linear Operators	850
14.4.1	Shift Invariant Causal Linear Operators	850
14.4.2	Realizations of Shift Invariant Causal Linear Operators	852
14.4.3	Toeplitz and Hankel Operators	853
14.5	Transformations and Interconnections of Well-Posed Input/State/Output Systems	857
14.5.1	Well-Posedness and Stability of Transformed I/S/O Systems	857
14.5.2	Well-Posedness and Stability of Interconnected I/S/O Systems	864
14.5.3	Stabilizable and Detectable I/S/O Systems	867
14.6	Dynamic Properties of Well-Posed Input/State/Output Systems	869
14.6.1	Strongly Invariant and Unobservably Invariant Subspaces	869
14.6.2	Intertwinements of Well-Posed I/S/O Systems	870
14.6.3	Restrictions, Projections, and Compressions	874
14.6.4	The General Structure of a Well-Posed I/S/O Compression	879
14.6.5	Compressions Into Minimal Well-Posed I/S/O Systems	886
14.7	Well-Posed Input/State/Output Systems in the Frequency Domain	887
14.7.1	Time and Frequency Domain External Equivalence	888
14.7.2	Time and Frequency Domain Invariance	888

14.7.3	Time and Frequency Domain Compressions and Intertwinements	889
14.7.4	Frequency Domain Stability	891
14.8	The Adjoint of a Well-Posed Input/State/Output System	893
14.9	Scattering Passive Input/State/Output Systems	897
14.9.1	Hilbert Space I/S/O Nodes and Systems	897
14.9.2	Scattering Passive I/S/O Systems	898
14.9.3	The Internal I/S/O Cayley Transform	901
14.9.4	The Adjoint of a Passive Scattering System	904
14.10	Notes and Comments	904
15	Well-Posed State/Signal Systems	907
15.1	Basic Properties of Well-Posed State/Signal Systems	907
15.1.1	Basic Definitions	907
15.1.2	Well-Posedness and Stability of Transformed I/S/O Systems	913
15.1.3	The Behaviors Induced by a Well-Posed State/Signal System	914
15.1.4	The Past/Present and Present/Future Maps of a Well-Posed State/Signal System	916
15.2	Stable State/Signal Systems	920
15.2.1	Stable State/Signal Trajectories	920
15.2.2	Stable State/Signal Behaviors	921
15.2.3	Stabilizable and Detectable State/Signal Systems	922
15.3	Realizations of Well-Posed Behaviors	925
15.3.1	Well-Posed Future, Past, and Two-Sided Behaviors	925
15.3.2	State/Signal Realizations of Well-Posed Behaviors	929
15.3.3	The Past/Future Map of a Well-Posed Behavior	929
15.4	Dynamic Properties of Well-Posed State/Signal Systems	930
15.4.1	Strongly Invariant and Unobservably Invariant Subspaces	930
15.4.2	Intertwinements of Well-Posed State/Signal Systems	932
15.4.3	Restrictions, Projections, and Compressions of Well-Posed State/Signal Systems	933
15.4.4	The General Structure of a Compression	936
15.4.5	Compressions into Minimal Well-Posed State/Signal Systems	939
15.5	Well-Posed State/Signal Systems in the Frequency Domain	940
15.6	The Adjoint of a Well-Posed State/Signal Node	942
15.7	Passive State/Signal Systems	944

<i>Contents</i>		xix
15.7.1	Kreĭn Spaces	944
15.7.2	The Kreĭn Node Space of a Scattering Passive I/S/O System	945
15.7.3	Passive State/Signal Systems	946
15.8	Notes and Comments	949
Appendix A	Operators and Analytic Vector Bundles in H-Spaces	950
A.1	H -Spaces	950
A.1.1	Using More than One Norm in a Vector Space	950
A.1.2	Introduction to H -Spaces	952
A.1.3	Linear Operators in H -Spaces	953
A.1.4	Closed Linear Operators in H -Spaces	955
A.1.5	Complementary Projections and Coordinate Representations of H -Spaces	956
A.1.6	Isomorphisms in H -Spaces	960
A.1.7	Partial Inverses of Bounded Linear Operators	961
A.1.8	Inversion of Block Matrix Operators	964
A.1.9	The Graph Norm and Graph Topology	965
A.1.10	Linear Multivalued Operators in H -Spaces	966
A.1.11	The Single-Valued and Injective Parts of a Multivalued Operator	969
A.1.12	On the Resolvent of a Bounded Operator	970
A.2	Duality in H -Spaces	971
A.2.1	The Dual of an H -Space	971
A.2.2	The Adjoint of a Bounded Linear Operator	973
A.2.3	Duals of Product Spaces	976
A.2.4	The Duals of the Components of a Direct Sum Decomposition	979
A.2.5	The Adjoint of a Linear Operator with Dense Domain	982
A.2.6	The Dual of a Continuous Dense Embedding	983
A.2.7	The Adjoint of a Multivalued Operator	984
A.3	Analytic Vector Bundles and Analytic Operator-Valued Functions	988
A.3.1	The Dual Vector Bundle	992
	<i>References</i>	994
	<i>Index</i>	1005

Cambridge University Press & Assessment
978-1-316-51967-7 — Linear State/Signal Systems
Damir Z. Arov , Olof J. Staffans
Frontmatter
[More Information](#)

Preface

The theory presented in this book arose as a product of a continued collaboration between the two authors during the years 2003–2021. The basis for this collaboration was our common interest in passive linear time-invariant input/state/output systems theory. At the time this project started, O. Staffans was preparing a joint article (Ball and Staffans, 2006) with Prof. J. Ball that, in particular, explored the connections between conservative input/state/output systems theory on the one hand and some results in the behavioral theory introduced by J. Willems in the late 1980s on the other hand. After extensive discussions on this approach, comparing it to the theory of passive electrical networks, we understood that this opens up a new direction in the study of passive linear time-invariant systems. We called the new class of systems that arose in this way passive *state/signal systems*. From the outset, it was clear that the notion of passivity with an arbitrary supply rate fits more naturally into the state/signal setting than in the input/state/output setting, and that the standard “diagonal transformation” of Livšic, the Potapov–Ginzburg transformation, and the Redheffer and chain-scattering transformations have natural interpretations as transformations between input/output resolvents of different input/state/output representations of a passive state/signal system. We also soon discovered that virtually all the standard control theory notions such as controllability and observability, minimality, stability, stabilizability, detectability, and well-posedness have natural state/signal counterparts.

Our first article (Arov and Staffans, 2005) on the state/signal system was completed and submitted for publication in the fall of 2003, and it was followed by many others. Some of the results presented in this book were obtained in collaboration with Ph.D. Mikael Kulula. The bulk of the work was done during D. Arov’s regular visits to Åbo Akademi during August–October 2003–2010 and to Aalto University during August–October 2011–2017, with an average length of almost three months. These visits were financed by the Academy of Finland, the Magnus Ehrnrooth Foundation, and the Finnish Society of Sciences and Letters.

In the fall of 2009, it was decided that the theory was sufficiently mature to be presented in terms of a book, and the writing of this book began on August 30, 2009. By the end of November 2009, a preliminary list of contents was ready. Two significant factors in this decision were the research grant from the Academy of Finland that relieved O. Staffans from teaching duties during the academic year 2009–2010 and the leave of absence for D. Arov for extensive periods of time from the South Ukrainian Pedagogical University based on a joint exchange agreement with Åbo Akademi.

The book we originally planned to write was supposed to be devoted to linear time-invariant systems in discrete time. In 2011, we realized that it would be more important to, instead, write a book on linear time-invariant systems in continuous time, and in 2013 it was clear that it was not feasible to write only one book on systems in continuous time. The continuous time theory contains a number of mathematical difficulties that must first be sorted out, and this is done in the present volume. The application of this theory to passive state/signal systems in continuous time remains to be written down.

We thank the Academy of Finland, the Magnus Ehrnrooth Foundation, and the Finnish Society of Sciences and Letters for their financial support, without which this work could not have been carried out. We also thank Åbo Akademi and Aalto University for excellent working facilities, and the South Ukrainian Pedagogical University for giving D. Arov ample time to devote to research.

Above all, we are grateful to our wives Nataliya and Satu-Marjatta for their constant support, understanding, and patience while this work was carried out.

Notations

Basic Sets and Symbols

\mathbb{C}	The complex plane.
\mathbb{C}_ω^+ , $\overline{\mathbb{C}}_\omega^+$	$\mathbb{C}_\omega^+ := \{z \in \mathbb{C} \mid \Re z > \omega\}$ and $\overline{\mathbb{C}}_\omega^+ := \{z \in \mathbb{C} \mid \Re z \geq \omega\}$.
\mathbb{C}_ω^- , $\overline{\mathbb{C}}_\omega^-$	$\mathbb{C}_\omega^- := \{z \in \mathbb{C} \mid \Re z < \omega\}$ and $\overline{\mathbb{C}}_\omega^- := \{z \in \mathbb{C} \mid \Re z \leq \omega\}$.
\mathbb{C}^+ , $\overline{\mathbb{C}}^+$	$\mathbb{C}^+ := \mathbb{C}_0^+$ and $\overline{\mathbb{C}}^+ := \overline{\mathbb{C}}_0^+$.
\mathbb{C}^- , $\overline{\mathbb{C}}^-$	$\mathbb{C}^- := \mathbb{C}_0^-$ and $\overline{\mathbb{C}}^- := \overline{\mathbb{C}}_0^-$.
\mathbb{D}_r^+ , $\overline{\mathbb{D}}_r^+$	$\mathbb{D}_r^+ := \{z \in \mathbb{C} \mid z > r\}$ and $\overline{\mathbb{D}}_r^+ := \{z \in \mathbb{C} \mid z \geq r\}$.
\mathbb{D}_r^- , $\overline{\mathbb{D}}_r^-$	$\mathbb{D}_r^- := \{z \in \mathbb{C} \mid z < r\}$ and $\overline{\mathbb{D}}_r^- := \{z \in \mathbb{C} \mid z \leq r\}$.
\mathbb{D}^+ , $\overline{\mathbb{D}}^+$	$\mathbb{D}^+ := \mathbb{D}_1^+$ and $\overline{\mathbb{D}}^+ := \overline{\mathbb{D}}_1^+$.
\mathbb{D}^- , $\overline{\mathbb{D}}^-$	$\mathbb{D}^- := \mathbb{D}_1^-$ and $\overline{\mathbb{D}}^- := \overline{\mathbb{D}}_1^-$.
\mathbb{R}	$\mathbb{R} := (-\infty, \infty)$.
\mathbb{R}^+ , $\overline{\mathbb{R}}^+$	$\mathbb{R}^+ := (0, \infty)$ and $\overline{\mathbb{R}}^+ := [0, \infty)$.
\mathbb{R}^- , $\overline{\mathbb{R}}^-$	$\mathbb{R}^- := (-\infty, 0)$ and $\overline{\mathbb{R}}^- := (-\infty, 0]$.
\mathbb{T}	The unit circle in the complex plane.
\mathbb{N}	\mathbb{N} is the set of natural numbers, i.e., $\mathbb{N} := \{1, 2, 3, \dots\}$.
\mathbb{Z}	\mathbb{Z} is the set of all integers, i.e., $\mathbb{Z} := \{\pm 1, \pm 2, \pm 3, \dots\}$.
\mathbb{Z}^+ , \mathbb{Z}^-	$\mathbb{Z}^+ := \{0, 1, 2, \dots\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \dots\}$.
j	$j := \sqrt{-1}$.
0	The number 0, or the zero vector in a vector space, or the zero operator.
1	The number 1 and also the identity operator.
Ω^* , Ω^\dagger	$\Omega^* = \{\bar{\lambda} \mid \lambda \in \Omega\}$ and $\Omega^\dagger = \{-\bar{\lambda} \mid \lambda \in \Omega\}$.

Operators and Related Symbols

A, B, C, D	In connection with an input/state/output system, A is usually the main operator, B is the control operator, C is the observation operator, and D is a feedthrough operator.
$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$	Often \mathfrak{A} is the evolution semigroup, \mathfrak{B} is the input map, \mathfrak{C} is the output map, and \mathfrak{D} is the input/output map of a well-posed linear input/state/output system. See Definition 14.1.14.
$\widehat{\mathfrak{A}}, \widehat{\mathfrak{B}}, \widehat{\mathfrak{C}}, \widehat{\mathfrak{D}}$	Often $\widehat{\mathfrak{A}}$ is the state/state resolvent, $\widehat{\mathfrak{B}}$ is the input/state resolvent, $\widehat{\mathfrak{C}}$ is the state/output resolvent, and $\widehat{\mathfrak{D}}$ is the input/output resolvent of an input/state/output node. See Definition 5.5.8.
$\widehat{\Sigma}(\lambda)$	If Σ is a state/signal node with a characteristic node bundle $\widehat{\mathfrak{C}}[= \mathfrak{C}]$, then $\widehat{\Sigma}(\lambda)$ is the state/signal node with generating subspace $\widehat{\mathfrak{C}}(\lambda)$, and if Σ is an input/state/output node with a (formal) input/state/output resolvent matrix $\widehat{\mathfrak{C}}$, then $\widehat{\Sigma}(\lambda)$ is the input/state/output node with a system operator $\widehat{\mathfrak{C}}(\lambda)$. See Definition 5.5.8 and Lemma 10.3.3.
$\mathcal{B}(\mathcal{U}; \mathcal{Y}), \mathcal{B}(\mathcal{U})$	The set of continuous linear operators from the H -space (or topological vector space) \mathcal{U} into the H -space (or topological vector space) \mathcal{Y} , respectively, from \mathcal{U} into itself. See Notation A.1.15.
$\mathcal{ISO}(\mathcal{U}; \mathcal{Y}), \mathcal{ISO}(\mathcal{U})$	The set of continuously invertible linear operators mapping the H -space (or topological vector space) \mathcal{U} one-to-one onto the H -space (or topological vector space) \mathcal{Y} , respectively, from \mathcal{U} into itself. See Definition 2.1.28.
$\mathcal{L}(\mathcal{U}; \mathcal{Y}), \mathcal{L}(\mathcal{U})$	The set of linear (single-valued) operators from the H -space (or topological vector space) \mathcal{U} into the H -space (or topological vector space) \mathcal{Y} , respectively, from \mathcal{U} into itself. See Definition A.1.13.
$\mathcal{ML}(\mathcal{U}; \mathcal{Y}), \mathcal{ML}(\mathcal{U})$	The set of multivalued linear operators from the H -space (or topological vector space) \mathcal{U} into the H -space (or topological vector space) \mathcal{U} into \mathcal{Y} , respectively, from \mathcal{U} into itself. See Definition A.1.51.
τ^t	The bilateral shift operator on R : $\tau^t u(s) := u(s + t)$, $t, s \in \mathbb{R}$ (this is a left shift when $t > 0$ and a right shift when $t < 0$).
τ^{*t}	$\tau^{*t} = \tau^{-t}$ (this is a right shift when $t > 0$ and a left shift when $t < 0$).

List of Notations

xxv

τ_+^t	The left shift operator on \mathbb{R}^+ : $\tau_+^t u(s) := u(s+t)$, $s \in \mathbb{R}^+$. Here $t \in \mathbb{R}^+$.
τ_+^{*t}	The right shift operator on \mathbb{R}^+ : $\tau_+^{*t} u(s) := 0$, $0 \leq s < t$ and $\tau_+^{*t} u(s) := u(s-t)$, $s \geq t$. Here $t \in \mathbb{R}^+$.
τ_-^t	The left shift operator on \mathbb{R}^- : $\tau_-^t u(s) := 0$, $-t < s \leq 0$ and $\tau_-^t u(s) := u(s+t)$, $s \leq -t$. Here $t \in \mathbb{R}^+$.
τ_-^{*t}	The right shift operator on \mathbb{R}^- : $\tau_-^{*t} u(s) := u(s-t)$, $s \in \mathbb{R}^-$. Here $t \in \mathbb{R}^+$.
ι_I	The embedding operator $L_{\text{loc}}^p(I) \hookrightarrow L_{\text{loc}}^p(\mathbb{R})$: $(\iota_I u)(t) := u(t)$, $t \in I$ and $(\iota_I u)(t) := 0$, $t \notin I$. Here $I \subset \mathbb{R}$.
ι_+ , ι_-	$\iota_+ := \iota_{[0, \infty)}$ and $\iota_- := \iota_{(-\infty, 0]}$.
ρ_I	The restriction operator $L_{\text{loc}}^p(\mathbb{R}) \rightarrow L_{\text{loc}}^p(I)$: $(\rho_I u)(t) := u(t)$, $t \in I$. Here $I \subset \mathbb{R}$. $\rho_I \iota_I = 1_{L_{\text{loc}}^p(I)}$ and $\iota_I \rho_I = \pi_I$.
ρ_+ , ρ_-	$\rho_+ := \rho_{[0, \infty)}$ and $\rho_- := \rho_{(-\infty, 0]}$.
π_I	The projection operator in $L_{\text{loc}}^p(\mathbb{R})$ with range $L_{\text{loc}}^p(I)$ and kernel $L_{\text{loc}}^p(\mathbb{R} \setminus I)$: $(\pi_I u)(s) := u(s)$ if $s \in I$ and $(\pi_I u)(s) := 0$ if $s \notin I$. Here $I \subset \mathbb{R}$. $\rho_I \pi_I = \rho_I$ and $\pi_I \iota_I = \iota_I$.
π_+ , π_-	$\pi_+ := \pi_{[0, \infty)}$ and $\pi_- := \pi_{(-\infty, 0]}$.
\mathbf{Y}	\mathbf{Y} is the time reflection operator in \mathbb{R} , i.e., $(\mathbf{Y} f)(t) = f(-t)$, $t \in \mathbb{R}$. See Definition 2.2.9.
\mathbf{Y}_s^t	\mathbf{Y}_s^t is the time reflection operator in the time interval $[s, t]$, i.e., $(\mathbf{Y}_s^t f)(v) = f(s+t-v)$, $v \in [s, t]$.
$\langle x, x^* \rangle$	The continuous linear functional x^* evaluated at x .
E^\perp	If $E \subset \mathcal{X}$, then $E^\perp = \{x^* \in \mathcal{X}^* \mid \langle x, x^* \rangle = 0 \text{ for all } x \in E\}$, and if $F^* \subset \mathcal{X}^*$, then $(F^*)^\perp = \{x \in \mathcal{X} \mid \langle x, x^* \rangle = 0 \text{ for all } x^* \in F^*\}$.
A^*	The (antilinear) adjoint of the operator A .
A^{-*}	$A^{-*} = (A^*)^{-1} = (A^{-1})^*$.
$A _{\mathcal{X}}$	The restriction of the operator A to the subspace \mathcal{X} .
$A \subset B$	If $A, B \in \mathcal{ML}(\mathcal{X}; \mathcal{Y})$ or $A, B \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$ and $\text{gph}(A) \subset \text{gph}(B)$, then we say that A is a restriction of B and that B is an extension of A , and write $A \subset B$.
$\text{dom}(A)$	The domain of the operator A .
$\text{rng}(A)$	The range of the operator A .
$\text{ker}(A)$	The null space (kernel) of the operator A .
$\text{mul}(A)$	The multivalued part of the operator A .

$\dim(\mathcal{X})$	The dimension of the space \mathcal{X} .
$\rho(A)$	The resolvent set of the operator A (see Definitions 3.4.27 and 10.1.3).
$\rho_\infty(A)$	The unbounded component of the resolvent set of the bounded operator A (see Notation 6.1.2).
$r_\infty(A)$	The spectral radius of the bounded operator A (see Notation 6.1.2).
$\rho_{i/s/o}(S)$	The input/state/output resolvent set of S (see Definition 5.5.8).
$\rho(\Sigma)$	The resolvent set of the input/state/output or state/signal system Σ (see Definitions 5.5.8 and 10.3.1).
$\rho^{\text{bnd}}(\Sigma)$	The union of the resolvent sets of all bounded input/state/output representations of the bounded state/signal system Σ (see Definition 7.1.1).
$\rho_\infty^{\text{bnd}}(\Sigma)$	The unbounded component of $\rho^{\text{bnd}}(\Sigma)$ (see Definition 7.1.1).
$\rho^{\text{sbd}}(\Sigma)$	The union of the resolvent sets of all semi-bounded input/state/output representations of the semi-bounded state/signal system Σ (see Definition 9.1.9).
$\rho_{+\infty}^{\text{sbd}}(\Sigma)$	The component of $\rho^{\text{sbd}}(\Sigma)$ that contains a right half-plane (see Definition 9.1.9).
$\omega(\mathfrak{A})$	The growth bound of the semigroup \mathfrak{A} . See (8.1.1).
TI, TIC	TI stands for the set of all shift invariant operators, and TIC stands for the set of all shift invariant and causal operators. See Definition 14.4.1 for details.

Vector Spaces

H -space	A topological vector space \mathcal{X} that is isomorphic to a Hilbert space, i.e., the topology in \mathcal{X} is induced by a norm induced by a Hilbert space inner product. See Definitions 2.1.2 and A.1.6.
B -space	A topological vector space \mathcal{X} that is isomorphic to a Banach space, i.e., the topology in \mathcal{X} is induced by a Banach space norm. See Definitions 2.1.2 and A.1.6.
\mathcal{U}	Frequently the input space of an input/state/output system.
\mathcal{X}	Frequently the state space of an input/state/output or state/signal system.
\mathcal{Y}	Frequently the output space of an input/state/output system.
\mathcal{W}	Frequently the signal space of a state/signal system.

List of Notations

$\mathcal{X}_\bullet, \mathcal{X}_\circ$	\mathcal{X}_\bullet is the interpolation space and \mathcal{X}_\circ is the extrapolation space induced by a closed operator A in \mathcal{X} with a dense domain. See Definitions 10.1.13 and 10.1.17.
A_\bullet, A_\circ	A_\bullet is the part of A in \mathcal{X}_\bullet and A_\circ is the extension of A to a closed operator in \mathcal{X}_\circ .
$\mathfrak{A}_\bullet, \mathfrak{A}_\circ$	\mathfrak{A}_\bullet is the restriction of the C_0 semigroup \mathfrak{A} in \mathcal{X} to a C_0 semigroup in \mathcal{X}_\bullet and \mathfrak{A}_\circ is the extension of $A\mathfrak{A}$ to a C_0 semigroup in \mathcal{X}_\circ .
$\mathcal{X} = \mathcal{X}_1 \dot{+} \mathcal{X}_2$	$\mathcal{X} = \mathcal{X}_1 \dot{+} \mathcal{X}_2$ means that \mathcal{X} is an H -space that is the direct sum of its two closed subspaces \mathcal{X}_1 and \mathcal{X}_2 , i.e., every $x \in \mathcal{X}$ has a unique representation of the form $x = x_1 + x_2$, where $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$.
$P_{\mathcal{Y}}^{\mathcal{Z}}$	If $\mathcal{X} = \mathcal{Y} \dot{+} \mathcal{Z}$, then $P_{\mathcal{Y}}^{\mathcal{Z}}$ is the projection in \mathcal{X} onto \mathcal{Y} along \mathcal{Z} , i.e., the range of $P_{\mathcal{Y}}^{\mathcal{Z}}$ is \mathcal{Y} and the kernel is \mathcal{U} .
$Q_{\mathcal{Y}}^{\mathcal{Z}}$	If $\mathcal{X} = \mathcal{Y} \dot{+} \mathcal{Z}$, then $Q_{\mathcal{Y}}^{\mathcal{Z}}x = y$, where $y \in \mathcal{Y}$ is the unique vector in \mathcal{Y} in the decomposition $x = y + z$ with $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$. Thus, $Q_{\mathcal{Y}}^{\mathcal{Z}}$ is equal to $P_{\mathcal{Y}}^{\mathcal{Z}}$, reinterpreted as an operator in $\mathcal{B}(\mathcal{X}; \mathcal{Y})$ (instead of an operator in $\mathcal{B}(\mathcal{X})$). See Definition A.1.29.
$\begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix}$	The cross-product of the two H -spaces \mathcal{U} and \mathcal{Y} . Thus, $\begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix} = \begin{bmatrix} \mathcal{U} \\ 0 \end{bmatrix} \dot{+} \begin{bmatrix} 0 \\ \mathcal{Y} \end{bmatrix}$. Also denoted by $\mathcal{U} \times \mathcal{Y}$.
$\mathcal{U} \times \mathcal{Y}$	The cross-product of the two H -spaces \mathcal{U} and \mathcal{Y} . Also denoted by $\begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix}$.

Special Functions

e_ω	$e_\omega(t) = e^{\omega t}$ for $\omega, t \in \mathbb{R}$.
log	The natural logarithm.

Function Spaces

$V(I; \mathcal{Z})$	Functions of type V ($= L^p, C, BC$, etc.) on the interval $I \subset \mathbb{R}$ with range in \mathcal{Z} .
$V_{\text{loc}}(I; \mathcal{Z})$	Functions that are locally of type V , i.e., they are defined on $I \subset \mathbb{R}$ with range in \mathcal{Z} , and they belong to $V(I'; \mathcal{Z})$ for every compact subinterval $I' \subset I$.
$V_\diamond(I; \mathcal{Z})$	Functions in $V(I; \mathcal{Z})$ with compact support.
$V_{\diamond, \text{loc}}(I; \mathcal{Z})$	Functions in $V_{\text{loc}}(I; \mathcal{Z})$ whose support is bounded to the left.
$V_{\text{loc}, \diamond}(I; \mathcal{Z})$	Functions in $V_{\text{loc}}(I; \mathcal{Z})$ whose support is bounded to the right.

xxviii

List of Notations

$V_\omega(I; \mathcal{Z})$	The set of functions u for which $(t \mapsto e^{-\omega t} u(t)) \in V(I; \mathcal{Z})$. See also the special cases listed below.
$V_{\diamond, \omega}(I; \mathcal{Z})$	Functions in $V_\omega(I; \mathcal{Z})$ whose support is bounded to the left.
$V_{\omega, \text{loc}}(I; \mathcal{Z})$	The set of functions $u \in V_{\text{loc}}(I; \mathcal{Z})$ that satisfy $\rho_{I \cap \mathbb{R}^-} u \in V_\omega(I \cap \mathbb{R}^-; \mathcal{Z})$.
$V_\circ(I; \mathcal{Z})$	The closure of $V_{\diamond}(I; \mathcal{Z})$ in $V(I; \mathcal{Z})$. Functions in $V_\circ(I; \mathcal{Z})$ “vanish at infinity.” See also the special cases listed below.
BC	The space of bounded continuous functions with the sup-norm.
BC_\circ	Functions in BC that tend to zero at $\pm\infty$.
BC_ω	Functions u for which $(t \mapsto e^{-\omega t} u(t)) \in BC$.
$BC_{\omega, \text{loc}}$	Continuous functions whose restrictions to \mathbb{R}^- belong to BC_ω .
$BC_{\circ, \omega}$	Functions u for which $(t \mapsto e^{-\omega t} u(t)) \in BC_\circ$.
$BC_{\circ, \omega, \text{loc}}$	Continuous functions whose restrictions to \mathbb{R}^- belong to $BC_{\circ, \omega}$.
BUC	Bounded uniformly continuous functions with the sup-norm.
BUC^n	Functions that together with their n first derivatives belong to BUC .
C	Continuous functions. The same space as BC_{loc} .
C^n	n times continuously differentiable functions. The same space as BC_{loc}^n .
$L^p, 1 \leq p < \infty$	See Notation 2.1.4.
L_{loc}^p	Functions that belong locally to L^p .
L_\diamond^p	Functions in L^p with compact support.
$L_{\diamond, \text{loc}}^p$	Functions in L_{loc}^p whose support is bounded to the left.
L_ω^p	Functions u for which $(t \mapsto e^{-\omega t} u(t)) \in L^p$.
$L_{\omega, \text{loc}}^p(\mathbb{R}; \mathcal{Z})$	Functions $u \in L_{\text{loc}}^p(\mathbb{R}; \mathcal{Z})$ that satisfy $\rho_- u \in L_\omega^p(\mathbb{R}^-; \mathcal{Z})$.
$W^{1,p}$	Functions in L^p that have a (distribution) derivative in L^p . See Notation 2.6.1.
$H^\infty(\Omega; \mathcal{X})$	The space of bounded analytic \mathcal{X} -valued functions on Ω .

Spaces of Sequences

$\ell^p, 1 \leq p < \infty$	Sequences $z = \{z_n\}_{n \in I}$ satisfying $\sum_I z_n ^p < \infty$. See Notation 6.6.3.
ℓ^∞	The vector space of bounded sequences $z = \{z_n\}_{n \in I}$. See Notation 6.6.3.