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Introduction and Overview

The class of linear time-invariant state/signal systems studied in this monograph is general enough to include most of the standard classes of linear time-invariant dynamical systems, and at the same time, it is small enough that standard control theory notions for input/state/output (i/s/o) systems have natural extensions to this class. This includes the notions of controllability and observability, minimality, stability, stabilizability and detectability, passivity, and optimal control. Like an i/s/o system, a state/signal system has a state component that can be used to model energy-storing elements and energy sources and sinks, and it also has a signal component that connects the system to the outside world and can be used to observe, control, and interconnect state/signal systems. In this chapter, we first discuss different mathematical approaches to the notion of a linear time-invariant dynamical system and explain the motivation behind our state/signal approach, and then continue with an overview of the contents of this monograph.

1.1 Linear Time-Invariant Dynamical Systems

There are many different mathematical approaches to the theory of dynamical systems. A dynamical system describes the evolution of some quantities as a function of a time variable, which can be discrete (i.e., the time variable takes integer values) or continuous (i.e., the time variable takes real values). In our case, this quantity will be a vector in a vector space, whose value changes with time. This varying value gives rise to a *trajectory* of the system, which is a vector-valued function of a scalar time variable. In the most general setting, the dynamical systems are allowed to be nonlinear and time dependent, but this monograph is devoted to the study of *linear* and *time-invariant* systems. Linearity means that the set of trajectories is invariant both under multiplications by scalars and under additions of trajectories defined on the same time interval, and time invariance means that trajectories that are shifted forward or backward in time remain trajectories of the same system. Most of the time, we take the time variable to be continuous (defined on a subinterval of the real line), but we also include a short discussion on bounded systems with a discrete time variable.

1.1.1 State Systems

In the simplest version of a linear time-invariant system in continuous time, the trajectories consist of a finite set of real or complex state variables that satisfy a finite system of differential equations. (If the time variable is discrete, then this system is replaced by a system of difference equations.) The linearity and time invariance of this system mean that the coefficients in the system of differential equations are independent of both the state and the time variables. We call a system of this type a (linear time-invariant) *finite-dimensional state system*. It can often be rewritten in the vector form

$$\dot{x}(t) = Ax(t), \qquad t \in \mathbb{R}, \tag{1.1.1}$$

where $x(t)$ is an n-dimensional real or complex vector (i.e., $x(t) \in \mathbb{R}^n$ or $x(t) \in \mathbb{C}^n$), $\dot{x}(t)$ is the time derivative of x, and A is an $n \times n$ matrix for a positive integer n. This system is well-posed (or well defined), i.e., it is true that for every initial state $x^0 \in \mathbb{R}^n$ or $x^0 \in \mathbb{C}^n$ and every initial time $t_0 \in \mathbb{R}$, the system has a unique trajectory, defined on the full real line $\mathbb{R} = (-\infty, \infty)$ with the given initial state x^0 at the given initial time t_0. Due to the time invariance of the system, the initial time is irrelevant in the sense that we can always take t_0 to be zero (by a simple time shift). Thus, the past and future evolution of such a system is determined completely by the state $x(0)$ at time zero.

If we replace the system of differential equations (1.1.1) with some other type of equations, such as a system of partial differential equations, or integral equations, or delay equations, or a mixture of such equations, then the dynamics of the system become more complicated. Such a system can often still be described by a linear first-order differential equation (of a very general type) in an infinite-dimensional vector space $\mathcal{X}$ with operator (possibly unbounded or multivalued) coefficients that depend neither on the space nor the time variable. In the sequel, we refer to $\mathcal{X}$ as the *state space* of the system. Depending on the situation, the state space $\mathcal{X}$ may be taken to be a Hilbert space, or a Banach space, or an even more general topological vector space. In this monograph, we concentrate our attention on the case where the state space is a Hilbert space (or strictly speaking, an H-space, as explained in Section 1.1.14). The well-posedness of a system of this type may depend on the direction of time, i.e., a system may be well-posed in the forward time direction without being well-posed in the backward time direction. Under suitable assumptions, a first-order system of this type can often be rewritten as an abstract differential equation

$$\dot{x}(t) = Ax(t), \qquad t \in I, \tag{1.1.2}$$

where I is a subinterval of the real line $\mathbb{R} = (-\infty, \infty)$ and A is a linear operator from its domain $\operatorname{dom}(A) \subset \mathcal{X}$ into $\mathcal{X}$. In some cases, equation (1.1.2) needs to be replaced with the even more general equation

$$\dot{x}(t) \in Ax(t), \qquad t \in I, \tag{1.1.3}$$

where A is a linear multivalued operator, and the inclusion $\dot{x}(t) \in Ax(t)$ is equivalent to the requirement that $\left[\begin{smallmatrix} \dot{x}(t) \\ x(t) \end{smallmatrix}\right] \in \operatorname{gph}(A)$, where $\operatorname{gph}(A)$ is the graph of A.

1.1.2 Systems That Interact with the Outside World

The dynamical systems that we have considered so far are "closed" (as opposed to being "open" in the sense of Livšic (1973)), i.e., they do not include any channels that can be used to interconnect the system with the outside world. Such channels are needed if one wants to monitor the system from the outside, or to guide the system to a desired state, or to interconnect two systems with each other, and they can be created in different ways.

(i) In the *input/state/output (i/s/o)* approach, one adds an input channel and an output channel to a state system of the type described in (1.1.3), through which information can enter and leave the state system. In this approach, each trajectory has three components, all of which are functions of the time variable t, namely a state component $x(t)$, an input component $u(t)$, and an output component $y(t)$.

(ii) In the *input/output (i/o)* approach, each trajectory consists of two components, namely an input component $u(t)$ and an output component $y(t)$. Here, the focus of attention is on how the output y depends on the input u. This can be thought of as a "black box" model of an i/s/o system of the type described in (i), where the underlying state system is not known (or ignored).

(iii) In classical *network theory*, one starts from a finite-dimensional state system and adds a bidirectional (multidimensional) interaction channel that connects this state system to the outside world and permits information to both enter and leave the state system. This channel is not a priori split into an input channel and an output channel. In this approach, each trajectory has two components, namely a state component $x(t)$ and an interaction signal $w(t)$.

(iv) In a *port-Hamiltonian system*, each trajectory consists of a state component and a signal component. The equations for the "internal dynamics" of the state component are energy preserving, and the interaction with the surroundings takes place through the same type of (finite- or infinite-dimensional) energy-preserving port structure as in network theory. Dissipative systems are modeled by terminating one of the ports with a dissipative element.

(v) In the *behavioral approach*, trajectories are functions with values in a "signal space," and the attention is focused on interactions between different parts of the signal without an explicit splitting of the signals into an "input part" and an "output part." This can be thought of as a "black box" model of a generalized version of a network of the type described in (iii), where the underlying state system is not known (or ignored). In this approach, each trajectory has only a signal component and no state component.

These different types of approaches are discussed in more detail in the following sections.

1.1.3 Input/State/Output Systems

In the finite-dimensional setting, it is easy to add inputs and outputs to a state system of the type (1.1.1) by adding input and output terms to (1.1.1) to get an i/s/o system of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \qquad t \in \mathbb{R}, \tag{1.1.4}$$

where $u(t)$ and $y(t)$ are p-dimensional and q-dimensional real or complex vectors, and B, C, and D are matrices of appropriate dimensions. The same approach works well in the infinite-dimensional setting, where $x(t)$, $u(t)$, and $y(t)$ take their values in some Hilbert spaces $\mathcal{X}, \mathcal{U}$, and $\mathcal{Y}$, and the operators A, B, C, and D are bounded linear operators between the appropriate spaces. We call the resulting system a *bounded i/s/o system*. We may even relax the condition for A and only require that it is the generator of a strongly continuous semigroup in $\mathcal{X}$, but keep the assumption that B, C, and D are bounded, in which case we end up with a *semi-bounded i/s/o system*. More general i/s/o systems will be encountered later in this monograph. In some cases, either the input u or the output y is missing, in which case we have a *state/output system* and an *input/state system*. Classical i/s/o systems are discussed, e.g., in Kalman et al. (1969).

1.1.4 Input/Output Systems

In the *i/o setting*, each trajectory has two components, namely an input component u whose values lie in an input space $\mathcal{U}$ and an output component y whose values lie in an output space $\mathcal{Y}$, but there is no explicit state component x. In this setting, one wants to know how the output component y depends on the input component u. In the finite-dimensional case, it is typically assumed that the input u and output y satisfy a finite-order system of differential equations of the type

$$P_{\text{out}}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)y = P_{\text{in}}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)u, \tag{1.1.5}$$

where P_{out} and P_{in} are matrix-valued polynomials with the same row dimension. Under suitable regularity assumptions, it is possible to construct an underlying i/s/o system with the property that to each i/o pair $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ satisfying the relation (1.1.5), there corresponds an i/s/o triple $\left[\begin{smallmatrix} x \\ y \\ u \end{smallmatrix}\right]$ satisfying (1.1.4). Such an i/s/o system is called a realization of the i/o relation (1.1.5). The state x of such an i/s/o representation is not unique, but there exists a realization with minimal state space dimension, and all realizations with the same minimal state space dimension are similar to each other. In addition, there also exist realizations with a nonminimal state dimension.

In the infinite-dimensional version of an i/o system, the differential equation (1.1.5) can be replaced with some other type of linear time-invariant relation (i.e., a linear relation between u and y, which commutes with time shifts). This relation may involve partial differential operators, or integral operators, or time delays, etc. The analysis of this more general type of i/o system is often based on the existing theory of linear operators acting on some space of functions of a time variable that are invariant under right shifts and/or left shifts, and the properties of the shift operator in various function spaces become important. Also, in this case, it is often possible to find some underlying i/s/o system for which the variables u and y play the role of inputs and outputs, respectively, but that system need not be bounded or semi-bounded (and it is, of course, not unique).

1.1.5 Classical (Sub)networks

A classical network (or strictly speaking, subnetwork) resembles a finite-dimensional i/s/o system in the sense that it has a state variable, and it can exchange information with the outside world, but this interchange of information does not take place through dedicated input and output channels. Instead, there is an interface consisting of $m \geq 1$ "ports," where each port supports two scalar signals, so that the total number of interaction signals is even ($= 2m$). In an electrical circuit, each port consists of two terminals, and the two port variables are the current entering the port through its "positive" terminal and the voltage between the positive and the "negative" terminals. The product of these two port variables is proportional to the power absorbed by the system through this particular port, where a positive value means that the power is absorbed by the network, and a negative value means that the power is emitted from the network. By combining port currents and voltages in different ways, one can group the $2m$-dimensional interaction signal into an m-dimensional input and an m-dimensional output. Some choices will lead to well-posed i/s/o systems, meaning that for each time interval $[0, T]$, the (final) state $x(T)$ at time T and the restriction of the output y to the interval $[0, T]$ depend continuously on the (initial) state $x(0)$ at time 0 and the restriction of the input u to the interval $[0, T]$. Other combinations of port currents and voltages into an m-dimensional input and an m-dimensional output may not lead to well-posed i/s/o systems. However, in order to connect two such (sub)networks to each other, there is no need to split the port currents and voltages into dedicated inputs and outputs; instead, one simply requires the connection to satisfy a certain energy-preserving algebraic condition – namely that the voltages over two connected ports are the same and that the sum of the current entering the two connected ports must be zero (i.e., the current entering one of the connected ports must be the same as the current leaving the other). Classical network theory is discussed in, e.g., Belevitch (1968), Fuhrmann and Helmke (2015), Kuh and Rohrer (1967), Seshu and Reed (1961), and Wohlers (1969).

1.1.6 Port-Hamiltonian Systems

Trajectories of a port-Hamiltonian system have both a state component and a signal component through which the system interacts with the outside world. A port-Hamiltonian system consists of several different components that are interconnected through an energy-preserving structure, called a Dirac structure. Two of these components are interpreted as "internal components," namely an energy-preserving dynamic component and a static dissipative component, and the interconnection to the outside world takes place through a third part of the Dirac structure that from the outside looks like the port of a network. In the network interpretation of a finite-dimensional port-Hamiltonian system, the state consists of a collection of capacitors and inductors that can store potential and magnetic energy, respectively, and energy is dissipated in resistors. The Dirac structure describes the interconnections of these elements, the signal "flows" correspond to currents entering the ports, and the "efforts" correspond to voltages over the ports. In an infinite-dimensional setting where the dynamics of a port-Hamiltonian system is described with a partial differential equation in a space domain, the signal part of the system is used to describe the flow of energy through the boundary. A port-Hamiltonian system can be interpreted (in the linear time-invariant case) as a special case of a passive state/signal system (a short introduction to passive state/signal systems is given in Section 15.7). For an introduction and further references to port-Hamiltonian systems, we refer the reader to Cervera et al. (2003, 2007), Le Gorrec et al. (2005), Jacob and Zwart (2012), Kurula et al. (2010), Ortega et al. (2002), van der Schaft (2000, 2006), van der Schaft and Jeltsema (2014), van der Schaft and Maschke (1994, 2002, 2018), and Wu et al. (2018).

1.1.7 Behavioral Systems

A behavioral system resembles an i/o system in the sense that it does not postulate the existence of an underlying state system; on the other hand, it differs from an i/o system in the sense that the trajectories of a behavioral system are not formally decomposed into an input component and an output component. This resembles the behavior of the port variables of a classical network, but there is no "port structure" imposed on the trajectories, i.e., the dimension of the signal space $\mathcal{W}$ in which the values of the trajectories may be even or odd, and there is no "power" associated with the trajectories. The easiest way to arrive at the notion of a finite-dimensional behavioral system is to combine the p-dimensional input u and the q-dimensional output y of a finite-dimensional i/o system into a $(p+q)$-dimensional signal w, and to require this signal to satisfy a simplified version of (1.1.5), namely

$$P\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)w = 0, \tag{1.1.6}$$

where P is a matrix-valued polynomial. Every relation of the form (1.1.5) can be put into the form (1.1.6) by defining the interaction signal w to be the i/o pair $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ and taking $P = \begin{bmatrix} P_{\mathrm{in}} & -P_{\mathrm{out}} \end{bmatrix}$. There also exist methods to go from (1.1.6) to (1.1.5), but since the splitting of the interaction signal w into an input u and an output y is not unique, to each signal relation of the type (1.1.6) there correspond infinitely many i/o relations of the type (1.1.5). It is further possible to develop i/s/o representations of the type (1.1.4) for a behavioral system by first splitting the signal w into an input and an output, and then applying known methods for getting an i/s/o representation of the i/o relation (1.1.5). Of course, there is now an additional free parameter in this construction: in addition to the nonuniqueness of the state space $\mathcal{X}$ of the system, the splitting of the signal space $\mathcal{W}$ into an input space $\mathcal{U}$ and an output space $\mathcal{Y}$ is highly nonunique. For an introduction to behavioral systems and further references, we refer the reader to Polderman and Willems (1998), Weiland and Willems (1991), Willems (1991, 2007), Willems and Yamamoto (2007), and Willems and Trentelman (1998, 2002).

The connection between behavioral systems and the class of state/signal systems will be discussed in Sections 1.1.8 and 1.1.12.

1.1.8 State/Signal Systems

The class of linear time-invariant dynamical systems that we introduce in this monograph under the name "state/signal systems" can be interpreted as a generalization of the notion of a classical network. As in the case of a classical network, each trajectory has two components: a state component $x(t)$ with values in a state space $\mathcal{X}$ and a signal component $w(t)$ with values in a signal space $\mathcal{W}$. These spaces are allowed to be (finite-dimensional or) infinite-dimensional Hilbert spaces (or more precisely, H-spaces, as will be explained in Section 1.1.14), and as in behavioral theory, there is no extra "port" structure imposed on the signal space $\mathcal{W}$. The main difference between the classes of i/s/o systems and state/signal (s/s) systems is that in a s/s system the interaction signal is not a priori split into an input and an output. We mentioned earlier that a behavioral system can be interpreted as a "black box" model of a generalized version of a network. A more precise statement would be that a behavioral system can be interpreted as a "black box" model of a s/s system.

The formal definition of a s/s system is very simple, and the same definition can be used in the finite- and infinite-dimensional settings. It does not involve any unbounded operators. To arrive at this definition, we take a closer look at equation (1.1.4), describing the evolution of the trajectories of a linear time-invariant finite-dimensional i/s/o system. This equation can be interpreted as a linear relation between the four variables $x(t)$, $\dot{x}(t)$, $u(t)$, and $y(t)$, where $x(t)$ and $\dot{x}(t)$ belong to the state space $\mathcal{X}$, $u(t)$ belongs to the input space $\mathcal{U}$, and $y(t)$ belongs to the output space $\mathcal{Y}$. If we remove the distinction between the input and the output, and consider both the input $u(t)$ and the output $y(t)$ to be parts of the interaction signal $w(t)$, then we end up with a linear relation between $x(t) \in \mathcal{X}$, $\dot{x}(t) \in \mathcal{X}$, and the signal $w(t) \in \mathcal{W}$, where $\mathcal{W}$ is the signal space. Every such linear relation can be written in the form

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in I, \tag{1.1.7}$$

where V is a subspace of $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$. We call V the *generating subspace* of the system. By a classical trajectory of (1.1.7), we mean a pair of functions $\begin{bmatrix} x \\ w \end{bmatrix}$, where x is continuously differentiable on I, w is continuous on I, and (1.1.7) holds.

It is, of course, possible to rewrite (1.1.7) into several other equivalent forms. If V is closed, then we can think about V as the kernel of a surjective bounded linear operator $\begin{bmatrix} -E & M & N \end{bmatrix}$ from $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ into an auxiliary space $\mathcal{Y}$ and rewrite (1.1.7) in the form

$$E\dot{x}(t) = Mx(t) + Nw(t), \qquad t \in I. \tag{1.1.8}$$

This representation is unique up to the multiplication with a bounded linear operator with bounded inverse from the left. We call this a *kernel representation* of (1.1.7). Another possibility is to interpret V as the range of an injective bounded linear operator $\left[\begin{smallmatrix} K \\ F \\ L \end{smallmatrix}\right]$ from an auxiliary space $\mathcal{U}$ into $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$, which can be used to rewrite (1.1.7) in the form

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} F v(t) &= K v(t), \\ x(t) &= F v(t), \qquad t \in I. \\ w(t) &= L v(t), \end{aligned} \tag{1.1.9}$$

This representation is unique up to the multiplication with a bounded linear operator with a bounded inverse from the right. We call this an *image representation* of (1.1.7). If the generating subspace V has the property that the first component $\dot{x}(t)$ in (1.1.7) is determined uniquely by the other two components $\left[\begin{smallmatrix} x(t) \\ w(t) \end{smallmatrix}\right]$, and if we let G be the linear operator mapping $\left[\begin{smallmatrix} x(t) \\ w(t) \end{smallmatrix}\right]$ into $\dot{x}(t)$ whose graph is equal to V, then (1.1.7) can alternatively be written in the form

$$\dot{x}(t) = G\left(\left[\begin{smallmatrix} x(t) \\ w(t) \end{smallmatrix}\right]\right), \qquad t \in I. \tag{1.1.10}$$

The domain of the operator G need not be the full space $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ or even dense in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$, which means that the implicit condition $\left[\begin{smallmatrix} x(t) \\ w(t) \end{smallmatrix}\right] \in \operatorname{dom}(G)$ hidden in (1.1.10) creates a linear dependence between $x(t)$ and $w(t)$.

In this monograph, we primarily use the representation (1.1.7), but certain results are easier to prove using the representation (1.1.8), (1.1.9), or (1.1.10).

1.1.9 State/Signal versus Input/State/Output Systems

Above we described how to convert the i/s/o system (1.1.4) into a s/s system (1.1.7) by combining the input $u(t)$ and the output $y(t)$ into an interaction signal $w(t) = \left[\begin{smallmatrix} y(t) \\ u(t) \end{smallmatrix}\right]$. This process can be reversed by splitting the signal space $\mathcal{W}$ of the s/s system (1.1.7) into $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$, and splitting the signal $w(t)$ accordingly into $w(t) = u(t) + y(t)$ where $u(t) \in \mathcal{U}$ and $y(t) \in \mathcal{Y}$. By doing so we can rewrite (1.1.7) in the form (where we have reordered the components, so that $y(t)$ comes before $x(t)$)

$$\left[\begin{smallmatrix} \dot{x}(t) \\ y(t) \\ x(t) \\ u(t) \end{smallmatrix}\right] \in V_{\mathrm{i/s/o}}, \qquad t \in I, \tag{1.1.11}$$

where $V_{\mathrm{i/s/o}}$ is the subset of $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ that we obtain from the generating subspace V in (1.1.7). If $V_{\mathrm{i/s/o}}$ has the property that the pair $\left[\begin{smallmatrix} \dot{x}(t) \\ y(t) \end{smallmatrix}\right]$ in (1.1.11) is defined uniquely by $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$, then we can rewrite (1.1.11) into a more familiar i/s/o form

$$\left[\begin{smallmatrix} \dot{x}(t) \\ y(t) \end{smallmatrix}\right] = S \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right], \qquad t \in I, \tag{1.1.12}$$

where the *system operator* S is the linear operator from $\left[\begin{smallmatrix}\mathcal{X}\\\mathcal{U}\end{smallmatrix}\right]$ into $\left[\begin{smallmatrix}\mathcal{X}\\\mathcal{Y}\end{smallmatrix}\right]$ whose graph is equal to $V_{\mathrm{i/s/o}}$. The advantage of the representation (1.1.12) compared with the representation (1.1.10) is that with a suitable choice of the decomposition $\mathcal{W}=\mathcal{U}\dotplus\mathcal{Y}$, it is usually possible to guarantee that the domain of S is dense in $\left[\begin{smallmatrix}\mathcal{X}\\\mathcal{U}\end{smallmatrix}\right]$. In the finite-dimensional case, after working out all the details, one ends up with an equation of the type (1.1.4) (with $\mathbb{R}$ being replaced by I). The i/s/o system (1.1.11) is called an *i/s/o representation* of the state/signal system (1.1.7). Later, we shall sometimes drop the condition that $\left[\begin{smallmatrix}\dot{x}(t)\\y(t)\end{smallmatrix}\right]$ in (1.1.11) is defined uniquely by $\left[\begin{smallmatrix}x(t)\\u(t)\end{smallmatrix}\right]$ and permit the operator S in (1.1.12) to be multivalued, in which case (1.1.12) should be replaced with the relation

$$\begin{bmatrix}\dot{x}(t)\\y(t)\end{bmatrix}\in S\begin{bmatrix}x(t)\\u(t)\end{bmatrix},\qquad t\in I. \tag{1.1.13}$$

The close relationship between i/s/o and state/signal systems expressed by (1.1.7), (1.1.11), and (1.1.13) makes it possible to transfer many standard system theoretic notions for i/s/o systems to the class of state/signal systems, provided we make a small change (with drastic consequences) in the standard definition of what one means by a classical trajectory of an i/s/o system. In the standard finite-dimensional setting (1.1.4), if we assume that x and u are continuous functions on an interval I, then it follows from (1.1.4) that x is continuously differentiable on I and that y is continuous on I. If we replace (1.1.4) with (1.1.12) or (1.1.13), then the continuity of y and continuous differentiability of x can no longer be taken for granted. Instead, we *therefore impose an a priori continuous differentiability assumption on x and an a priori continuity assumption on y in* (1.1.12) *or* (1.1.13), in addition to the assumption that u is continuous on I. (In the finite-dimensional well-posed case, this extra condition is redundant.) With this added smoothness condition, there is a one-to-one correspondence between classical trajectories of (1.1.7) and those of (1.1.12) or (1.1.13), as soon as the decomposition $\mathcal{W}=\mathcal{U}\dotplus\mathcal{Y}$ is fixed. This makes it possible to transfer all the standard "dynamic" notions for i/s/o systems that can be defined in terms of the behavior of trajectories into corresponding notions for state/signal systems. This includes notions related to stability, stabilizability and detectability, controllability and observability, minimality, compressions and dilation, and various transformations and interconnections. Depending on the particular i/s/o notion, the corresponding state/signal notion falls into one of the following two categories:

(i) In some cases, if one of the i/s/o representations of a s/s system Σ has a particular i/s/o property, then *every i/s/o representation of* Σ has the same property, in which case we say that the corresponding state/signal system has the analogous s/s property. For example, the property that the system operator S in (1.1.12) is closed is of this type, i.e., if V has at least one representation as the graph of a closed system operator, then V is closed, and every other system operator in a graph representation of V is also closed. Thanks to our slightly nonstandard definition of the notion of a trajectory, also the notions of controllability, observability, minimality, stabilizability, and detectability are of the same type. There even exist some weak existence and uniqueness properties (where the input and the output are treated in a symmetrical way), which belong to the same class. We call this class of i/s/o properties *i/o invariant*.

(ii) If it is instead true that the s/s system has a particular s/s property as soon as *at least one of its i/s/o representations* has the analogous i/s/o property (but the s/s system may also have i/s/o representations that do not have this property), then we say that this i/s/o property is *i/o dependent*.

Thus, in order to show that a particular i/s/o representation has a property of the type (i), it suffices to show that some other i/s/o representation has the same property, and in order to show that a state/signal system has a property of type (ii), it suffices to show that it has at least one i/s/o representation that has the corresponding i/s/o property. For example, the i/s/o notions of boundedness, semi-boundedness, well-posedness, and stability are i/o dependent, i.e., every bounded, or semi-bounded, or well-posed, or stable s/s system has at least one bounded, or semi-bounded, or well-posed, or stable i/s/o representation, but it may also have i/s/o representations that do not have these properties. This will be explained in more detail in Section 1.2.

1.1.10 Frequency Domain Systems

In the existing literature, nonlinear and time-dependent systems are primarily discussed in the time domain (as we have done above). Linear time-invariant i/o and i/s/o systems also have a rich frequency domain theory that complements the time domain theory. In this monograph, we develop an analogous frequency domain theory for state/signal systems, and in addition, we expand the frequency domain theory for i/s/o systems by introducing the notion of a frequency domain trajectory of an i/s/o system. A time domain trajectory is a vector-valued function of a time variable, whereas a frequency domain trajectory is an analytic vector-valued function of a frequency variable. In the i/s/o setting, both the time domain and frequency domain trajectories have an initial state, a "final" state, an input, and an output, and in our state/signal setting, both the time domain and frequency domain trajectories have an initial state, a "final" state, and an interaction signal. Under additional regularity assumptions, frequency domain trajectories can be interpreted as Laplace transforms of time domain trajectories in the case where the time variable is continuous, or as Z-transforms of time domain trajectories when the time variable is discrete. A time domain trajectory is defined in some time interval (finite or infinite), whereas a frequency domain trajectory is defined in an open subset of the complex plane. The choice of which particular frequency domain to use depends on the situation at hand. For example, for a passive discrete time system, the natural frequency domain is either the outside or the inside of the unit disk, depending on whether we are looking for the evolution in the forward or backward time direction, and for a passive continuous time system, the natural frequency domain is either the right or the left half-plane, again depending on the direction of time.

Basically, all the standard frequency domain i/s/o notions have state/signal counterparts, although these counterparts often appear in a different form. For example, the standard i/o "transfer function" or "characteristic function," which is an analytic operator-valued function, is replaced by an analytic vector bundle that we call the *characteristic signal*