From: The Evolution of Modern Analysis, R. G. Douglas

Coburn had shown that there was more than one extension of the compact operators by the algebra of complex-valued continuous functions on the circle: a trivial one and the one I had exploited to calculate the index of continuous Toeplitz operators. Both Atiyah and Singer asked about the possibility of others and whether they could somehow be classified. Ultimately, investigating this question as well as several others, especially the classification of essentially normal operators, led to my joint work with Larry Brown and Peter Fillmore. … The heart of the BDF results involves a functor from compact metric spaces to abelian groups defined by equivalence classes of C^*-extensions. Fundamentally, our work involved identifying this functor as defining the odd group in K-homology [48, pp. 283].

Overview

A normal operator on a finite dimensional inner product space can be diagonalised and the eigenvalues together with their multiplicities are a complete set of unitary invariants for the operator, while on an infinite dimensional Hilbert space the spectral theorem provides a model and a complete set of unitary invariants for such operators (refer to Appendix C). Thus, we view the theory of normal operators to be well understood. It is natural to study operators that may be thought of to be nearly normal in some sense. One hope is that it would be possible to provide canonical models and a complete set of invariants for such operators. Since an operator is normal if $[T,T^*] := TT^* - T^*T$ is 0, one may say an operator is nearly normal if $[T,T^*]$ is small in some appropriate sense, for example, finite rank, trace class, or compact. In these notes, we will take the last of these three measures of smallness for $[T,T^*]$ and make the following definition.

An operator T in $\mathcal{L}(\mathcal{H})$ is *essentially normal* if the self-commutator $[T,T^*]$ of T is compact. We say that T is *essentially unitary* if T is essentially normal and $T^*T - I$ is compact. Let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on a complex separable Hilbert space $\mathcal{H}$ and $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{C}(\mathcal{H})$ be the natural quotient map. Set $\mathcal{Q}(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{C}(\mathcal{H})$. An operator T in $\mathcal{L}(\mathcal{H})$ is essentially normal if and only if $\pi(T)$ is normal in the C^*-algebra $\mathcal{Q}(\mathcal{H})$. Further, an operator U in $\mathcal{L}(\mathcal{H})$ is essentially unitary if and only if $\pi(U)$ is unitary.

One of the main goals of these notes is to describe a complete set of invariants for the essentially normal operators with respect to a suitable notion of equivalence. As we are considering compact operators to be small, the correct notion of equivalence would seem to be the following.

Two operators T_1 and T_2 in $\mathcal{L}(\mathcal{H})$ are said to be *essentially equivalent* if there exist an essentially unitary operator U and a compact operator K such that $UT_1U^* = T_2 + K$. In this case, we write, $T_1 \sim T_2$. However, it turns out that one may replace the essentially unitary operator in this definition with a unitary operator without any loss of generality. The BDF theorem describes, among other things, the equivalence classes {essentially normal operators}/$\sim$.

The *essential spectrum* $\sigma_e(T)$, of an operator T in $\mathcal{L}(\mathcal{H})$ is the spectrum $\sigma(\pi(T))$ of $\pi(T)$ in the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{C}(\mathcal{H})$. Let

$$\mathcal{N} + \mathcal{C} = \{N + K : N \text{ is normal and } K \text{ is compact}\}.$$

For an operator T in $\mathcal{N}+\mathcal{C}$ of the form $N+K$, note that $\sigma_e(T) = \sigma_e(N)$. The Weyl–von Neumann–Berg–Sikonia theorem states that the essential spectrum is a complete invariant for essential equivalence, that is, unitary equivalence modulo compact of operators in the class $\mathcal{N}+\mathcal{C}$. Moreover, if X is any compact subset of the complex plane $\mathbb{C}$, then there is a normal operator N such that $\sigma_e(N) = X$.

Not all essentially normal operators are in $\mathcal{N}+\mathcal{C}$. To give an example of an essentially normal operator not in $\mathcal{N}+\mathcal{C}$, consider the Toeplitz operator T_z on the Hardy space $H^2(\mathbb{T})$. Note that $I - T_zT_z^* = P$ and $I - T_z^*T_z = 0$, where P is a rank one projection; therefore, T_z is an essentially unitary operator. An operator T is called *Fredholm* if it has a closed range and the dimension of its kernel and cokernel are finite. For a Fredholm operator T, the index $\operatorname{ind}(T)$ is an integer given by the formula

$$\operatorname{ind}(T) = \dim\ker(T) - \dim\ker(T^*).$$

If T is Fredholm and K is compact, then $\operatorname{ind}(T+K) = \operatorname{ind}(T)$. Moreover, the index of a Fredholm operator remains invariant under essential equivalence. Finally, if N is a normal operator that is also Fredholm, then its index is zero. It is easy to see that the Toeplitz operator T_z is Fredholm with $\operatorname{ind}(T_z) = -1$. If T_z is also in $\mathcal{N}+\mathcal{C}$, then the index of T_z must be zero. This contradiction shows that the essentially normal operator T_z is not in $\mathcal{N}+\mathcal{C}$.

First, the essential spectrum is a complete invariant modulo essential equivalence in the claas $\mathcal{N}+\mathcal{C}$. Second, note that the normal operator M_z of multiplication by the coordinate function on $L^2(\mathbb{T})$ and the Toeplitz operator T_z both have the same essential spectrum, namely the unit circle $\mathbb{T}$. If these two operators were essentially equivalent via the unitary U, then we must have

$$-1 = \operatorname{ind}(T_z) = \operatorname{ind}(T_z+K) = \operatorname{ind}(U^*M_zU) = \operatorname{ind}(M_z) = 0.$$

This contradiction shows that the essential spectrum is not the only invariant for essential equivalence. The remarkable theorem of Brown, Douglas, and Fillmore says that the essential spectrum together with the index data is a complete set of invariants for essential equivalence.

Brown, Douglas, and Fillmore considered the problem of classifying the extension of compact operators by the C^*-algebra $C(X)$ of continuous functions on a compact set X induced by essentially normal operators with essential spectrum equal to X. Some of the details of this correspondence are provided in the following text. First, observe that if $\mathcal{S}_T$ is the C^*-algebra generated by the essentially normal operator T, the compact operators $C(\mathcal{H})$ and the identity operator I on the Hilbert space $\mathcal{H}$, then $\mathcal{S}_T/C(\mathcal{H})$ is isomorphic to the C^*-algebra generated by 1 and $\pi(T)$ in the Calkin algebra $\mathcal{Q}(\mathcal{H})$. Since T is essentially normal, it follows that $\mathcal{S}_T/C(\mathcal{H})$ is commutative and we have

$$\begin{array}{ccc} \mathcal{S}_T & \longrightarrow & C(\sigma_{\mathrm{e}}(T)) \\ \big\downarrow \pi & & \big\uparrow \Gamma_{\mathcal{S}_T/C(\mathcal{H})} \\ \mathcal{S}_T/C(\mathcal{H}) & \hookrightarrow & \mathcal{S}_T/C(\mathcal{H}) \subseteq \mathcal{Q}(\mathcal{H}) \end{array}$$

where $\Gamma_{\mathcal{S}_T/C(\mathcal{H})}$ is the Gelfand map and we have an extension, that is

$$0 \to C(\mathcal{H}) \to \mathcal{S}_T \xrightarrow{\varphi_T} C(\sigma_e(T)) \to 0$$

is exact. Conversely, if $\mathcal{S}$ is any C^*-algebra of operators on the Hilbert space $\mathcal{H}$ containing compact operators, that is $C(\mathcal{H}) \subseteq \mathcal{S} \subseteq \mathcal{L}(\mathcal{H})$ and X is any compact subset of the complex plane $\mathbb{C}$ such that

$$0 \to C(\mathcal{H}) \xrightarrow{i} \mathcal{S} \xrightarrow{\varphi} C(X) \to 0$$

is exact, then for any T in $\mathcal{S}$, $\varphi(TT^* - T^*T) = 0$ and it follows that T is essentially normal. Fix any T in $\mathcal{S}$ such that $\varphi(T) = \mathrm{id}|_X$. Let $\mathcal{S}_T$ be the C^*-algebra generated by the operator T, the compact operators and the identity on $\mathcal{H}$. Now, $\varphi(\mathcal{S}_T)$ is a C^*-subalgebra of $C(X)$ containing the identity function and therefore must be all of $C(X)$. If S is any operator in $\mathcal{S}$, then there is always an operator S' in $\mathcal{S}_T$ such that $\varphi(S) = \varphi(S')$ so that $\varphi(S - S') = 0$, or equivalently, $S - S'$ is compact and hence, S is in $\mathcal{S}_T$. Since $\mathcal{S}_T \subseteq \mathcal{S}$, it follows that $\mathcal{S}_T = \mathcal{S}$. Thus, there is a natural correspondence between essentially normal operators T with essential spectrum, a compact set $X \subseteq \mathbb{C}$ and extensions of $C(\mathcal{H})$ by $C(X)$.

Let us now relate unitary equivalence modulo the compacts of essentially normal operators to such extensions. If $(\mathcal{S}_1, \varphi_1)$ and $(\mathcal{S}_2, \varphi_2)$ are two extensions corresponding to equivalent essentially normal operators T_1 and T_2, that is, $U^*T_2U = T_1 + K$ for some unitary operator U and compact operator K, then $U^*\mathcal{S}_2U = \mathcal{S}_1$ by continuity of the map $T \mapsto U^*TU$ and $\varphi_2(T) = \varphi_1(U^*TU)$ for all T in $\mathcal{S}_2$. It is therefore natural to say that two extensions $(\mathcal{S}_1, \varphi_1)$ and $(\mathcal{S}_2, \varphi_2)$ are *equivalent* if there exists a unitary operator U such that $U^*\mathcal{S}_2U = \mathcal{S}_2$ and $\varphi_2(T) = \varphi_1(U^*TU)$. Thus, if two essentially normal operators T_1 and T_2 are equivalent modulo the compacts, then the corresponding extensions are equivalent. Conversely, if the extensions are equivalent, then

$$\varphi_1(U^*T_2U) = \varphi_2(T_2) = \mathrm{id}|_X = \varphi_1(T_1)$$

and we see that $U^*T_2U - T_1$ is compact.

The classification problem for essentially normal operators and for extensions of $C(\mathcal{H})$ by $C(X)$ are identical for any compact subset X of $\mathbb{C}$. The extension point of view, of course, has many advantages. For any compact metrizable space X, let $\mathrm{Ext}(X)$ denote the *equivalence classes of the extensions* of $C(\mathcal{H})$ by $C(X)$. If X is a compact subset of the complex plane $\mathbb{C}$, then $\mathrm{Ext}(X)$ is just the *equivalence classes of essentially normal operators* N with $\sigma_e(N) = X$. Note that if Δ is a subset of the real line $\mathbb{R}$ and if S is any operator in $\mathcal{L}(\mathcal{H})$ such that $\pi(S)$ is normal with spectrum Δ, then $\pi(S)$ is self-adjoint,

$$\pi(S - S^*) = 0 \text{ implies that } S = \Re(S) + \text{compact},$$

where $\Re(S)$ denotes the real part of S. By the Weyl–von Neumann theorem, any two of these operators are equivalent modulo the compacts or in other words, $\mathrm{Ext}(\Delta) = 0$, for $\Delta \subseteq \mathbb{R}$.

The very deep and powerful theorem of Brown, Douglas, and Fillmore says that $\mathrm{Ext}(X)$ is a group for any compact metric space X and in particular, if X is a planar set, then $\mathrm{Ext}(X)$ is isomorphic to $\mathrm{Hom}(\pi^1(X), \mathbb{Z})$. Here, $\pi^1(X)$ is the first cohomotopy group of X. These homomorphisms can be realized as follows: Let T be an essentially normal operator with essential spectrum X and $\{X_i\}_{i \in I}$ be the connected components of $\mathbb{C} \setminus X$. Suppose that the index of $T - \lambda$, $\lambda \in X_i$ be equal to n_i, $i \in I$. Define the map $\gamma_T : I \to \mathbb{Z}$ by $\gamma_T(i) = n_i$. This is the index data of the operator T and the BDF theorem says that the set $\{X, \gamma_T\}$ is a complete invariant for the operator T. In particular, taking $X = \mathbb{T}$, one obtains the following classification of essentially normal operators with essential spectrum $\mathbb{T}$. If T is an essentially normal operator and $\sigma_e(T) = \mathbb{T}$, then T is essentially equivalent to the bilateral shift, the n-fold direct sum of the unilateral forward shift or the m-fold direct sum of the unilateral backward shift according as the index of T is 0, $-n, n > 0$, or $m, m > 0$.

1

Spectral Theory for Hilbert Space Operators

The purpose of this chapter is two-fold. First, it serves as a rapid introduction to some of the basics of modern operator theory. Second, it has all the prerequisites needed to prove the Brown–Douglas–Fillmore theorem.

Unless otherwise stated, all Hilbert spaces considered in this text are assumed to be complex and separable. Whenever separability is not needed or it does not simplify the situation, the same is mentioned. Throughout this text, $\mathcal{H}$ denotes a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ stands for the algebra of bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$. Note that $\mathcal{L}(\mathcal{H})$ is a unital C^*-algebra, where the identity operator I is the unit, composition of operators is the multiplication and the uniquely defined adjoint T^* of a bounded linear operator T on Hilbert space $\mathcal{H}$ is the involution (the reader is referred to Appendix B for the definition of a C^*-algebra). To avoid ambiguity, whenever necessary, we let $I_{\mathcal{H}}$ denote the identity operator on $\mathcal{H}$. Given $T \in \mathcal{L}(\mathcal{H})$, the symbols $\ker T$ and $\operatorname{ran} T$ stand for the kernel and range of the operator T, respectively. As usual, we let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norm and the inner product in the Hilbert space $\mathcal{H}$.

1.1 Partial Isometries and Polar Decomposition

If λ is a nonzero complex number, then $\lambda = |\lambda|e^{i\theta}$ for some real number θ; this is the polar decomposition of λ. Theorem 1.1.3 provides, for operators in $\mathcal{L}(\mathcal{H})$, a similar decomposition. The challenge is to find the two factors analogous to $|\lambda|$ and $e^{i\theta}$ in $\mathcal{L}(\mathcal{H})$. A natural choice for $|\lambda|$ quickly presents itself, namely, the operator $(T^*T)^{1/2}$. The choice for $e^{i\theta}$ would seem to be either a unitary or an isometry; however, none of these choices is quite correct for an operator on a Hilbert space of dimension greater than one as we will see here.

Definition 1.1.1

An operator $V \in \mathcal{L}(\mathcal{H})$ is a *partial isometry* if $\|Vf\| = \|f\|$ for all $f \in \mathcal{H}$ that are orthogonal to $\ker V$. If, in addition, the kernel of V is $\{0\}$, then V is said to be an *isometry*. The *initial space* of a partial isometry V is defined as the orthogonal complement $(\ker V)^\perp$ of $\ker V$, whereas the *final space* of V is the range $\operatorname{ran} V$ of V.

***Remark* 1.1.2** Let $V \in \mathcal{L}(\mathcal{H})$ be a partial isometry. Let $f \in \mathcal{H}$ and write $f = f_1 + f_2$, where $f_1 \in \ker V$ and $f_2 \in (\ker V)^\perp$. Then

$$\langle (I - V^*V)f, f\rangle = \|f\|^2 - \|Vf\|^2 = \|f_1\|^2.$$

Thus, $I - V^*V$ is a positive operator, and by Remark C.1.6, it has a unique positive square root, say, $(I - V^*V)^{1/2}$. Moreover, for any $f \in (\ker V)^\perp$,

$$\|(I - V^*V)^{1/2} f\|^2 = \langle (I - V^*V)f, f\rangle = 0.$$

Consequently, $(I - V^*V)^{1/2} f = 0$ or $V^*Vf = f$. In particular, V^*V is the orthogonal projection of $\mathcal{H}$ onto the initial space of V. Now, $VV^*(Vf) = Vf$ and if $g \perp \operatorname{ran} V$, then $VV^*g = 0$. So, VV^* is the orthogonal projection of $\mathcal{H}$ onto the final space of V. Thus, V^* is a partial isometry with initial space $\operatorname{ran} V$. It also follows that the range of a partial isometry is closed.

It can be seen that the correct analogy for $e^{i\theta}$ is a partial isometry. For $T \in \mathcal{L}(\mathcal{H})$, we let $|T|$ denote the positive square root $(T^*T)^{1/2}$ of the operator T^*T (see Remark C.1.6).

Theorem 1.1.3 (Polar Decomposition)

If $T \in \mathcal{L}(\mathcal{H})$, then there exists a positive operator P and a partial isometry V (with initial space $\overline{\operatorname{ran} P}$ and final space $\overline{\operatorname{ran} T}$) such that

$$T = VP \text{ and } \ker V = \ker P.$$

Moreover, such a pair (V, P) is uniquely determined. Furthermore, P can be chosen to be $|T|$.

Proof Note that for any $f \in \mathcal{H}$,

$$\||T|f\|^2 = \langle |T|f, |T|f\rangle = \langle |T|^2 f, f\rangle = \langle T^*Tf, f\rangle = \|Tf\|^2. \tag{1.1.1}$$

Thus, if we define $\tilde{V} : \operatorname{ran}|T| \to \mathcal{H}$ by

$$\tilde{V}(|T|f) = Tf, \quad f \in \mathcal{H},$$

then $\tilde{V}$ is well defined. Moreover, it is isometric and extends uniquely to an isometric mapping from $\overline{\operatorname{ran}|T|}$ to $\mathcal{H}$. If we further define $V : \mathcal{H} \to \mathcal{H}$ by

$$Vf = \begin{cases} \tilde{V}f & \text{if } f \in \overline{\operatorname{ran}|T|}, \\ 0 & \text{if } f \in (\operatorname{ran}|T|)^\perp, \end{cases}$$

then V is a partial isometry satisfying $T = V|T|$ and

$$\ker V = (\text{ran } |T|)^{\perp} = \ker|T|.$$

Thus, choosing $P = |T|$, we have $T = VP$.

For uniqueness, suppose that $T = WQ$, where W is a partial isometry, Q is positive and $\ker W = \ker Q$. By Remark 1.1.2, W^*W is the orthogonal projection onto $(\ker W)^{\perp} = (\ker Q)^{\perp} = \overline{\text{ran } Q}$. It follows that

$$|T|^2 = T^*T = QW^*WQ = Q^2.$$

Thus, by the uniqueness of the positive square root, we have $Q = |T|$. Consequently, $W|T| = V|T|$, and hence, $W = V$ on the range of $|T|$. However

$$(\text{ran } |T|)^{\perp} = \ker|T| = \ker W = \ker V,$$

and hence, $W = V$ on $(\text{ran } |T|)^{\perp}$. Therefore, $V = W$ and the proof is complete. □

A polar decomposition in which the order of the factors are reversed is useful in certain instances.

Corollary 1.1.1

If $T \in \mathcal{L}(\mathcal{H})$, then there exist a positive operator Q and partial isometry W such that

$$T = QW \text{ and } \text{ran } W = (\ker Q)^{\perp}.$$

Moreover, such a pair (Q, W) is uniquely determined.

Proof An application of Theorem 1.1.3 to the operator T^* shows that there exists a partial isometry V such that

$$T^* = V|T^*| \text{ and } \ker V = \ker|T^*|.$$

Setting $Q = |T^*|$, we have $T = |T^*|W$, where $W = V^*$ and $\ker W^* = \ker|T^*|$. By Remark 1.1.2, W is a partial isometry with closed range. Further, $\ker W^* = \ker|T^*|$ if and only if $\text{ran } W = (\ker|T^*|)^{\perp}$. The uniqueness part now follows from the preceding theorem. □

***Remark* 1.1.4** If W is chosen such that $\text{ran } W = (\ker|T^*|)^{\perp}$, then $\text{ran } W = \overline{\text{ran } |T^*|}$, and hence,

$$\text{ran } |T^*| = |T^*|(\mathcal{H}) = |T^*|(\overline{\text{ran } |T^*|}) = |T^*|W(\mathcal{H}) = \text{ran } T.$$

1.2 Compact and Fredholm Operators

In this section, we discuss some basic properties of compact and Fredholm operators on a Hilbert space. In particular, we show that the subset $C(\mathcal{H})$ of compact operators forms a closed two-sided $*$-ideal in the C^*-algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$. Furthermore, we show that the quotient $\mathcal{L}(\mathcal{H})/C(\mathcal{H})$ is a C^*-algebra. The invertible elements of this quotient algebra naturally give rise to the notion of Fredholm operators.

Compact Operators

In this section, it is shown that an operator is compact if and only if it is the norm limit of a sequence of finite rank operators. Thus, compact operators are natural generalizations of finite dimensional operators in a topological sense. We first show that any closed subspace of the Hilbert space $\mathcal{H}$ in the range of a compact operator $T : \mathcal{H} \to \mathcal{H}$ must be finite dimensional. Moreover, any operator whether compact or not, possessing this property can be approximated in norm by a sequence of finite rank operators. Thus, we obtain another characterization of compact operators, namely an operator in $\mathcal{L}(\mathcal{H})$ is compact if and only if the only closed subspaces of the Hilbert space $\mathcal{H}$ in its range are finite dimensional.

Let $\mathbb{B} = \{x \in \mathcal{H} : \|x\| < 1\}$ denote the open unit ball in $\mathcal{H}$ and let $\overline{\mathbb{B}}$ denote the closure of $\mathbb{B}$ in $\mathcal{H}$.

Definition 1.2.1

A bounded linear transformation $T : \mathcal{H} \to \mathcal{K}$ is of *finite rank* if the dimension of its range is finite and *compact* if the image of the closed unit ball $\overline{\mathbb{B}}$ under T is relatively compact in $\mathcal{K}$. Let $\mathcal{T}(\mathcal{H})$ and $C(\mathcal{H})$ denote the set of all finite rank and compact operators on $\mathcal{H}$ respectively.

By the Heine–Borel theorem, finite rank operators are compact. The converse is certainly not true (see Exercise 1.7.11). Most of the elementary properties of finite rank operators are collected together in the following proposition.

Proposition 1.2.1

The collection $\mathcal{T}(\mathcal{H})$ is a minimal two-sided nonzero $*$-ideal in $\mathcal{L}(\mathcal{H})$.

Proof Note that

$$\operatorname{ran}(S + T) \subseteq \operatorname{ran} S + \operatorname{ran} T \text{ and } \operatorname{ran}(ST) \subseteq \operatorname{ran} S.$$

These inclusions show that $\mathcal{T}(\mathcal{H})$ is a right ideal in $\mathcal{L}(\mathcal{H})$. Further, the equality

$$\operatorname{ran} T^* = T^*(\ker T^*)^{\perp} = T^*(\overline{\operatorname{ran} T})$$

shows that $T \in \mathcal{T}(\mathcal{H})$ if and only if $T^* \in \mathcal{T}(\mathcal{H})$. Finally, if $S \in \mathcal{L}(\mathcal{H})$ and $T \in \mathcal{T}(\mathcal{H})$, then $T^*S^* \in \mathcal{T}(\mathcal{H})$ and hence, $ST = (T^*S^*)^* \in \mathcal{T}(\mathcal{H})$. Therefore, $\mathcal{T}(\mathcal{H})$ is a two-sided $*$-ideal in $\mathcal{L}(\mathcal{H})$.

To show that $\mathcal{T}(\mathcal{H})$ is minimal, assume that $\mathcal{J}$ is a nonzero ideal in $\mathcal{L}(\mathcal{H})$. Thus, there exists a nonzero operator $T \in \mathcal{J}$, and hence, there is a nonzero vector f and a unit vector g in $\mathcal{H}$ such that $Tf = g$. For $k, h \in \mathcal{H}$, let $k \otimes h$ denote the rank one bounded linear operator defined by

$$k \otimes h(\ell) = \langle \ell, h \rangle k, \quad \ell \in \mathcal{H}. \tag{1.2.2}$$

Note that for any $\ell \in \mathcal{H}$,

$$(k \otimes g)T(f \otimes h)(\ell) = k \otimes g(\langle \ell, h \rangle Tf) = \langle \ell, h \rangle \langle Tf, g \rangle k = \langle \ell, h \rangle k = k \otimes h(\ell),$$

and therefore, $k \otimes h \in \mathcal{J}$ for any pair of vectors h and k in $\mathcal{H}$. However,

$$\{T \in \mathcal{L}(\mathcal{H}) : T \text{ is of rank one}\} = \{k \otimes h : h \text{ and } k \text{ belong to } \mathcal{H}\}.$$

Thus, $\mathcal{J}$ contains all rank one operators and hence, all finite rank operators. This completes the proof. □

Next, we obtain a very useful alternative characterization of compact operators.

Lemma 1.2.1

If T belongs to $\mathcal{L}(\mathcal{H})$, then T is compact if and only if for every sequence $\{f_n\}_{n\geqslant 0}$, which converges to f weakly, it is true that $\{Tf_n\}_{n\geqslant 0}$ converges to Tf in norm.

Proof Let $T \in \mathcal{L}(\mathcal{H})$ and suppose that T is compact. Let $\{f_n\}_{n\geqslant 0}$ be a sequence in $\mathcal{H}$ converging weakly to $f \in \mathcal{H}$. One may assume that $\{f_n\}_{n\geqslant 0}$ is contained in the unit ball after multiplying by a nonzero scalar if required (see Remark B.1.12). Assume contrary to the hypothesis that $\{Tf_n\}_{n\geqslant 0}$ does not converge in norm to Tf. After passing to a subsequence, if necessary, we may assume that for some $\epsilon > 0$,

$$\|Tf_n - Tf\| \geqslant \epsilon \text{ for all } n \geqslant 0. \tag{1.2.3}$$

On the other hand, since $\{Tf_n\}_{n\geqslant 0}$ is a relatively compact subset of $\mathcal{H}$, it has a convergent subsequence $\{Tf_{n_k}\}_{k\geqslant 0}$ converging to $g \in \mathcal{H}$. By the uniqueness of the weak limit, we must have $g = Tf$. This is not possible in view of (1.2.3), and hence, this contradiction proves the necessary part.

To prove the converse, it suffices to check that every sequence in $T(\overline{\mathbb{B}})$ has a convergent subsequence. To verify this, let $\{f_n\}_{n\geqslant 0}$ be a sequence in $\mathbb{B}$. By Theorem B.1.13 $\{f_n\}_{n\geqslant 0}$ has a weakly convergent subsequence, and hence by assumption, $\{Tf_n\}_{n\geqslant 0}$ has a convergent subsequence. This completes the proof. □

In the following Lemma, we need not apply our standing assumption of separability to the Hilbert space $\mathcal{H}$. The proof below works for any complex Hilbert space.

Lemma 1.2.2

The closed unit ball in a Hilbert space $\mathcal{H}$ is compact if and only if $\mathcal{H}$ is finite dimensional.

Proof If $\mathcal{H}$ is finite dimensional, then it is isometrically isomorphic to $\mathbb{C}^n$ for some positive integer n, and hence, by the Heine–Borel theorem, its closed unit ball is compact. Conversely, if $\mathcal{H}$ is infinite dimensional, then there exists an infinite orthonormal sequence $\{e_n\}_{n\geqslant 0}$ in $\overline{\mathbb{B}}$. The fact that

$$\|e_n - e_m\| = \sqrt{2} \text{ for } n \neq m$$

shows that the sequence $\{e_n\}_{n\geqslant 0}$ has no convergent subsequence. Thus, the closed unit ball $\overline{\mathbb{B}}$ is not compact. □

The following proposition says that the range of a compact operator cannot contain a closed subspace of infinite dimension.