1 Introduction and Notation

A particle (from the Latin *particula*, little part) is a minute portion of matter.

1.1 Subatomic Particles

When we first observe the Universe, it might appear to us as a very complex object. One of the primary goals of the philosophy of Nature (or simply Physics) is to "reduce" ("simplify") this picture in order to find out what the most fundamental constituents of matter (i.e., the atoms from the Greek word *indivisible*) are and to understand the basic forces by which they interact in the otherwise *void* space, along the line of thinking of **Demokritos**[1] who wrote *"Nothing exists except atoms and empty space."*

In this context, **subatomic particles** are physical objects smaller than atoms. In particle physics, particles are objects that are localized in space and that are characterized by intrinsic properties. As we will see later, the set of intrinsic properties used to classify particles is chosen from those which behave in a well-defined way under the action of a transformation. As a matter of fact, we would expect some of these properties not to change at all under particular transformation. For instance, we expect some of its properties to be independent of the velocity of the particle or the direction in which it is traveling. This procedure of classification was actually initiated by **Eugene Wigner**[2] in his seminal paper of 1939 [1]. We will come back to this later.

Elementary particles are particles which, according to current theories, are not made of other particles, or we should rather say, *whose substructure, if any, is unknown*. They are thus considered as **point-like objects**. **Composite particles**, on the other hand, are composed of other particles, in general of elementary particles, and are thus extended in space. All subatomic particles are classified according to their properties and given *common names* such as electrons, muons, taus, protons, neutrons, neutrinos, etc. Particles are indistinguishable in the sense that, for example, one electron is identical to another electron.

Elementary particles can be **massless** or **massive**. Massless particles have a zero **rest mass**, while massive particles have a finite rest mass. Following **Albert Einstein**,[3] energy is equivalent to (inertial) mass and vice versa [2,3]. Consequently, composite particles (i.e., bound states of sub-elementary objects) cannot be massless, as their rest mass should at least account for the binding energy among their constituents.

Particles can be **electrically neutral** or **charged**. An electrically neutral particle has no charge, while charged particles possess an electric charge which is a fraction or multiple of the **elementary unit charge** e. In 2018, the kilogram, Ampère, Kelvin, and mole were redefined in terms of new permanently fixed values of the elementary charge, Planck, Boltzmann, and Avogadro constants (see www.bipm.org). Accordingly, the **exact** value of the elementary charge is given by [4]:

$$e = 1.602\,176\,634 \times 10^{-19}\ \mathrm{C} \quad \text{(exact)} \tag{1.1}$$

1 Demokritos (c.460–c.370 BC), Ancient Greek philosopher.
2 Eugene Paul "E. P." Wigner (1902–1995), Hungarian–American theoretical physicist, engineer, and mathematician.
3 Albert Einstein (1879–1955), German-born theoretical physicist.

Other than rest mass and charge properties, particles can possess additional **internal degrees of freedom**, such as the **spin**. Such degrees of freedom are characterized by specific integer or half-integer **quantum numbers**. Other internal characteristics such as isospin, strangeness, etc. will be defined as well. Some of these quantities, such as the electric charge, are always conserved, while others, such as strangeness, are violated under some circumstances, as will be discussed later on.

The properties of all of the known particles (e.g., mass, charge, and spin) and the experimental results that measured them are collected in the **Particle Data Group's (PDG) Review of Particle Physics**. The review is published every two years. It can be accessed online at http://pdg.lbl.gov or one can order the book directly from the website. Here, we will most frequently refer to the Particle Data Group's Review published in 2020 [5].

As far as we know, the Universe is composed of 12 elementary (fundamental) spin-1/2 fermions, divided into six **leptons** and six **quarks**. These are listed in Table 1.1. Apart from the differences in their rest masses, the particles of each generation possess the same fundamental interactions. It is natural to ask the origin of this pattern and if there are further generations. This seems not to be the case and consequently the matter content of the Universe of this type can be reduced to these 12 elements. Astronomical observations point

Family	**Leptons**		**Quarks**	
	Name	Q	Name	Q
First generation	electron (e)	-1	down	$-1/3$
	electron neutrino (ν_e)	0	up	$+2/3$
Second generation	muon (μ)	-1	strange	$-1/3$
	muon neutrino (ν_μ)	0	charm	$+2/3$
Third generation	tau (τ)	-1	bottom	$-1/3$
	tau neutrino (ν_τ)	0	top	$+2/3$

Table 1.1 A summary of the known fundamental spin-1/2 fermions in Nature.

to the existence of an additional type of matter (called **dark matter**) (see Section 32.5), but the elementary particle content of this latter is unknown as of today.

It is difficult to completely objectively define an elementary particle. We can imagine them as point-like objects with particular characteristics, however, such a picture will never be completely satisfactory. For instance, **Bruno Pontecorvo**[4] stated that a good model to represent some of the neutrino properties (and other spin-1/2 fermions) is with a screw. The helix of the thread of a screw can twist in two possible directions, which is known as **handedness**. See Figure 1.1. Similar considerations apply to the neutrino handedness, and in general to the chiral components of Dirac particles, as will be discussed in Section 8.24! Such macroscopic illustrations can help us visualize the properties of elementary particles.

Identical particles obey certain statistics and are classified accordingly: **fermions** obey Fermi–Dirac statistics while **bosons** obey Bose–Einstein statistics. The **spin-statistics theorem** was first formulated by **Markus Fierz**[5] in 1939 in his Habilitation paper where he noted that, from the very general requirements that the exchange relations of the field quantities should be relativistically invariant and infinitesimal, and that the energy should be positive, it follows that particles with integer spin must always follow Bose statistics, and particles with half-integer spin must follow Fermi statistics.[6] The result was then demonstrated in a more systematic way by **Wolfgang Pauli**[7] in a famous 1940 paper [6]. In other words, in a relativistically invariant wave equation, the requirements of causality and that the energy is positive definite, imply that the intrinsic

4 Bruno Pontecorvo (1913–1993), Italian nuclear physicist, early assistant of Enrico Fermi. A convinced communist, he defected to the Soviet Union in 1950.

5 Markus Fierz (1912–2006), Swiss physicist.

6 "*Aus den sehr allgemeinen Forderungen, dass die Vertauschungsrelationen der Feldgrössen relativistisch invariant und infinitesimal sein sollen, und dass die Energie positiv sein soll, folgt nämlich, dass Teilchen mit ganzzahligem Spin stets Bose-Statistik, Teilchen mit halbzahligem Spin Fermi-Statistik haben müssen.*" Reprinted from M. Fierz, "Über die relativistische Theorie kräftefreier Teilchen mit beliebigem Spin," *Helv. Phys. Acta*, vol. 12, pp. 3–37, 1939.

7 Wolfgang Ernst Pauli (1900–1958), Austrian-born Swiss and American theoretical physicist.

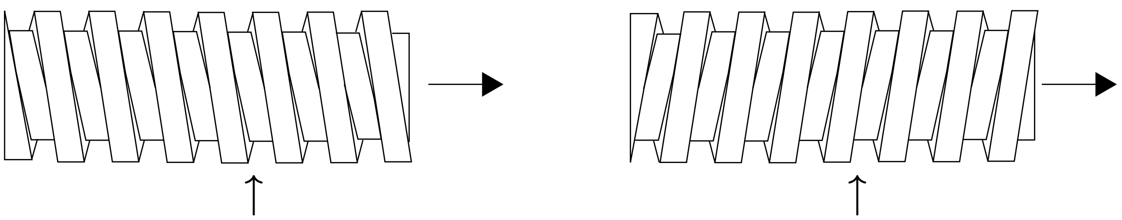

Figure 1.1 Right- and left-handed screw threads. Adapted from Eugene Wycliffe Kerr (1908) *Power and Power Transmission*, 2nd ed., John Wiley & Sons, New York, p. 79, fig. 61 on Google Books, Public Domain, https://commons.wikimedia.org/w/index.php?curid=31218881.

spin of a particle is related to the statistics, and that they obey: *spinless or integer spin particles are bosons while half-integer spin particles are fermions.*

As first postulated by **Louis de Broglie**[8] [7], all particles possess a **dual particle–wave nature** (see Section 4.1). Hence, the rules of **quantum mechanics** apply in the description of subatomic particles and their physical behavior. Actually this has been verified not only for subatomic particles but also for composite objects like atoms and even molecules, and in principle also applies for macroscopic objects. One of the challenges that will be addressed in this book is the need to develop a relativistic formulation of quantum mechanics beyond the "classical" non-relativistic Schrödinger theory.

1.2 Action at Distance

Particles exhibit **interactions at a distance**, described by forces whose strengths are characterized by **coupling constants**. The four **fundamental interactions** – gravitation (related to mass/energy), electromagnetic (related to electric charge), weak (related to the weak isospin and hypercharge), and strong (related to the color charge) – also known as fundamental forces – are the ones that do not appear to be reducible to more basic interactions. As we will see later, interactions among particles are actually *local*. Therefore, forces acting at a distance are thought to be transmitted via **intermediate gauge bosons**. Each force is characterized by a particular set of gauge bosons, with specific properties. The forces, the associated intermediate gauge bosons, and some of their properties are summarized in Table 1.2.

Interaction	**Gauge boson**			**Fermion coupling**
	Type	Spin	Mass	
Strong	8 gluons	1	0	q, $\bar{q}$
Electromagnetic (EM)	1 photon	1	0	ℓ^-, ℓ^+, q, $\bar{q}$
Weak	$W^\pm$ boson	1	80 GeV	ℓ^-, ℓ^+, ν, $\bar{\nu}$, q, $\bar{q}$
	Z^0 boson	1	91 GeV	ℓ^-, ℓ^+, ν, $\bar{\nu}$, q, $\bar{q}$
Gravity	1 graviton (?)	2	0	all

Table 1.2 A summary of the fundamental forces in Nature; ℓ stands for the charged leptons e, μ, or τ.

The strength of a force actually depends on the **energy scale**, which represents the characteristic energy at which the particular interaction is taking place. This fact is illustrated in Figure 1.2, where energy is expressed

8 Louis Victor Pierre Raymond de Broglie (1892–1987), French physicist.

in units of **electronvolt**, with 1 GeV = 1 Giga-electronvolt = 10^9 eV and therefore the (exact) value of the unit of energy is given by (see [4]):

$$1 \text{ GeV} = 10^9\ e \text{ J} = 1.602\,176\,634 \times 10^{-10} \text{ J} \quad \text{(exact)} \tag{1.2}$$

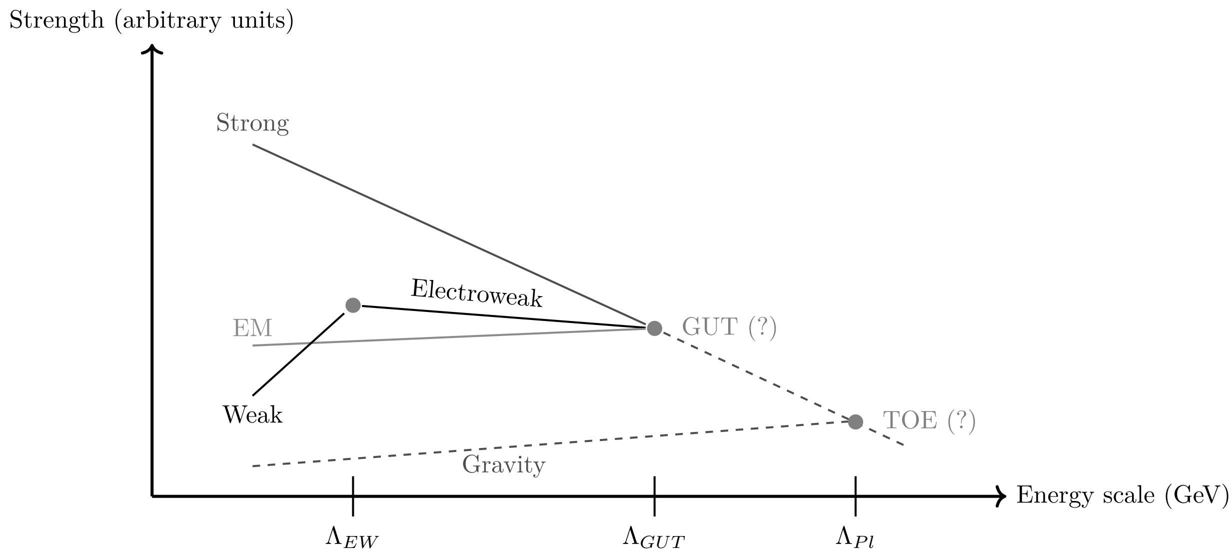

Figure 1.2 Illustration of the strength of the fundamental forces as a function of the energy scale. At the highest energies all forces should unify.

The four forces are believed to be fundamentally related. Special relativity was derived from the attempt to unify Newtonian mechanics with Maxwell's electromagnetism. Later, **Einstein attempted to unify his general theory of relativity with electromagnetism**. The weak and electromagnetic forces have been unified within the **electroweak theory** to be discussed in Chapter 25. These two forces actually behave with a similar strength at the **electroweak scale**, that is determined by the vacuum expectation value (its exact meaning will be discussed in Section 24.5):

$$\Lambda_{EW} \simeq v = \left(\sqrt{2}G_F\right)^{-1/2} \simeq 246 \text{ GeV} \tag{1.3}$$

where G_F is the Fermi coupling constant (see Chapter 21). The hypothesized **Grand Unified Theories (GUT)** unite the electroweak and strong forces at the much higher GUT scale (see Chapter 31), which extrapolating from current experimental data should be around

$$\Lambda_{GUT} \simeq 10^{15\pm1} \text{ GeV} \tag{1.4}$$

(this value will be discussed in Section 31.5). All four forces are believed to unite into a single force at the **Planck[9] scale**, characterized by the **Planck mass** M_{Pl}, which corresponds to the simplest combination of the fundamental constants $\hbar$, and c (see Section 1.8), and the Newtonian universal gravitational constant G:

$$\Lambda_{Pl} \simeq M_{Pl} = \sqrt{\frac{\hbar c}{G}} \simeq 2.2 \times 10^{-8} \text{ kg} = 1.2 \times 10^{19} \text{ GeV/c}^2 \tag{1.5}$$

9 Max Karl Ernst Ludwig Planck (1858–1947), German theoretical physicist.

One can also characterize the Planck scale considering the Planck length $l_{Pl} = \sqrt{\hbar G/c^3} \simeq 1.6 \times 10^{-35}$ m or the Planck time $t_{Pl} = l_{Pl}/c \simeq \sqrt{\hbar G/c^5} = 5.4 \times 10^{-44}$ s, which are indeed small numbers compared to our everyday scales! Imagining what happens in those regimes is far from trivial. At present, the development of a unified field theory able to describe all forces in a unified way and up to the Planck scale, the so-called **Theory of Everything** (TOE), remains an open research topic.

1.3 Symmetries and Conservation Laws

Symmetry in physics has been generalized to mean **invariance** – that is, lack of change – under any kind of transformation. Symmetries play an *extremely important role* in particle physics. To illustrate this fact, we for instance quote **Dirac**[10] who wrote on Einstein's theory:

> *"There is one strong reason in support of the theory. It appears as one of the fundamental principles of Nature that the equations expressing basic laws should be invariant under the widest possible group of transformations. The confidence that one feels in Einstein's theory arises because its equations are invariant under a very wide group, the group of transformations of curvilinear coordinates in Riemannian space. Of course even if space were flat, the equations of physics could still be expressed in terms of curvilinear coordinates and would still be invariant under transformations of the coordinates, but there would then exist preferred systems of coordinates, the rectilinear ones, and it would be only the group of transformations of the preferred coordinates that would be physically significant. The wider group of transformations of curvilinear coordinates would then be just a mathematical extension, of no importance for the discussion of physical laws."*[11]

The importance of symmetries is related to the existence of the fundamental **theorem of Noether**[12] (first derived in 1915 and published in 1918 [8], valid for any Lagrangian theory, classical or quantum) which relates such symmetries to *conserved quantities* of the physical system. It states:

> *For every continuous symmetry, there exists a conservation law. For every conservation law, there exists a continuous symmetry.*

Conserved quantities are at the heart of *conservation laws*, which are so fundamental and crucial to describe the behavior of physical systems. Conservation laws highly restrict the possible outcomes of an experiment. In a quantum system, quantum numbers describe quantized properties of the system and are usually subject to constraints. Conserved quantities are the basis to define such good quantum numbers. In particle physics there are many examples of symmetries and their associated conservation laws. Identifying the symmetries of a physical system allows us to express its governing laws in a very elegant and fundamental way. Last but not least, there are also cases where a symmetry is *broken*, and the mechanism has to be understood. The reason why and the mechanism by which a symmetry is broken (or partially broken) can give us insight on the underlying physics. In the case of spontaneous symmetry breaking (see Chapter 24), the underlying theory possesses a fundamental symmetry, however, the dynamical choice of vacuum breaks this symmetry. In all cases, symmetries, broken or not, are fundamental.

Noether's theorem is valid for **continuous transformations**. Specifically, translations in time or in space are examples of such continuous transformations. As a classical case, let us consider two mechanical masses m_1 and

10 Paul Adrien Maurice Dirac (1902–1984), English theoretical physicist.
11 Republished with permission of The Royal Society (UK) from P. A. M. Dirac, "Long range forces and broken symmetries," *Proc. R. Soc. Lond. A* 333: 403–418 (1973); permission conveyed through Copyright Clearance Center, Inc.
12 Amalie Emmy Noether (1882–1935), German mathematician and theoretical physicist.

m_2 and a potential of the system that depends only on the *relative position* of the two masses $V \equiv V(\vec{x}_1 - \vec{x}_2)$. The Newtonian equations of motion are:

$$m_i \frac{\mathrm{d}^2\vec{x}_i}{\mathrm{d}t^2} = -\frac{\partial}{\partial\vec{x}_i} V(\vec{x}_1 - \vec{x}_2) \qquad (i = 1, 2) \tag{1.6}$$

where $\partial/\partial\vec{x}_i$ is the **3-gradient** of V with respect to the $\vec{x}_i$ coordinates (see Appendix A.9). Transforming with the translation $\vec{x} \to \vec{x}' = \vec{x} + \vec{a}$, where $\vec{a}$ is a given constant vector, transforms

$$\vec{x}_i \to \vec{x}'_i = \vec{x}_i + \vec{a}: \quad \begin{cases} m_i & \to m_i \\ \dfrac{\mathrm{d}^2\vec{x}_i}{\mathrm{d}t^2} & \to \dfrac{\mathrm{d}^2\vec{x}'_i}{\mathrm{d}t^2} = \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}(\vec{x}_i + \vec{a}) = \dfrac{\mathrm{d}^2\vec{x}_i}{\mathrm{d}t^2} \\ V(\vec{x}_1 - \vec{x}_2) & \to V(\vec{x}'_1 - \vec{x}'_2) = V(\vec{x}_1 + \vec{a} - \vec{x}_2 - \vec{a}) = V(\vec{x}_1 - \vec{x}_2) \end{cases}$$

We note that the masses m_i are unchanged quantities during the translation and are hence labeled **scalars**, to define a number quantity which does not change under the transformation. For the gradient we have (dropping the i index and setting $\vec{x} = (x, y, z)$ and $\vec{x}' = (x', y', z')$):

$$\frac{\partial}{\partial\vec{x}} \to \frac{\partial}{\partial\vec{x}'} = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right) = \left(\frac{\partial}{\partial x}\frac{\partial x}{\partial x'}, \frac{\partial}{\partial y}\frac{\partial y}{\partial y'}, \frac{\partial}{\partial z}\frac{\partial z}{\partial z'}\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial\vec{x}} \tag{1.7}$$

The equations of motion in the translated system are then given by:

$$\text{Eq. (1.6)} \quad \longrightarrow \quad m_i \frac{\mathrm{d}^2\vec{x}'_i}{\mathrm{d}t^2} = -\frac{\partial}{\partial\vec{x}'_i} V(\vec{x}'_1 - \vec{x}'_2) \qquad (i = 1, 2) \tag{1.8}$$

Consequently, the **equations of motion exhibit the same form in the two systems and are hence considered as translation-invariant**. If we now calculate the total force acting on the masses, we get

$$\vec{F}_{tot} = \vec{F}_1 + \vec{F}_2 = -\frac{\partial}{\partial\vec{x}_1} V(\vec{x}_1 - \vec{x}_2) - \frac{\partial}{\partial\vec{x}_2} V(\vec{x}_1 - \vec{x}_2) = -\frac{\partial}{\partial\vec{x}_1} V(\vec{x}_1 - \vec{x}_2) + \frac{\partial}{\partial\vec{x}_1} V(\vec{x}_1 - \vec{x}_2) = \vec{0} \tag{1.9}$$

hence

$$\boxed{\frac{\mathrm{d}\vec{P}_{tot}}{\mathrm{d}t} \equiv \vec{F}_{tot} = \vec{0}} \tag{1.10}$$

and **the total momentum is constant in time and is hence conserved**.

The above derivation can be generalized to a special class of continuous transformations of space-time, called proper orthochronous Lorentz transformations (see Section 5.4). The **homogeneity in time** corresponds to the invariance under translation in time (this basically implies that there is no absolute time, and that one can define the origin of time $t = 0$ at any time, without changing the physical laws). Similarly, one can define the **homogeneity in space** for the invariance under space translation, and the isotropy of space for the invariance under rotation in space. These transformations lead to the following well-known conserved quantities:

translation in time	$\longrightarrow$	conservation of energy
translation in space	$\longrightarrow$	conservation of momentum
rotation in space	$\longrightarrow$	conservation of angular momentum

We can interpret these results as first principles: it is *natural* (or *aesthetically pleasing*) to *assume* that time and space are homogeneous and that space is isotropic. The conserved quantity corresponding to the time-translation symmetry is defined as energy, the conserved quantity associated with the space-translation symmetry is called momentum, and the conserved quantity related to the rotational symmetry is called angular momentum. **The symmetries of the system guide us in the choice of physical quantities to describe our system**.

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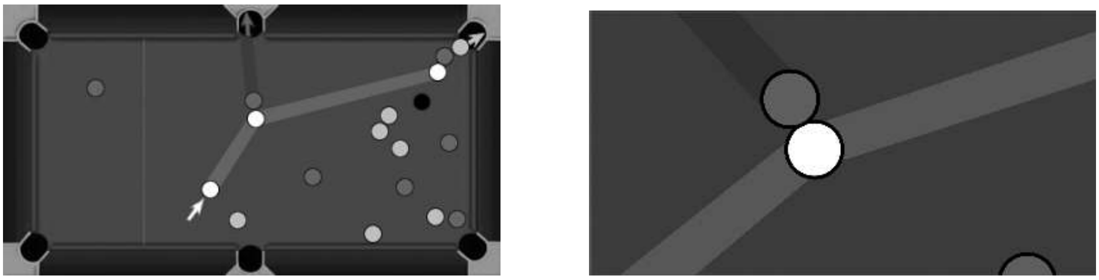

Figure 1.3 Illustration of collisions on a billiard table: (left) view of the entire table; (right) zooming in on one collision.

Transformations of a system are conveniently organized using the mathematical algebra tools of **group theory** (see Appendix B). Continuous transformations form a **Lie group** (see Appendix B.5) and can depend on one or many continuous parameters. For example, translation in time depends on one parameter while translation in space can be represented by three independent parameters corresponding to the three degrees of freedom of space. One can consider in these cases **infinitesimal transformations**, when the parameters are made infinitesimal. For example, the system can undergo an infinitesimal translation in space $\delta\vec{a}$ with $|\delta\vec{a}| \ll 1$ or in time $|\delta t| \ll 1$. A *finite* transformation can then be obtained from *a successive application of infinitesimal transformations*. We say that the finite transformation $T(\alpha)$ can be derived from the product of infinitesimal ones $T(\delta\alpha)$, where $\delta\alpha = \alpha/n$. The net effect of the finite transformation can consequently be written as:

$$T(\alpha) = \underbrace{T(\delta\alpha)\ldots T(\delta\alpha)}_{n} = [T(\delta\alpha)]^n = \left[T\left(\frac{\alpha}{n}\right)\right]^n \tag{1.11}$$

1.4 Discrete Symmetries

Discrete transformations describe non-continuous changes in a system. They *cannot* be derived from the product of infinitesimal transformations, such as Eq. (1.11). These transformations are associated with *discrete symmetry groups*. There are three important discrete transformations of space and time that we will use throughout the chapters: parity (P), charge conjugation[13] (C), and time reversal (T).

- The **parity transformation** P inverts every spatial coordinate through the origin, while the **time inversion** T (or **time reversal**) means to reverse the direction of time:

$$P : \begin{cases} \vec{x} \to -\vec{x} \\ t \to t \end{cases} \quad ; \qquad T : \begin{cases} \vec{x} \to \vec{x} \\ t \to -t \end{cases} \tag{1.12}$$

  The parity operation is equivalent to a reflection in the $x - y$ plane followed by a rotation about the z-axis.[14] There are a number of ways in which we can consider time reversal. For example, if we look at collisions on a billiard table, as illustrated in Figure 1.3(left), it would clearly violate our sense of how "moving spheres behave" if time were reversed. It is very unlikely that we would have a set of billiard balls moving in just the directions and speeds necessary for them to collect and form a perfect triangle at rest, with the cue ball moving away. However, if we look *at an individual collision* (see Figure 1.3(right)),

13 It will become clearer later why the C transformation is associated with a space-time operation.
14 Often one associates parity with the reflection in a mirror but this is not precisely correct.

reversing time results in a perfectly "normal-looking collision" (if we ignore the small loss in kinetic energy due to inelasticity in the collision). The classical equations of dynamics are invariant under T because they are of second order in time. At the microscopic level, there is time invariance in mechanics. At the macroscopic level, the arrow of time is selected statistically due to entropy. Hence, the former lack of time-reversal invariance has to do with the laws of thermodynamics; in particle physics we are generally interested in individual microscopic processes for which the laws of thermodynamics are not important, and we will consider whether particular interactions between elementary particles are invariant under T or not.

- The **charge conjugation** C is a discrete symmetry that exchanges matter with antimatter and vice versa. Hence, it reverses the sign of the electric charge, color charge, and magnetic moment of a particle (it also reverses the values of the weak isospin and hypercharge charges associated with the weak force).

Figure 1.4 illustrates the actions of P, T, and C on a fundamental (spin-1/2) fermion, for instance an electron or a positron, where the arrow through the sphere corresponds to the direction of the momentum $\vec{p}$ and where the spin $\vec{s}$ (see Section 4.11) is indicated as well as an arrow next to the particle. The parity operator

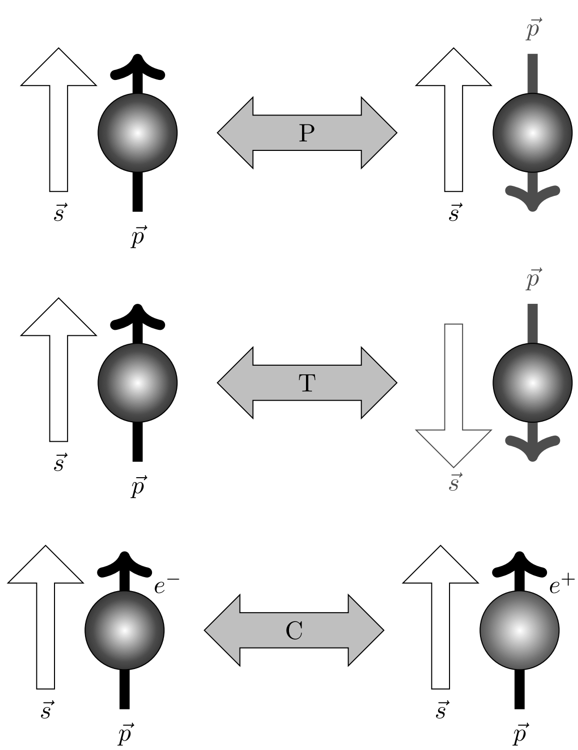

Figure 1.4 Effect of the discrete symmetries P (top), T (middle), and C (bottom) on an electron or positron. Note that the action of the operator applied on the state twice results in the original particle.

changes the direction of motion of a particle, but not the direction of the spin vector. Time reversal changes the sign of both the spin and momentum. Charge conjugation reverses the charge but does not change the direction of the spin nor the momentum of a particle. The action of any operator on a state twice gives back the original particle (up to an arbitrary quantum phase). **Understanding the behavior of a physical system or**

its interactions under discrete transformations is an important tool, as described in more detail in the next section and in later chapters.

1.5 Quantities with Well-Defined Transformation Properties

In general, we want to introduce quantities that best describe the properties of the *physical reality*. **To understand the physical properties of a given observable and hence its fundamental meaning, it is important to understand how it behaves under specific transformations of space-time or any other transformation**.

A **scalar** is a physical quantity such as a mass or a volume, which only has a magnitude and does not depend on direction. It is not affected by space-time transformations. A **vector** is a physical object that has a magnitude and a direction. A translation in space-time leaves a vector unchanged. A rotation in space is *defined* as a transformation that changes the direction but keeps the magnitude of the vector unchanged. An ordinary vector $\vec{x}$ in space changes sign under parity transformation P:

$$\text{Vector}: \qquad P(\vec{x}) = -\vec{x} \tag{1.13}$$

From any vector $\vec{x}$, one can construct a *scalar* quantity s with the **scalar product** $s \equiv \vec{x} \cdot \vec{x}$. Under parity, we find that s is an invariant scalar, since it remains unchanged:

$$\text{Scalar}: \qquad P(s) = P(\vec{x} \cdot \vec{x}) = P(\vec{x}) \cdot P(\vec{x}) = (-\vec{x}) \cdot (-\vec{x}) = \vec{x} \cdot \vec{x} = +s \tag{1.14}$$

Similarly, with the **cross-product** of two vectors, we can construct the **pseudovector** or **axial vector** $\vec{a} \equiv \vec{x_1} \times \vec{x_2}$, and this new vector does *not* change sign under parity:

$$\text{Pseudovector (or axial vector)}: \qquad P(\vec{a}) = P(\vec{x_1} \times \vec{x_2}) = -\vec{x_1} \times (-\vec{x_2}) = +\vec{a} \tag{1.15}$$

Finally, from an ordinary and pseudovector we can construct the **pseudoscalar** $p \equiv \vec{x} \cdot \vec{a}$, which keeps its magnitude but changes sign under parity. So, ultimately, we have derived quantities that have specific properties under continuous (i.e., translation, rotation) and discrete (parity) transformations of space-time:

$$\begin{array}{rcl} s \text{ scalar} & \Leftrightarrow & P(s) = +s \\ p \text{ pseudoscalar} & \Leftrightarrow & P(p) = -p \\ \vec{x} \text{ vector} & \Leftrightarrow & P(\vec{x}) = -\vec{x} \\ \vec{a} \text{ pseudovector} & \Leftrightarrow & P(\vec{a}) = +\vec{a} \end{array} \tag{1.16}$$

The above expressions can be summarized in the form of an eigenvector equation of the parity with eigenvalues $\lambda_p = \pm 1$:

$$P(c) = \lambda_p c \text{ where } \lambda_p = \begin{cases} +1 \text{ if } c \text{ is scalar/pseudovector} \\ -1 \text{ if } c \text{ is pseudoscalar/vector} \end{cases} \tag{1.17}$$

We can now consider the transformation properties of the common physical properties. A physical position $\vec{x}$ transforms as a vector ($\lambda_p = -1$). So do the velocity $\vec{v} = \mathrm{d}\vec{x}/\mathrm{d}t$ and the linear momentum vector $\vec{p} = m\vec{v}$, $P(\vec{p}) = P(m\vec{v}) = m(-\vec{v}) = -\vec{p}$, $\lambda_p = -1$. The angular momentum vector $\vec{L} = \vec{x} \times \vec{p}$ is an axial vector since it is the cross-product of a position and a momentum vector. Hence, the quantity $\vec{L}$ transforms as $P(\vec{L}) = (-\vec{x}) \times (-\vec{p}) = \vec{L}$, $\lambda_p = +1$, and so does the spin $\vec{s}$ of a particle (see also Figure 1.4).

The transformation property of the classical electric field $\vec{E}$ can be derived applying P on Gauss's law:

$$\frac{\partial}{\partial \vec{x}} \cdot \vec{E}(\vec{x}, t) = \rho(\vec{x}, t) \quad \stackrel{P}{\to} \quad \frac{\partial}{\partial \vec{x}'} \cdot \vec{E}'(\vec{x}', t) = \rho(\vec{x}', t) \tag{1.18}$$

or

$$-\frac{\partial}{\partial \vec{x}} \cdot \vec{E}'(-\vec{x}, t) = \frac{\partial}{\partial \vec{x}} \cdot \left[-\vec{E}'(-\vec{x}, t)\right] = \rho(-\vec{x}, t) \tag{1.19}$$

which yields the original equation if the electric field transforms as $\vec{E} \to -\vec{E}$ under parity. Similarly, one can show that the magnetic field must transform as $\vec{B} \to +\vec{B}$ under parity.

Other important quantities can be derived by considering the product of two quantities with well-defined transformation properties. For example, we can mention the **electric dipole moment** $\vec{\sigma} \cdot \vec{E}$, the **magnetic dipole moment** $\vec{\sigma} \cdot \vec{B}$, the **longitudinal polarization** $\vec{\sigma} \cdot \vec{p}$, or the **transverse polarization** $\vec{\sigma} \cdot (\vec{p}_1 \times \vec{p}_2)$ (see Section 4.14). Their transformations under parity are summarized in Table 1.3. Similar arguments can also be repeated for the other two discrete transformations T and C, and are also listed in this table. For example, time reversal reverses linear and angular momenta and also spin, and so on. Such quantities are important to understand the invariance properties of fundamental interactions, as will be shown later.

Quantity	**Notation**	P	C	T
Position	$\vec{x}$	$-\vec{x}$	$+\vec{x}$	$+\vec{x}$
Velocity	$\vec{v} = \mathrm{d}\vec{x}/\mathrm{d}t$	$-\vec{v}$	$+\vec{v}$	$-\vec{v}$
Linear momentum	$\vec{p} = m\vec{v}$	$-\vec{p}$	$+\vec{p}$	$-\vec{p}$
Angular momentum	$\vec{L} = \vec{r} \times \vec{p}$	$+\vec{L}$	$+\vec{L}$	$-\vec{L}$
Spin	$\vec{S}$ or $\vec{\sigma}$	$+\vec{\sigma}$	$+\vec{\sigma}$	$-\vec{\sigma}$
Helicity	$h = \vec{\sigma} \cdot \vec{p}/\lvert p \rvert$	$-h$	$+h$	$+h$
Electric field	$\vec{E}$	$-\vec{E}$	$-\vec{E}$	$+\vec{E}$
Magnetic field	$\vec{B}$	$+\vec{B}$	$-\vec{B}$	$-\vec{B}$
Electric dipole moment	$\vec{\sigma} \cdot \vec{E}$	$-\vec{\sigma} \cdot \vec{E}$	$-\vec{\sigma} \cdot \vec{E}$	$-\vec{\sigma} \cdot \vec{E}$
Magnetic dipole moment	$\vec{\sigma} \cdot \vec{B}$	$+\vec{\sigma} \cdot \vec{B}$	$-\vec{\sigma} \cdot \vec{B}$	$+\vec{\sigma} \cdot \vec{B}$
Longitudinal polarization	$\vec{\sigma} \cdot \vec{p}$	$-\vec{\sigma} \cdot \vec{p}$	$+\vec{\sigma} \cdot \vec{p}$	$+\vec{\sigma} \cdot \vec{p}$
Transverse polarization	$\vec{\sigma} \cdot (\vec{p}_1 \times \vec{p}_2)$	$+\vec{\sigma} \cdot (\vec{p}_1 \times \vec{p}_2)$	$+\vec{\sigma} \cdot (\vec{p}_1 \times \vec{p}_2)$	$-\vec{\sigma} \cdot (\vec{p}_1 \times \vec{p}_2)$

Table 1.3 A summary of the transformations of various physical quantities with well-defined transformation properties under C, P, and T.

1.6 Global and Local Gauge Symmetries

Up to now we have discussed the continuous transformations of space-time (part of the so-called proper orthochronous Lorentz group) and the discrete symmetries of space-time (see Chapter 5). One considers also **internal symmetries**, which do not act on the space-time coordinates but rather on internal degrees of freedom of the quantum systems. **Gauge invariance** is equivalent to the fact that absolute changes in the complex phase of a wave function of a quantum system are unobservable, and only changes in the magnitude of the wave function result in changes to the probabilities (see Section 4.2).

As will be discussed in later chapters, it has become quite evident that the fundamental interactions of Nature can be shown to originate from some sort of *local* gauge symmetry. In this sense local gauge symmetry is considered as a primary concept – a first principle! In a quantum field theory (see Chapter 7) with a **local gauge symmetry** (see Chapter 24), there is an **absolutely** conserved quantity (a charge) and a **long-range force** coupled to the charge. This corresponds for instance to the electric charge Q or e and the long-range electromagnetic interaction, and is the basis of quantum electrodynamics (see Chapter 10). Other examples are the strong (see Chapter 18) and the electroweak (see Chapter 25) interactions.

Global gauge symmetries are also very important, although they seem to lead to approximate laws of Nature. This is, for example, the case for conserved charges such as the lepton number L (see Chapter 21)