Notes on the Brown–Douglas–Fillmore Theorem

The Brown–Douglas–Fillmore (BDF) theorem provides a remarkable solution to a long-standing open question in operator theory. The solution is obtained by first finding an equivalent formulation of the problem involving extensions of C^*-algebras. This does not make the original problem any simpler. None the less, Brown, Douglas, and Fillmore used new tools from algebraic topology and homological algebra to solve the problem of extensions of commutative C^*-algebras. This new approach also inspired the study of extensions of non-commutative C^*-algebras. The spurt of activity around the BDF theorem in the seventies was the precursor to the important fields of non-commutative geometry and K-theory.

The book presents the original proof of BDF in its entirety. Some parts of this proof, which were only briefly indicated in the BDF paper "Unitary equivalence modulo the compact operators and extensions of C*-algebras" published in the Springer Lecture Notes 345, appear here in complete detail. These notes have attempted to preserve the crucial connection of the BDF theorem with concrete problems in operator theory. This connection, for instance, is manifest in the study of the Arevson–Douglas conjecture.

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To the memory of R. G. Douglas with great admiration,
much affection and deep respect.

The second named author is grateful to R. G. Douglas for the mathematical training he has received from him, first as a doctoral student and then while working together on several joint research projects.

Contents

Preface

A well-known theorem of Weyl says that every Hermitian operator on a separable complex Hilbert space is the sum of a diagonal and a compact operator. Halmos raised the question: Is every normal operator the sum of a diagonal and compact operator? Berg and Sikonia, independent of each other, proved that one may replace the Hermitain operators in Weyl's theorem by normal operators without changing the conclusion. However, this is not true if the operator is *essentially normal*, that is, its self-commutator is compact. The unilateral shift provides an example of such an operator. The obstruction to the decomposition of an essentially normal operator as a diagonal plus compact operator is a certain index data. This was established in a remarkable theorem by Brown, Douglas, and Fillmore that says that the index is the only obstruction to expressing, up to unitary equivalence, an essentially normal operator as the sum of a diagonal plus compact operator. They showed that the classification of essentially normal operators modulo compact is the same as an equivalence problem of $*$-monomorphisms of C^*-algebras. What has come to be known as the "BDF" theorem provides a solution to this problem. Their proofs are imaginative, novel and use a vast array of techniques and intuition from algebraic topology and homological algebra. This theory, cutting across several different areas, is quite sophisticated. Not surprisingly, therefore, many alternative proofs for different building blocks in the original proof were found soon afterward. So much so that the ingenious original proof in [26], which is a bit terse at some places, was all but forgotten.

These notes follow very closely the initial exposition of Brown, Douglas, and Fillmore [26]. In particular, we have resisted the temptation of giving simpler proofs to many of the arguments, which are now available. Our main goal was to occasionally fill in some details and explain few of the obscure points in the original proof of Brown, Douglas, and Fillmore, which appeared in [26], taking advantage of the exposition in the monograph [47]. In the process, we hope that the reader would develop an interest in modern operator theory and be exposed to some non-trivial techniques.

The first chapter provides standard preparatory material in basic operator theory; we largely follow the exposition of [44]. To get to the main points of the proof of the BDF theorem as fast as possible, a number of topics that aid in the understanding of the proof but are either not central to the proof or are expected to be known by the reader have been included in the appendices rather than in the main body of the text.

We make no claim to originality in preparing this material. All of it already exists in one form or the other. We reiterate that our goal is merely to make the original proof of Brown, Douglas, and Fillmore available to beginning doctoral students and young researchers in its entirety.

These notes are an expanded version of the "lecture notes" prepared by the second named author on the Brown, Douglas, and Fillmore (BDF) theorem for the Instructional Conference on Operator Algebras, ISI-Bangalore, 1990. This deep and non-trivial theorem, from the early seventies, used tools from algebraic topology and homological algebra to solve an outstanding problem in operator theory. In the following years, many new and novel approaches to this topic have been discovered. These have considerably expanded the scope of the original theorem and have succeeded in making inroads into many other closely related topics. It is in the backdrop of this theorem that operator theory and operator algebras achieved their separate identities. Although new proofs have been found for different parts of the BDF theorem, the elegant and somewhat intriguing original proof remains an enigma.

There are several excellent expositions on the BDF theroem. However, these use many tools and advanced techniques from the theory of operator algebras. For instance, the exposition by Higson and Roe [77] provides a complete proof of the BDF theorem taking advantage of some of the later developments in operator theory and operator algebras. On the other hand, a number of intricate operator theoretic aspects of the original proof are apparent in the book of Davidson [40].

The goal we have set ourselves is to bring the original and beautiful proof of the BDF theorem to the notice of beginning doctoral students and researchers. Therefore, we have chosen to provide only brief explanations, where necessary, while following the original proof very closely. These notes attempt to provide an opportunity to those who might wish to learn what the BDF theorem says and what it takes to prove it. The proof of the BDF theorem in these notes occupies chapters two through four. The last chapter includes applications to operator theory. It contains, among other things, a discussion of essentially normal essentially homogeneous operators and reductive Hilbert modules which is at the heart of the Arveson–Douglas conjecture. We conclude with an Epilogue consisting of three short sections. In the first of these three sections, we discuss simpler proofs of different parts of the BDF theorem. The two main components here are the proof that Ext(X) is a group and Ext is a homotopy invariant functor. In the second section, we discuss some of the later developments related to the BDF theory. Here we discuss extensions of non-commutative C^*-algebras and describe the general notion of an index and relate it to K-theory. Finally, in the third section, we discuss some open problems concerning the complete set of invariants for essentially normal tuples and essential normality of essentially homogeneous tuples. We also discuss the Arveson–Douglas conjecture for semi-sub modules of a class of Hilbert modules.

The intended readership being the beginning doctoral students, we have added some preparatory material in the first chapter. To ensure that the original proof of the BDF theorem always remains the main focus of these notes, we have decided not to clutter the text with additional material except for occasionally filling in some details. Some of the topics which might give a greater understanding of the proof have been discussed in the Appendix. Again, the attempt here is not to be encyclopaedic but only to briefly recall some notions from topology and functional analysis that one may not easily find in one place. Consequently, the topics discussed in the Appendix appear in no particular order; they are somewhat unrelated to one another but serve the purpose of providing familiarity to a large variety of tools used in the proof and the flexibility of reading them independently or not at all.

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