LINEAR STATE/SIGNAL SYSTEMS

The authors explain in this work a new approach to observing and controlling linear systems whose inputs and outputs are not fixed in advance. They cover a class of linear time-invariant state/signal systems that is general enough to include most of the standard classes of linear time-invariant dynamical systems, but simple enough to make it easy to understand the fundamental principles. They begin by explaining the basic theory of finite-dimensional and bounded systems in a way suitable for graduate courses in systems theory and control. They then proceed to more advanced infinite-dimensional settings, opening up new ways for researchers to study distributed parameter systems, including linear port-Hamiltonian systems and boundary triplets. They include the general nonpassive part of the theory in continuous and discrete time, and provide a short introduction to the passive situation. Numerous examples from circuit theory are used to illustrate the theory.

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Linear State/Signal Systems

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Preface

The theory presented in this book arose as a product of a continued collaboration between the two authors during the years 2003–2021. The basis for this collaboration was our common interest in passive linear time-invariant input/state/output systems theory. At the time this project started, O. Staffans was preparing a joint article (Ball and Staffans, 2006) with Prof. J. Ball that, in particular, explored the connections between conservative input/state/output systems theory on the one hand and some results in the behavioral theory introduced by J. Willems in the late 1980s on the other hand. After extensive discussions on this approach, comparing it to the theory of passive electrical networks, we understood that this opens up a new direction in the study of passive linear time-invariant systems. We called the new class of systems that arose in this way passive *state/signal systems*. From the outset, it was clear that the notion of passivity with an arbitrary supply rate fits more naturally into the state/signal setting than in the input/state/output setting, and that the standard "diagonal transformation" of Livšic, the Potapov–Ginzburg transformation, and the Redheffer and chain-scattering transformations have natural interpretations as transformations between input/output resolvents of different input/state/output representations of a passive state/signal system. We also soon discovered that virtually all the standard control theory notions such as controllability and observability, minimality, stability, stabilizability, detectability, and well-posedness have natural state/signal counterparts.

Our first article (Arov and Staffans, 2005) on the state/signal system was completed and submitted for publication in the fall of 2003, and it was followed by many others. Some of the results presented in this book were obtained in collaboration with Ph.D. Mikael Kulula. The bulk of the work was done during D. Arov's regular visits to Åbo Akademi during August–October 2003–2010 and to Aalto University during August–October 2011–2017, with an average length of almost three months. These visits were financed by the Academy of Finland, the Magnus Ehrnrooth Foundation, and the Finnish Society of Sciences and Letters.

In the fall of 2009, it was decided that the theory was sufficiently mature to be presented in terms of a book, and the writing of this book began on August 30, 2009. By the end of November 2009, a preliminary list of contents was ready. Two significant factors in this decision were the research grant from the Academy of Finland that relieved O. Staffans from teaching duties during the academic year 2009–2010 and the leave of absence for D. Arov for extensive periods of time from the South Ukrainian Pedagogical University based on a joint exchange agreement with Åbo Akademi.

The book we originally planned to write was supposed to be devoted to linear time-invariant systems in discrete time. In 2011, we realized that it would be more important to, instead, write a book on linear time-invariant systems in continuous time, and in 2013 it was clear that it was not feasible to write only one book on systems in continuous time. The continuous time theory contains a number of mathematical difficulties that must first be sorted out, and this is done in the present volume. The application of this theory to passive state/signal systems in continuous time remains to be written down.

We thank the Academy of Finland, the Magnus Ehrnrooth Foundation, and the Finnish Society of Sciences and Letters for their financial support, without which this work could not have been carried out. We also thank Åbo Akademi and Aalto University for excellent working facilities, and the South Ukrainian Pedagogical University for giving D. Arov ample time to devote to research.

Above all, we are grateful to our wives Nataliya and Satu-Marjatta for their constant support, understanding, and patience while this work was carried out.

Notations

Basic Sets and Symbols

$\mathbb{C}$	The complex plane.
$\mathbb{C}_\omega^+$, $\overline{\mathbb{C}}_\omega^+$	$\mathbb{C}_\omega^+ := \{z \in \mathbb{C} \mid \Re z > \omega\}$ and $\overline{\mathbb{C}}_\omega^+ := \{z \in \mathbb{C} \mid \Re z \geq \omega\}$.
$\mathbb{C}_\omega^-$, $\overline{\mathbb{C}}_\omega^-$	$\mathbb{C}_\omega^- := \{z \in \mathbb{C} \mid \Re z < \omega\}$ and $\overline{\mathbb{C}}_\omega^- := \{z \in \mathbb{C} \mid \Re z \leq \omega\}$.
$\mathbb{C}^+$, $\overline{\mathbb{C}}^+$	$\mathbb{C}^+ := \mathbb{C}_0^+$ and $\overline{\mathbb{C}}^+ := \overline{\mathbb{C}}_0^+$.
$\mathbb{C}^-$, $\overline{\mathbb{C}}^-$	$\mathbb{C}^- := \mathbb{C}_0^-$ and $\overline{\mathbb{C}}^- := \overline{\mathbb{C}}_0^-$.
$\mathbb{D}_r^+$, $\overline{\mathbb{D}}_r^+$	$\mathbb{D}_r^+ := \{z \in \mathbb{C} \mid \lvert z\rvert > r\}$ and $\overline{\mathbb{D}}_r^+ := \{z \in \mathbb{C} \mid \lvert z\rvert \geq r\}$.
$\mathbb{D}_r^-$, $\overline{\mathbb{D}}_r^-$	$\mathbb{D}_r^- := \{z \in \mathbb{C} \mid \lvert z\rvert < r\}$ and $\overline{\mathbb{D}}_r^- := \{z \in \mathbb{C} \mid \lvert z\rvert \leq r\}$.
$\mathbb{D}^+$, $\overline{\mathbb{D}}^+$	$\mathbb{D}^+ := \mathbb{D}_1^+$ and $\overline{\mathbb{D}}^+ := \overline{\mathbb{D}}_1^+$.
$\mathbb{D}^-$, $\overline{\mathbb{D}}^-$	$\mathbb{D}^- := \mathbb{D}_1^-$ and $\overline{\mathbb{D}}^- := \overline{\mathbb{D}}_1^-$.
$\mathbb{R}$	$\mathbb{R} := (-\infty, \infty)$.
$\mathbb{R}^+$, $\overline{\mathbb{R}}^+$	$\mathbb{R}^+ := (0, \infty)$ and $\overline{\mathbb{R}}^+ := [0, \infty)$.
$\mathbb{R}^-$, $\overline{\mathbb{R}}^-$	$\mathbb{R}^- := (-\infty, 0)$ and $\overline{\mathbb{R}}^- := (-\infty, 0]$.
$\mathbb{T}$	The unit circle in the complex plane.
$\mathbb{N}$	$\mathbb{N}$ is the set of natural numbers, i.e., $\mathbb{N} := \{1, 2, 3, \ldots\}$.
$\mathbb{Z}$	$\mathbb{Z}$ is the set of all integers, i.e., $\mathbb{Z} := \{\pm 1, \pm 2, \pm 3, \ldots\}$.
$\mathbb{Z}^+$, $\mathbb{Z}^-$	$\mathbb{Z}^+ := \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \ldots\}$.
j	$j := \sqrt{-1}$.
0	The number 0, or the zero vector in a vector space, or the zero operator.
1	The number 1 and also the identity operator.
Ω^*, $\Omega^\dagger$	$\Omega^* = \{\overline{\lambda} \mid \lambda \in \Omega\}$ and $\Omega^\dagger = \{-\overline{\lambda} \mid \lambda \in \Omega\}$.

Operators and Related Symbols

$A,\ B,\ C,\ D$	In connection with an input/state/output system, A is usually the main operator, B is the control operator, C is the observation operator, and D is a feedthrough operator.
$\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{D}$	Often $\mathfrak{A}$ is the evolution semigroup, $\mathfrak{B}$ is the input map, $\mathfrak{C}$ is the output map, and $\mathfrak{D}$ is the input/output map of a well-posed linear input/state/output system. See Definition 14.1.14.
$\widehat{\mathfrak{A}},\ \widehat{\mathfrak{B}},\ \widehat{\mathfrak{C}},\ \widehat{\mathfrak{D}}$	Often $\widehat{\mathfrak{A}}$ is the state/state resolvent, $\widehat{\mathfrak{B}}$ is the input/state resolvent, $\widehat{\mathfrak{C}}$ is the state/output resolvent, and $\widehat{\mathfrak{D}}$ is the input/output resolvent of an input/state/output node. See Definition 5.5.8.
$\widehat{\boldsymbol{\Sigma}}(\lambda)$	If Σ is a state/signal node with a characteristic node bundle $\widehat{\mathfrak{E}}[=\mathfrak{E}]$, then $\widehat{\boldsymbol{\Sigma}}(\lambda)$ is the state/signal node with generating subspace $\widehat{\mathfrak{E}}(\lambda)$, and if Σ is an input/state/output node with a (formal) input/state/output resolvent matrix $\widehat{\mathfrak{S}}$, then $\widehat{\boldsymbol{\Sigma}}(\lambda)$ is the input/state/output node with a system operator $\widehat{\mathfrak{S}}(\lambda)$. See Definition 5.5.8 and Lemma 10.3.3.
$\mathcal{B}(\mathcal{U};\mathcal{Y}),\ \mathcal{B}(\mathcal{U})$	The set of continuous linear operators from the H-space (or topological vector space) $\mathcal{U}$ into the H-space (or topological vector space) $\mathcal{Y}$, respectively, from $\mathcal{U}$ into itself. See Notation A.1.15.
$\mathcal{ISO}(\mathcal{U};\mathcal{Y}),\ \mathcal{ISO}(\mathcal{U})$	The set of continuously invertible linear operators mapping the H-space (or topological vector space) $\mathcal{U}$ one-to-one onto the H-space (or topological vector space) $\mathcal{Y}$, respectively, from $\mathcal{U}$ into itself. See Definition 2.1.28.
$\mathcal{L}(\mathcal{U};\mathcal{Y}),\ \mathcal{L}(\mathcal{U})$	The set of linear (single-valued) operators from the H-space (or topological vector space) $\mathcal{U}$ into the H-space (or topological vector space) $\mathcal{Y}$, respectively, from $\mathcal{U}$ into itself. See Definition A.1.13.
$\mathcal{ML}(\mathcal{U};\mathcal{Y}),\ \mathcal{ML}(\mathcal{U})$	The set of multivalued linear operators from the H-space (or topological vector space) $\mathcal{U}$ into the H-space (or topological vector space) $\mathcal{U}$ into $\mathcal{Y}$, respectively, from $\mathcal{U}$ into itself. See Definition A.1.51.
τ^t	The bilateral shift operator on R: $\tau^t u(s) := u(s+t),\ t,\ s\in\mathbb{R}$ (this is a left shift when $t>0$ and a right shift when $t<0$).
τ^{*t}	$\tau^{*t}=\tau^{-t}$ (this is a right shift when $t>0$ and a left shift when $t<0$).

τ_+^t	The left shift operator on $\mathbb{R}^+$: $\tau_+^t u(s) := u(s+t)$, $s \in \mathbb{R}^+$. Here $t \in \mathbb{R}^+$.
τ_+^{*t}	The right shift operator on $\mathbb{R}^+$: $\tau_+^{*t} u(s) := 0$, $0 \le s < t$ and $\tau_+^{*t} u(s) := u(s-t)$, $s \ge t$. Here $t \in \mathbb{R}^+$.
τ_-^t	The left shift operator on $\mathbb{R}^-$: $\tau_-^t u(s) := 0$, $-t < s \le 0$ and $\tau_-^t u(s) := u(s+t)$, $s \le -t$. Here $t \in \mathbb{R}^+$.
τ_+^{*t}	The right shift operator on $\mathbb{R}^-$: $\tau_+^{*t} u(s) := u(s-t)$, $s \in \mathbb{R}^-$. Here $t \in \mathbb{R}^+$.
ι_I	The embedding operator $L^p_{\mathrm{loc}}(I) \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R})$: $(\iota_I u)(t) := u(t)$, $t \in I$ and $(\iota_I u)(t) := 0$, $t \notin I$. Here $I \subset \mathbb{R}$.
$\iota_+,\ \iota_-$	$\iota_+ := \iota_{[0,\infty)}$ and $\iota_- := \iota_{(-\infty,0]}$.
ρ_I	The restriction operator $L^p_{\mathrm{loc}}(R) \to L^p_{\mathrm{loc}}(I)$: $(\rho_I u)(t) := u(t)$, $t \in I$. Here $I \subset \mathbb{R}$. $\rho_I \iota_I = 1_{L^p_{\mathrm{loc}}(I)}$ and $\iota_I \rho_I = \pi_I$.
$\rho_+,\ \rho_-$	$\rho_+ := \rho_{[0,\infty)}$ and $\rho_- := \rho_{(-\infty,0]}$.
π_I	The projection operator in $L^p_{\mathrm{loc}}(\mathbb{R})$ with range $L^p_{\mathrm{loc}}(I)$ and kernel $L^p_{\mathrm{loc}}(\mathbb{R} \setminus I)$: $(\pi_I u)(s) := u(s)$ if $s \in I$ and $(\pi_I u)(s) := 0$ if $s \notin I$. Here $I \subset \mathbb{R}$. $\rho_I \pi_I = \rho_I$ and $\pi_I \iota_I = \iota_I$.
$\pi_+,\ \pi_-$	$\pi_+ := \pi_{[0,\infty)}$ and $\pi_- := \pi_{(-\infty,0]}$.
Я	**Я** is the time reflection operator in $\mathbb{R}$, i.e., $(\text{Я} f)(t) = f(-t)$, $t \in \mathbb{R}$. See Definition 2.2.9.
Я_s^t	Я_s^t is the time reflection operator in the time interval $[s, t]$, i.e., $(\text{Я}_s^t f)(v) = f(s+t-v)$, $v \in [s, t]$.
$\langle x, x^* \rangle$	The continuous linear functional x^* evaluated at x.
$E^\perp$	If $E \subset \mathcal{X}$, then $E^\perp = \{x^* \in \mathcal{X}^* \mid \langle x, x^* \rangle = 0 \text{ for all } x \in E\}$, and if $F^* \subset \mathcal{X}^*$, then $(\mathcal{F}^*)^\perp = \{x \in \mathcal{X} \mid \langle x, x^* \rangle = 0 \text{ for all } x^* \in F^*\}$.
A^*	The (antilinear) adjoint of the operator A.
A^{-*}	$A^{-*} = (A^*)^{-1} = (A^{-1})^*$.
$A\|_{\mathcal{X}}$	The restriction of the operator A to the subspace $\mathcal{X}$.
$A \subset B$	If $A, B \in \mathcal{ML}(\mathcal{X}; \mathcal{Y})$ or $A, B \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$ and gph $(A) \subset$ gph (B), then we say that A is a restriction of B and that B is an extension of A, and write $A \subset B$.
dom (A)	The domain of the operator A.
rng (A)	The range of the operator A.
ker (A)	The null space (kernel) of the operator A.
mul (A)	The multivalued part of the operator A.

$\dim(\mathcal{X})$	The dimension of the space $\mathcal{X}$.
$\rho(A)$	The resolvent set of the operator A (see Definitions 3.4.27 and 10.1.3).
$\rho_\infty(A)$	The unbounded component of the resolvent set of the bounded operator A (see Notation 6.1.2).
$r_\infty(A)$	The spectral radius of the bounded operator A (see Notation 6.1.2).
$\rho_{\mathrm{i/s/o}}(S)$	The input/state/output resolvent set of S (see Definition 5.5.8).
$\rho(\Sigma)$	The resolvent set of the input/state/output or state/signal system Σ (see Definitions 5.5.8 and 10.3.1).
$\rho^{\mathrm{bnd}}(\Sigma)$	The union of the resolvent sets of all bounded input/state/output representations of the bounded state/signal system Σ (see Definition 7.1.1).
$\rho^{\mathrm{bnd}}_\infty(\Sigma)$	The unbounded component of $\rho^{\mathrm{bnd}}(\Sigma)$ (see Definition 7.1.1).
$\rho^{\mathrm{sbd}}(\Sigma)$	The union of the resolvent sets of all semi-bounded input/state/output representations of the semi-bounded state/signal system Σ (see Definition 9.1.9).
$\rho^{\mathrm{sbd}}_{+\infty}(\Sigma)$	The component of $\rho^{\mathrm{sbd}}(\Sigma)$ that contains a right half-plane (see Definition 9.1.9).
$\omega(\mathfrak{A})$	The growth bound of the semigroup $\mathfrak{A}$. See (8.1.1).
TI, *TIC*	*TI* stands for the set of all shift invariant operators, and *TIC* stands for the set of all shift invariant and causal operators. See Definition 14.4.1 for details.

Vector Spaces

H-space	A topological vector space $\mathcal{X}$ that is isomorphic to a Hilbert space, i.e., the topology in $\mathcal{X}$ is induced by a norm induced by a Hilbert space inner product. See Definitions 2.1.2 and A.1.6.
B-space	A topological vector space $\mathcal{X}$ that is isomorphic to a Banach space, i.e., the topology in $\mathcal{X}$ is induced by a Banach space norm. See Definitions 2.1.2 and A.1.6.
$\mathcal{U}$	Frequently the input space of an input/state/output system.
$\mathcal{X}$	Frequently the state space of an input/state/output or state/signal system.
$\mathcal{Y}$	Frequently the output space of an input/state/output system.
$\mathcal{W}$	Frequently the signal space of a state/signal system.

$\mathcal{X}_\bullet$, $\mathcal{X}_\circ$	$\mathcal{X}_\bullet$ is the interpolation space and $\mathcal{X}_\circ$ is the extrapolation space induced by a closed operator A in $\mathcal{X}$ with a dense domain. See Definitions 10.1.13 and 10.1.17.
$A_\bullet$, $A_\circ$	$A_\bullet$ is the part of A in $\mathcal{X}_\bullet$ and $A_\circ$ is the extension of A to a closed operator in $\mathcal{X}_\circ$.
$\mathfrak{A}_\bullet$, $\mathfrak{A}_\circ$	$\mathfrak{A}_\bullet$ is the restriction of the C_0 semigroup $\mathfrak{A}$ in $\mathcal{X}$ to a C_0 semigroup in $\mathcal{X}_\bullet$ and $\mathfrak{A}_\circ$ is the extension of $A\mathfrak{A}$ to a C_0 semigroup in $\mathcal{X}_\circ$.
$\mathcal{X} = \mathcal{X}_1 \dotplus \mathcal{X}_2$	$\mathcal{X} = \mathcal{X}_1 \dotplus \mathcal{X}_2$ means that $\mathcal{X}$ is an H-space that is the direct sum of its two closed subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$, i.e., every $x \in \mathcal{X}$ has a unique representation of the form $x = x_1 + x_2$, where $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$.
$P_{\mathcal{Y}}^{\mathcal{Z}}$	If $\mathcal{X} = \mathcal{Y} \dotplus \mathcal{Z}$, then $P_{\mathcal{Y}}^{\mathcal{Z}}$ is the projection in $\mathcal{X}$ onto $\mathcal{Y}$ along $\mathcal{Z}$, i.e., the range of $P_{\mathcal{Y}}^{\mathcal{U}}$ is $\mathcal{Y}$ and the kernel is $\mathcal{U}$.
$Q_{\mathcal{Y}}^{\mathcal{Z}}$	If $\mathcal{X} = \mathcal{Y} \dotplus \mathcal{Z}$, then $Q_{\mathcal{Y}}^{\mathcal{Z}} x = y$, where $y \in \mathcal{Y}$ is the unique vector in $\mathcal{Y}$ in the decomposition $x = y + z$ with $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$. Thus, $Q_{\mathcal{Y}}^{\mathcal{Z}}$ is equal to $P_{\mathcal{Y}}^{\mathcal{Z}}$, reinterpreted as an operator in $\mathcal{B}(\mathcal{X}; \mathcal{Y})$ (instead of an operator in $\mathcal{B}(\mathcal{X})$). See Definition A.1.29.
$\left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right]$	The cross-product of the two H-spaces $\mathcal{U}$ and $\mathcal{Y}$. Thus, $\left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right] = \left[\begin{smallmatrix}\mathcal{U}\\ 0\end{smallmatrix}\right] \dotplus \left[\begin{smallmatrix}0\\ \mathcal{Y}\end{smallmatrix}\right]$. Also denoted by $\mathcal{U} \times \mathcal{Y}$.
$\mathcal{U} \times \mathcal{Y}$	The cross-product of the two H-spaces $\mathcal{U}$ and $\mathcal{Y}$. Also denoted by $\left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right]$.

Special Functions

e_ω	$\mathrm{e}_\omega(t) = \mathrm{e}^{\omega t}$ for $\omega, t \in \mathbb{R}$.
log	The natural logarithm.

Function Spaces

$V(I; \mathcal{Z})$	Functions of type V ($= L^p$, C, BC, etc.) on the interval $I \subset \mathbb{R}$ with range in $\mathcal{Z}$.
$V_{\mathrm{loc}}(I; \mathcal{Z})$	Functions that are locally of type V, i.e., they are defined on $I \subset \mathbb{R}$ with range in $\mathcal{Z}$, and they belong to $V(I'; \mathcal{Z})$ for every compact subinterval $I' \subset I$.
$V_\diamond(I; \mathcal{Z})$	Functions in $V(I; \mathcal{Z})$ with compact support.
$V_{\diamond,\mathrm{loc}}(I; \mathcal{Z})$	Functions in $V_{\mathrm{loc}}(I; \mathcal{Z})$ whose support is bounded to the left.
$V_{\mathrm{loc},\diamond}(I; \mathcal{Z})$	Functions in $V_{\mathrm{loc}}(I; \mathcal{Z})$ whose support is bounded to the right.

$V_\omega(I;\mathcal{Z})$	The set of functions u for which $(t \mapsto \mathrm{e}^{-\omega t}u(t)) \in V(I;\mathcal{Z})$. See also the special cases listed below.
$V_{\diamond,\omega}(I;\mathcal{Z})$	Functions in $V_\omega(I;\mathcal{Z})$ whose support is bounded to the left.
$V_{\omega,\mathrm{loc}}(I;\mathcal{Z})$	The set of functions $u \in V_{\mathrm{loc}}(I;\mathcal{Z})$ that satisfy $\rho_{I\cap\mathbb{R}^-}u \in V_\omega(I\cap\mathbb{R}^-;\mathcal{Z})$.
$V_\circ(I;\mathcal{Z})$	The closure of $V_\diamond(I;\mathcal{Z})$ in $V(I;\mathcal{Z})$. Functions in $V_\circ(I;\mathcal{Z})$ "vanish at infinity." See also the special cases listed below.
BC	The space of bounded continuous functions with the sup-norm.
$BC_\circ$	Functions in BC that tend to zero at $\pm\infty$.
BC_ω	Functions u for which $(t \mapsto \mathrm{e}^{-\omega t}u(t)) \in BC$.
$BC_{\omega,\mathrm{loc}}$	Continuous functions whose restrictions to $\mathbb{R}^-$ belong to BC_ω.
$BC_{\circ,\omega}$	Functions u for which $(t \mapsto \mathrm{e}^{-\omega t}u(t)) \in BC_\circ$.
$BC_{\circ,\omega,\mathrm{loc}}$	Continuous functions whose restrictions to $\mathbb{R}^-$ belong to $BC_{\circ,\omega}$.
BUC	Bounded uniformly continuous functions with the sup-norm.
BUC^n	Functions that together with their n first derivatives belong to BUC.
C	Continuous functions. The same space as BC_{loc}.
C^n	n times continuously differentiable functions. The same space as BC^n_{loc}.
$L^p,\ 1 \le p < \infty$	See Notation 2.1.4.
L^p_{loc}	Functions that belong locally to L^p.
$L^p_\diamond$	Functions in L^p with compact support.
$L^p_{\diamond,\mathrm{loc}}$	Functions in L^p_{loc} whose support is bounded to the left.
L^p_ω	Functions u for which $(t \mapsto \mathrm{e}^{-\omega t}u(t)) \in L^p$.
$L^p_{\omega,\mathrm{loc}}(\mathbb{R};\mathcal{Z})$	Functions $u \in L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{Z})$ that satisfy $\rho_- u \in L^p_\omega(\mathbb{R}^-;\mathcal{Z})$.
$W^{1,p}$	Functions in L^p that have a (distribution) derivative in L^p. See Notation 2.6.1.
$H^\infty(\Omega;\mathcal{X})$	The space of bounded analytic $\mathcal{X}$-valued functions on Ω.

Spaces of Sequences

$\ell^p,\ 1 \le p < \infty$	Sequences $z = \{z_n\}_{n\in I}$ satisfying $\sum_I \lvert z_n\rvert^p_{\mathcal{Z}} < \infty$. See Notation 6.6.3.
ℓ^∞	The vector space of bounded sequences $z = \{z_n\}_{n\in I}$. See Notation 6.6.3.