

Contents

	<i>Preface</i>	page xxiii
	<i>User's Guide</i>	xxv
	<i>List of Topics and Phenomena</i>	xxviii
1	A Panorama of Lebesgue Integration	1
1.1	Modern Integration. 'Also zuerst: Was hat man unter $\int_a^b f(x) dx$ zu verstehen?'	1
1.2	The Idea Behind Lebesgue Integration	4
1.3	Lebesgue Essentials – Measures and σ -Algebras	6
1.4	Lebesgue Essentials – Integrals and Measurable Functions	10
1.5	Spaces of Integrable Functions	13
1.6	Convergence Theorems	17
1.7	Product Measure, Fubini and Tonelli	21
1.8	Transformation Theorems	24
1.9	Extension of Set Functions and Measures	27
1.10	Signed Measures and Radon–Nikodým	29
1.11	A Historical Aperçu From the Beginnings Until 1854	31
1.12	Appendix: H. Lebesgue's Seminal Paper	33
2	A Refresher of Topology and Ordinal Numbers	36
2.1	A Modicum of Point-Set Topology	36
2.2	The Axiom of Choice and Its Relatives	41
2.3	Cardinal and Ordinal Numbers	43
2.4	The Ordinal Space	46
2.5	The Cantor Set: A Nowhere Dense, Perfect Set	47
2.6	The Cantor Function and Its Inverse	49
3	Riemann Is Not Enough	55
3.1	The Riemann–Darboux upper integral is not additive	57

vi	<i>Contents</i>	
	3.2 Why one should define Riemann integrals on bounded intervals	58
	3.3 There are no unbounded Riemann integrable functions	58
	3.4 A function which is not Riemann integrable	59
	3.5 Yet another function which is not Riemann integrable	59
	3.6 A non-Riemann integrable function where a sequence of Riemann sums is convergent	60
	3.7 A Riemann integrable function without a primitive	61
	3.8 A Riemann integrable function whose discontinuity points are dense	62
	3.9 Semicontinuity does not imply Riemann integrability	63
	3.10 A function which has the intermediate value property but is not Riemann integrable	64
	3.11 A Lipschitz continuous function g and a Riemann integrable function f such that $f \circ g$ is not Riemann integrable	65
	3.12 The composition of Riemann integrable functions need not be Riemann integrable	65
	3.13 An increasing sequence of Riemann integrable functions $0 \leq f_n \leq 1$ such that $\sup_n f_n$ is not Riemann integrable	65
	3.14 A decreasing sequence of Riemann integrable functions $0 \leq f_n \leq 1$ such that $\inf_n f_n$ is not Riemann integrable	65
	3.15 Limit theorems for Riemann integrals are sub-optimal	66
	3.16 The space of Riemann integrable functions is not complete	67
	3.17 An example where integration by substitution goes wrong	68
	3.18 A Riemann integrable function which is not Borel measurable	68
	3.19 A non-Riemann integrable function f which coincides a.e. with a continuous function	69
	3.20 A Riemann integrable function on \mathbb{R}^2 whose iterated integrals are not Riemann integrable	69
	3.21 Upper and lower integrals do not work for the Riemann–Stieltjes integral	71
	3.22 The Riemann–Stieltjes integral does not exist if integrand and integrator have a common discontinuity	72
4	Families of Sets	73
	4.1 A Dynkin system which is not a σ -algebra	76
	4.2 A monotone class which is not a σ -algebra	77
	4.3 A σ -algebra which contains all singletons but no non-trivial interval	77

<i>Contents</i>		vii
4.4	There is no σ -algebra with $\#\mathcal{A} = \#\mathbb{N}$	78
4.5	A σ -algebra which has no non-empty atoms	78
4.6	An increasing family of σ -algebras whose union fails to be a σ -algebra	79
4.7	The union of countably many strictly increasing σ -algebras is never a σ -algebra	80
4.8	A countably generated σ -algebra containing a sub- σ -algebra which is not countably generated	81
4.9	Two countably generated σ -algebras whose intersection is not countably generated	82
4.10	A Borel σ -algebra which is not countably generated	83
4.11	$\sigma(\mathcal{G})$ can only separate points if \mathcal{G} does	84
4.12	A family \mathcal{G} of intervals whose endpoints form a dense subset of \mathbb{R} but $\sigma(\mathcal{G}) \subsetneq \mathcal{B}(\mathbb{R})$	84
4.13	Intersection and the σ -operation do not commute: $\sigma\left(\bigcap_{n \in \mathbb{N}} \mathcal{G}_n\right) \subsetneq \bigcap_{n \in \mathbb{N}} \sigma(\mathcal{G}_n)$	84
4.14	A metric space such that the σ -algebra generated by the open balls is smaller than the Borel σ -algebra	85
4.15	The σ -algebra generated by the compact sets can be larger than the Borel σ -algebra (compact sets need not be Borel sets)	85
4.16	The σ -algebra generated by the compact sets can be smaller than the Borel σ -algebra	86
4.17	A topology such that every non-empty Borel set has uncountably many elements	87
4.18	A metrizable and a non-metrizable topology having the same Borel sets	87
4.19	A σ -algebra which is not generated by any topology	89
4.20	A σ -algebra which is strictly between the Borel and the Lebesgue sets	91
4.21	The Borel sets cannot be constructed by induction	91
4.22	The Borel sets can be constructed by transfinite induction	95
4.23	(Non-)equivalent characterizations of the Baire σ -algebra	96
4.24	The Baire σ -algebra can be strictly smaller than the Borel σ -algebra	98
5	Set Functions and Measures	100
5.1	A class of measures where the $\mu(\emptyset) = 0$ is not needed in the definition	102
5.2	A set function which is additive but not σ -additive	102

5.3	A finite set function which is additive but not σ -additive	103
5.4	Another finite set function which is additive but not σ -additive	104
5.5	A set function with infinitely many extensions	105
5.6	A measure that cannot be further extended	105
5.7	A measure defined on the open balls which cannot be extended to the Borel sets	106
5.8	A signed pre-measure on an algebra \mathcal{R} which cannot be extended to a signed measure on $\sigma(\mathcal{R})$	106
5.9	A measure defined on a non-measurable set	107
5.10	A measure which is not continuous from above	108
5.11	A σ -finite measure which is not σ -finite on a smaller σ -algebra	108
5.12	A σ -finite measure μ on $\mathcal{B}(\mathbb{R})$ such that $\mu(I) = \infty$ for every non-trivial interval	108
5.13	A σ -finite measure μ on $\mathcal{B}(\mathbb{R})$ which is not a Lebesgue–Stieltjes measure	108
5.14	Infinite sums of finite measures need not be σ -finite	109
5.15	The image measure of a σ -finite measure is not necessarily σ -finite	109
5.16	A locally finite measure need not be σ -finite	109
5.17	Two measures on $\sigma(\mathcal{G})$ such that $\mu _{\mathcal{G}} \leq \nu _{\mathcal{G}}$ but $\mu \leq \nu$ fails	110
5.18	Two measures on $\sigma(\mathcal{G})$ such that $\mu _{\mathcal{G}} = \nu _{\mathcal{G}}$ but $\mu \neq \nu$	110
5.19	Two measures $\mu \neq \nu$ such that $\int p d\mu = \int p d\nu$ for all polynomials	111
5.20	Two finite measures $\mu \neq \nu$ whose Fourier transforms coincide on an interval containing zero	113
5.21	(Non)Equivalent definitions of the convolution of measures	114
5.22	The convolution of σ -finite measures need not be σ -finite	115
5.23	$\mu * \nu = \mu$ does not imply $\nu = \delta_0$	116
5.24	The push forward ‘disaster’ (image measures behaving badly)	117
5.25	The pull-back of a measure need not be a measure	118
5.26	A finite Borel measure which is not tight	119
5.27	A translation-invariant Borel measure which is not a multiple of Lebesgue measure	120
5.28	There is no Lebesgue measure in infinite dimension	121
6	Range and Support of a Measure	123
6.1	A measure where $\text{supp } \mu \neq \bigcap \{B; \mu(B^c) = 0\}$	124

<i>Contents</i>		ix
6.2	A measure which has no minimal closed support	124
6.3	Measures may have very small support	125
6.4	A measure μ such that the support of $\mu _{\text{supp } \mu}$ is strictly smaller than $\text{supp } \mu$	125
6.5	A measure with $\text{supp } \mu = \{c\}$ but $\mu \neq \delta_c$	126
6.6	Measures such that $\text{supp } \mu + \text{supp } \nu \subsetneq \text{supp } \mu * \nu$	126
6.7	Measures such that $\text{supp } \mu * \nu \subsetneq \overline{\text{supp } \mu + \text{supp } \nu}$	127
6.8	A signed measure such that $\text{supp } \mu^+ = \text{supp } \mu^-$	128
6.9	A two-valued measure which is not a point mass	128
6.10	A two-valued measure on a countably generated σ -algebra must be a point mass	129
6.11	(Non-)equivalent characterizations of atoms of a measure	130
6.12	A purely atomic measure such that $\mu \neq \sum_x \mu(\{x\})\delta_x$	131
6.13	A measure such that every set with positive measure is an atom	131
6.14	An infinite sum of atomic measures which is non-atomic	131
6.15	Any non-atomic finite σ -additive measure defined on $\mathcal{P}(\mathbb{R})$ is identically zero	132
6.16	A measure on a discrete space which attains all values in $[0, \infty]$	133
6.17	A measure whose range is not a closed set	133
6.18	A measure with countable range	134
6.19	A vector measure which is non-atomic but whose range is not convex	134
6.20	A non-trivial measure which assigns measure zero to all open balls	135
6.21	A signed measure $\mu: \mathcal{A} \rightarrow (-\infty, \infty]$ is uniformly bounded below	136
7	Measurable and Non-Measurable Sets	138
7.1	A dense open set in $(0, 1)$ with arbitrarily small Lebesgue measure	139
7.2	A set of positive Lebesgue measure which does not contain any interval	140
7.3	A Cantor-like set with arbitrary measure	140
7.4	An uncountable set of zero measure	141
7.5	A Lebesgue null set $A \subseteq \mathbb{R}$ such that for every $\delta \in [0, 1]$ there exist $x, y \in A$ with $\delta = x - y $	141
7.6	A dense open set whose complement has positive measure	144

x	<i>Contents</i>	
7.7	A compact set whose boundary has positive Lebesgue measure	144
7.8	A set of first category in $[0, 1]$ with measure one	144
7.9	A set of second category with measure zero	145
7.10	An uncountable, dense set of measure zero such that the complement is of first category	145
7.11	A null set which is not an F_σ -set	145
7.12	A Borel set which is neither F_σ nor G_δ	146
7.13	Each Borel set is the union of a null set and a set of first category	147
7.14	A set $B \subseteq \mathbb{R}$ such that $B \cap F \neq \emptyset$ and $B^c \cap F \neq \emptyset$ for any uncountable closed set F	147
7.15	A Borel set $B \subseteq \mathbb{R}$ such that $\lambda(B \cap I) > 0$ and $\lambda(B^c \cap I) > 0$ for all open intervals $I \neq \emptyset$	148
7.16	There is no Borel set B with $\lambda(B \cap I) = \frac{1}{2}\lambda(I)$ for all intervals I	148
7.17	A non-Borel set B such that $K \cap B$ is Borel for every compact set K	149
7.18	A convex set which is not Borel	149
7.19	A Souslin set which is not Borel	150
7.20	A Lebesgue measurable set which is not Borel measurable	153
7.21	A Lebesgue measurable set which is not a Souslin set	153
7.22	A non-Lebesgue measurable set	153
7.23	Arbitrary unions of non-trivial closed balls need not be Borel measurable	155
7.24	The image of a Borel set under a continuous mapping need not be Borel	155
7.25	The image of a Lebesgue set under a continuous mapping need not be Lebesgue measurable	157
7.26	The Minkowski sum $A + B$ of two Borel sets is not necessarily Borel	158
7.27	A Lebesgue null set B such that $B + B = \mathbb{R}$	158
7.28	The difference of fat Cantor sets contains an interval	159
7.29	The sum of scaled Cantor sets is sometimes an interval	161
7.30	The difference of fat Cantor sets is exactly $[-1, 1]$	161
7.31	The Banach–Tarski paradox	162
8	Measurable Maps and Functions	164
8.1	A measurable space where every map is measurable	165
8.2	A measurable space where only constant functions are measurable	165

<i>Contents</i>		xi
8.3	A non-measurable function whose modulus $ f $ is measurable	165
8.4	A non-measurable function whose level sets $\{x; f(x) = \alpha\}$ are measurable	165
8.5	A measurable function which is not μ -a.e. constant on any atom	165
8.6	A function $f(x, y)$ which is Borel measurable in each variable, but fails to be jointly measurable	166
8.7	Another function $f(x, y)$ which is Borel measurable in each variable, but fails to be jointly measurable	167
8.8	A function $f = (f_1, f_2)$ which is not measurable but whose components are measurable	168
8.9	The set of continuity points of any function f is Borel measurable	168
8.10	A set D for which there exists no function having D as its discontinuity set	170
8.11	A bijective measurable function f such that f^{-1} is not measurable	171
8.12	A continuous bijective function $f : [0, 1] \rightarrow [0, 1]$ which is not Lebesgue measurable	171
8.13	A Lebesgue measurable bijective map $f : \mathbb{R} \rightarrow \mathbb{R}$ whose inverse is not Lebesgue measurable	172
8.14	Borel measurable bijective maps have Borel measurable inverses	173
8.15	Sums and products of measurable functions need not be measurable	173
8.16	The limit of a sequence of measurable functions need not be measurable	174
8.17	A sequence of measurable functions such that the set $\{x; \lim_{n \rightarrow \infty} f_n(x) \text{ exists}\}$ is not measurable	175
8.18	The supremum of measurable functions need to be measurable	175
8.19	Measurability is not preserved under convolutions	176
8.20	The factorization lemma fails for general measurable spaces	177
8.21	A Lebesgue measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ for which there is no Borel measurable function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f \leq g$	178
8.22	A positive Borel measurable function which cannot be approximated a.e. from below by step functions	179

xii	<i>Contents</i>	
	8.23 $\mathbb{1}_{\mathbb{R} \setminus \mathbb{Q}}$ cannot be the pointwise limit of continuous functions	180
9	Inner and Outer Measure	182
	9.1 An explicit construction of a non-measurable set	185
	9.2 A set which is not Lebesgue measurable with strictly positive outer and zero inner measure	186
	9.3 A decreasing sequence $A_n \downarrow \emptyset$ such that $\lambda^*(A_n) = 1$	186
	9.4 A set such that $\lambda_*(E) = 0$ and $\lambda^*(E \cap B) = \lambda(B) = \lambda^*(B \setminus E)$ for all $B \in \mathcal{B}(\mathbb{R})$	187
	9.5 Lebesgue measure beyond the Lebesgue sets	188
	9.6 The Carathéodory extension λ^* of $\lambda _{[0,1]}$ is not continuous from above	189
	9.7 An outer measure which is not continuous from below	189
	9.8 A measure μ such that its outer measure μ^* is not additive	190
	9.9 A measure space such that $(X, \mathcal{A}^*, \mu^* _{\mathcal{A}^*})$ is not the completion of (X, \mathcal{A}, μ)	190
	9.10 A measure space where $\mu_*(E) = \mu^*(E)$ does not imply measurability of E	190
	9.11 A non-Lebesgue measurable set with identical inner and outer measure	191
	9.12 A measure such that every set is μ^* measurable	191
	9.13 A measure μ relative to \mathcal{S} such that every non-empty set in \mathcal{S} fails to be μ^* measurable	192
	9.14 An additive set function μ on a semi-ring such that μ^* is not an extension of μ	193
	9.15 An outer measure constructed on the intervals $[a, b)$ such that not all Borel sets are measurable	193
	9.16 There exist non- μ^* measurable sets if, and only if, μ^* is not additive on $\mathcal{P}(X)$	194
	9.17 An outer regular measure which is not inner compact regular	195
	9.18 An inner compact regular measure which is neither inner nor outer regular	195
	9.19 A measure which is neither inner nor outer regular	196
	9.20 A measure which is inner regular but not inner compact regular	197
	9.21 The regularity of a measure depends on the topology	197
	9.22 A regular Borel measure whose restriction to a Borel set is not regular	198

	<i>Contents</i>	xiii
10	Integrable Functions	202
10.1	An integrable function which is unbounded in every interval	203
10.2	A continuous integrable function such that $\lim_{ x \rightarrow \infty} f(x) \neq 0$	204
10.3	A continuous function vanishing at infinity which is not in L^p for any $p > 0$	205
10.4	A non-integrable function such that $\lim_{r \rightarrow \infty} r\mu(\{ f > r\}) = 0$	205
10.5	Characterizing integrability in terms of series	206
10.6	A non-integrable function such that $f(x - 1/n)$ is integ- rable for all $n \in \mathbb{N}$	207
10.7	An integrable function such that $f(x - 1/n)$ fails to be integrable for all $n \in \mathbb{N}$	208
10.8	An improperly Riemann integrable function which is not Lebesgue integrable	208
10.9	A function such that $\lim_{n \rightarrow \infty} \int_0^n f \, d\lambda$ exists and is finite but $\int_0^\infty f \, d\lambda$ does not exist	209
10.10	A function which is nowhere locally integrable	210
10.11	Integrable functions f, g such that $f \cdot g$ is not integrable	210
10.12	A function such that $f \notin L^p$ for all $p \in [1, \infty)$ but $fg \in L^1$ for all $g \in L^q, q \geq 1$	210
10.13	$f \in L^p$ for all $p < q$ does not imply $f \in L^q$	211
10.14	A function such that $f \in L^p$ for all $p < \infty$ but $f \notin L^\infty$	211
10.15	A function such that $f \in L^\infty$ but $f \notin L^p$ for all $p < \infty$	212
10.16	A function which is in exactly one space L^p	212
10.17	Convolution is not associative	213
10.18	An example where integration by substitution goes wrong	214
10.19	There is no non-constant function such that $\int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{R}^d} f(x+y) - f(x) y ^{-d-1} \, dx \, dy < \infty$	215
10.20	A measure space which has no strictly positive function $f \in L^1$	217
10.21	In infinite measure spaces there is no function $f > 0$ with $f \in L^1$ and $1/f \in L^1$	217
10.22	There is no continuous function $f \geq 0$ with $\int f^n \, d\lambda = 1$ for all $n \in \mathbb{N}$	218
10.23	A measure space where $\int_A f \, d\mu = \int_A g \, d\mu$ (for all A) does not entail $f = g$ a.e.	219
10.24	A vector function which is weakly but not strongly integrable	220

xiv	<i>Contents</i>	
11	Modes of Convergence	221
11.1	Classical counterexamples to a.e. convergence vs. convergence in probability	222
11.2	Pointwise convergence does not imply convergence in measure	223
11.3	L^p -convergence does not imply L^r -convergence for $r \neq p$	224
11.4	Classical counterexamples related to weak convergence in L^p	224
11.5	The convergence tables	225
11.6	The limit in probability is not necessarily unique	225
11.7	A sequence converging in probability without having an a.e. converging subsequence	227
11.8	A sequence converging in probability without having any subsequence converging in measure	228
11.9	A sequence such that $\int f_n(x) dx \rightarrow 0$ but $(f_n)_{n \in \mathbb{N}}$ has no convergent subsequence	229
11.10	A sequence converging a.e. and in measure but not almost uniformly	229
11.11	Egorov's theorem fails for infinite measures	229
11.12	Egorov's theorem does not hold for nets	229
11.13	A uniformly convergent sequence of L^1 -functions which is not convergent in L^1	231
11.14	Convergence in measure is not stable under products	231
11.15	A measure space where convergence in measure and uniform convergence coincide	232
11.16	A measure space where strong and weak convergence of sequences in L^1 coincide	233
11.17	Convergence a.e. is not metrizable	233
12	Convergence Theorems	235
12.1	Classical counterexamples to dominated convergence	236
12.2	Fatou's lemma may fail for non-positive integrands	236
12.3	Fatou's lemma may lead to a strict inequality	237
12.4	The monotone convergence theorem needs a lower integrable bound	237
12.5	A series of functions such that integration and summation do not interchange	238
12.6	Riesz's convergence theorem fails for $p = \infty$	239
12.7	A sequence such that $f_n \rightarrow 0$ pointwise but $\int_I f_n d\lambda \rightarrow \lambda(I)$ for all intervals	239

<i>Contents</i>		xv
12.8	$\int_I f_n d\lambda \rightarrow \int_I f d\lambda$ for all intervals I does not imply $\int_B f_n d\lambda \rightarrow \int_B f d\lambda$ for all Borel sets B	240
12.9	The classical convergence theorems fail for nets	242
12.10	The continuity lemma ‘only’ proves sequential continuity	243
12.11	A sequence f_n converging to 0 in L^1 without integrable envelope – the ‘sliding hump’	244
12.12	A sequence $(f_n)_{n \in \mathbb{N}}$ which is uniformly integrable but $\sup_n f_n $ is not integrable	244
12.13	A sequence which is not uniformly integrable but $f_n \rightarrow 0$ and $\int f_n d\lambda \rightarrow 0$	245
12.14	An L^1 -bounded sequence which is not uniformly integrable	245
12.15	A uniformly integrable sequence which does not converge in L^1	245
12.16	An L^1 -bounded sequence which fails to be uniformly integrable on any set of positive measure	246
13	Continuity and a.e. Continuity	247
13.1	An a.e. continuous function which does not coincide a.e. with any continuous function	248
13.2	A nowhere continuous function which equals a.e. a continuous function	248
13.3	A function f such that every g with $f = g$ a.e. is nowhere continuous	248
13.4	A function which is everywhere sequentially continuous but nowhere continuous	249
13.5	An a.e. continuous function whose discontinuity points are dense	249
13.6	An a.e. discontinuous function whose continuity points are dense	249
13.7	The composition of two a.e. continuous functions which is nowhere continuous	250
13.8	An a.e. continuous function which is not Borel measurable	251
13.9	A bounded Borel measurable function such that $f(x + 1/n) \rightarrow f(x)$ fails to hold on a set of positive measure	251
13.10	A nowhere constant function which is a.e. continuous and has countable range	252
13.11	A continuous function such that $f(x) \in \mathbb{Q}$ a.e. and f is not constant on any interval	252
13.12	A continuous function which is strictly positive on \mathbb{Q} but fails to be strictly positive almost everywhere	253

xvi	<i>Contents</i>	
	13.13 A measurable function which is zero almost everywhere but whose graph is dense	254
	13.14 A continuous function $f : [0, 1] \rightarrow \mathbb{R}^2$ whose image has positive Lebesgue measure	255
	13.15 The image of a Lebesgue null set under a continuous bijective mapping need not have Lebesgue measure zero	257
	13.16 Lusin's theorem fails for non-regular measures	257
	13.17 The convolution of two integrable functions may be discontinuous	258
14	Integration and Differentiation	261
	14.1 A non-Riemann integrable function f which has a primitive	262
	14.2 A function f which is differentiable, but f' is not integrable	263
	14.3 Volterra's version of Example 14.2	264
	14.4 A continuous function such that f' exists almost everywhere and is integrable but the fundamental theorem of calculus fails	265
	14.5 A continuous strictly increasing function with $f' = 0$ Lebesgue almost everywhere	266
	14.6 A continuous function f such that $f' > 1$ a.e. but f is not increasing on any interval	266
	14.7 A function which is Lebesgue almost everywhere differentiable but f' does not exist on a dense subset of \mathbb{R}	268
	14.8 $f_n \rightarrow f$ and $f'_n \rightarrow g$ pointwise does not imply $f' = g$ a.e.	268
	14.9 A function $f(t, x)$ for which $\partial_t \int f(t, x) dx$ and $\int \partial_t f(t, x) dx$ exist but are not equal	271
	14.10 A function such that $\partial_t \int f(t, x) dx$ exists but $\int \partial_t f(t, x) dx$ does not	271
	14.11 A function such that $\int \partial_t f(t, x) dx$ exists but $\partial_t \int f(t, x) dx$ does not	272
	14.12 A bounded function such that $t \mapsto f(t, x)$ is continuous but $t \mapsto \int f(t, x) \mu(dx)$ is not continuous	272
	14.13 An increasing continuous function ϕ and a continuous function f such that $\int_0^1 f(x) d\alpha(x) \neq \int_0^1 f(x) \alpha'(x) dx$	272
	14.14 A nowhere continuous function whose Lebesgue points are dense	273
	14.15 A discontinuous function such that every point is a Lebesgue point	273

	<i>Contents</i>	xvii
14.16	An integrable function f such that $x \mapsto \int_0^x f(t) dt$ is differentiable at $x = x_0$ but x_0 is not a Lebesgue point of f	274
14.17	Lebesgue points of f need not be Lebesgue points of f^2	275
14.18	Functions $f \in L^p$, $0 < p < 1$, without Lebesgue points	275
14.19	Lebesgue's differentiation theorem fails for sets which are not shrinking nicely	276
14.20	A measure for which Lebesgue's differentiation theorem fails	278
15	Measurability on Product Spaces	280
15.1	A function which is Borel measurable but not Lebesgue measurable	281
15.2	The product of complete σ -algebras need not be complete	281
15.3	$\mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R}^2)$	282
15.4	Sigma algebras $\mathcal{A} = \sigma(\mathcal{G})$ and $\mathcal{B} = \sigma(\mathcal{H})$ such that $\sigma(\mathcal{G} \times \mathcal{H})$ is strictly smaller than $\mathcal{A} \otimes \mathcal{B}$	282
15.5	An example where $\mathcal{P}(X) \otimes \mathcal{P}(X) \neq \mathcal{P}(X \times X)$	283
15.6	The product of Borel σ -algebras is not always the Borel σ -algebra of the product	284
15.7	Topological spaces X, Y such that $\mathcal{B}(X) = \mathcal{B}(Y)$ but $\mathcal{B}(X \times X) \neq \mathcal{B}(Y \times Y)$	285
15.8	$\mathcal{B}(X)^{\otimes I}$ is strictly smaller than $\mathcal{B}(X^I)$ for uncountable I	286
15.9	The diagonal $\Delta = \{(x, x); x \in X\}$ need not be measurable	288
15.10	A metric which is not jointly measurable	289
15.11	A non-measurable set whose projections are measurable	289
15.12	A measurable set whose projection is not measurable	289
15.13	A non-measurable set whose slices are measurable	290
15.14	A measurable function with a non-measurable graph	291
15.15	A non-measurable function with a measurable graph	291
15.16	A function $f(x, y)$ which is measurable in each variable but fails to be jointly measurable	291
15.17	A function $f(x, y)$ which is separately continuous in each variable but fails to be Borel measurable	292
15.18	An $\mathcal{A} \otimes \mathcal{B}$ measurable function $f \geq 0$ which cannot be approximated from below by simple functions of product form	293
16	Product Measures	295
16.1	Non-uniqueness of product measures	298
16.2	A measure on a product space which is not a product measure	299

16.3	The product of complete measure spaces need not be complete	299
16.4	A Lebesgue null set in $[0, 1]^2$ which intersects any set $A \times B$ whose Lebesgue measure is positive	299
16.5	A set $A \subseteq \mathbb{R}^2$ of positive Lebesgue measure which does not contain any rectangle	300
16.6	A set $A \subseteq \mathbb{R}^2$ of positive Lebesgue measure such that the intersection of every non-degenerate rectangle with A^c has positive measure	300
16.7	A set $A \subseteq \mathbb{R}^2$ of positive Lebesgue measure which is not a countable union of rectangles	302
16.8	A jointly measurable function such that $x \mapsto \int f(x, y) \mu(dy)$ is not measurable	302
16.9	A function $f(x, y)$ such that $f(\cdot, y)$ is \mathcal{A} measurable but $\int f(\cdot, y) dy$ is not \mathcal{A} measurable	302
16.10	Tonelli's theorem fails for non-positive integrands	304
16.11	A positive function with $f(x, y) = f(y, x)$ such that the iterated integrals do not coincide	305
16.12	A positive function $f(x, y)$ whose iterated integrals do not coincide	305
16.13	A finite measure μ and a Borel set B such that $\iint \mathbb{1}_B(x + y) \mu(dx) \lambda(dy) \neq \iint \mathbb{1}_B(x + y) \lambda(dy) \mu(dx)$	306
16.14	A non-measurable function $f(x, y)$ such that the iterated integral $\iint f(x, y) dx dy$ exists and is finite	307
16.15	A function $f(x, y)$ whose iterated integrals exist but do not coincide	308
16.16	A function $f(x, y)$ which is not integrable but whose iterated integrals exist and coincide	309
16.17	Yet another example where the iterated integrals exist, but the double integral doesn't	310
16.18	An a.e. continuous function $f(x, y)$ where only one iterated integral exists	311
16.19	Classical integration by parts fails for Lebesgue–Stieltjes integrals	311
16.20	A function which is $K(x, dy)$ -integrable but fails to be $\mu K(dy)$ -integrable	313
16.21	A consistent family of marginals which does not admit a projective limit	315

<i>Contents</i>		xix
17	Radon–Nikodým and Related Results	317
17.1	An absolutely continuous measure without a density	317
17.2	Another absolutely continuous measure without density	318
17.3	Yet another absolutely continuous measure without density	318
17.4	A not-absolutely continuous measure given by a density	319
17.5	A measure $\mu \ll \lambda$ such that $\lambda(A_n) \rightarrow 0$ does not imply $\mu(A_n) \rightarrow 0$	320
17.6	A measure μ which is absolutely continuous w.r.t. Lebesgue measure and $\mu(a, b) = \infty$ for any $(a, b) \neq \emptyset$	320
17.7	A continuous measure which is not absolutely continuous	321
17.8	An absolutely continuous function whose inverse is not absolutely continuous	321
17.9	A continuous measure with atoms	321
17.10	The Radon–Nikodým density $f = d\nu/d\mu$ does not necessarily satisfy $f(x) = \lim_{r \downarrow 0} \nu(B_r(x))/\mu(B_r(x))$	322
17.11	Lebesgue’s decomposition theorem fails without σ -finiteness	322
17.12	Two mutually singular measures which have the same support	322
17.13	A probability measure μ with full support such that μ and $\mu(c \cdot)$ are mutually singular for $c \neq 1$	322
17.14	The convolution of two singular measures may be absolutely continuous	324
17.15	Singular measures with full support – the case of Bernoulli convolutions	325
17.16	The maximum of two measures need not be the maximum of its values	329
18	Function Spaces	330
18.1	Relations between L^r, L^s, L^t if $r < s < t$	332
18.2	One may have $\ell^p(\mu) \subseteq \ell^q(\mu)$, or $\ell^p(\mu) \supseteq \ell^q(\mu)$, or no inclusion at all	334
18.3	A measure space where $L^p = \{0\}$ for all $0 \leq p < \infty$	336
18.4	A measure space where all spaces $L^p, 1 \leq p \leq \infty$ coincide	336
18.5	A measure space where $L^1 \not\subseteq L^\infty$	337
18.6	$L^1(\mu) = L^\infty(\mu)$ if, and only if, $1 \leq \dim(L^1(\mu)) < \infty$	337
18.7	A function where $\sup_{x \in U} f(x) \neq \ f\ _{L^\infty(U)}$ for any open set U	340
18.8	One cannot compare L^p -norms on $C[0, 1]$	341
18.9	The spaces L^p with $0 < p < 1$ are only quasi-normed spaces	341
18.10	The spaces L^p with $0 < p < 1$ are not locally convex	343

18.11	The dual of $L^p(\lambda)$ with $0 < p < 1$ is trivial	344
18.12	Functions $f \in L^p$, $0 < p < 1$, need not be locally integrable	345
18.13	The spaces L^q with $q < 0$ are not linear spaces	346
18.14	A measure space where L^p is not separable	346
18.15	Separability of the space L^∞	347
18.16	$C_b(X)$ need not be dense in $L^p(\mu)$	349
18.17	A subset of L^p which is dense in L^r , $r < p$, but not dense in L^p	350
18.18	L^p is not an inner product space unless $p = 2$ or $\dim(L^p) \leq 1$	351
18.19	The condition $\sup_{\ g\ _{L^q} \leq 1} \int fg d\mu < \infty$ need not imply that $f \in L^p(\mu)$	352
18.20	Identifying the dual of L^p with L^q is a tricky business	354
18.21	The dual of L^1 can be larger than L^∞	355
18.22	The dual of L^1 can be isometrically isomorphic to a space which is strictly smaller than L^∞	357
18.23	A measure space such that the dual of L^1 is L^1	358
18.24	The dual of L^∞ can be larger than L^1	358
18.25	A measure space where the dual of L^∞ is L^1	359
18.26	Non-uniqueness in the Riesz representation theorem	360
18.27	Non-uniqueness in the Riesz representation theorem II	360
18.28	A measure space where L^∞ is not weakly sequentially complete	361
18.29	Uniform boundedness does not imply weak compactness in L^1	363
18.30	The algebra $L^1(\lambda^d)$ does not have a unit element	364
18.31	The algebra $L^1(\lambda^d)$ contains non-trivial divisors of zero	364
18.32	Uniform convexity/rotundity of L^p	365
18.33	An absolutely continuous measure such that the translation operator is not continuous in L^1	366
18.34	There is no Bochner integral in spaces which are not locally convex	367
19	Convergence of Measures	370
19.1	Classical counterexamples related to vague and weak convergence	373
19.2	Vague convergence does not preserve mass	375
19.3	Vague convergence of positive measures $\mu_n \rightarrow \mu$ does not imply $ \mu_n - \mu \rightarrow 0$	375
19.4	Vague convergence $\mu_n \rightarrow 0$ does not entail vague convergence $ \mu_n \rightarrow 0$	375

<i>Contents</i>	xxi
19.5 Vague convergence does not imply $\mu_n(B) \rightarrow \mu(B)$ for all Borel sets	376
19.6 A sequence of absolutely continuous measures which converges weakly to λ on $[0, 1]$ but $\mu_n(B) \rightarrow \lambda(B)$ fails for some Borel set $B \subseteq [0, 1]$	376
19.7 A sequence of measures μ_n such that $\lim_{n \rightarrow \infty} \int f d\mu_n$ exists, but is not of the form $\int f d\mu$	376
19.8 Weakly convergent sequences need not be tight	377
19.9 Signed measures μ_n such that $\int f d\mu_n \rightarrow \int f d\mu$ for all $f \in C(\mathbb{R})$ but $\mu_n(B) \rightarrow \mu(B)$ fails for sets with $\mu(\partial B) = 0$	377
19.10 Signed measures μ_n such that $\int f d\mu_n \rightarrow \int f d\mu$ for all bounded uniformly continuous functions f but μ_n does not converge weakly to μ	378
19.11 A sequence of measures which does not converge weakly but whose Fourier transforms converge pointwise	378
19.12 Lévy's continuity theorem fails for nets	379
19.13 A sequence of non-atomic measures converging weakly to a purely atomic measure	380
19.14 A sequence of purely atomic measures converging weakly to a non-atomic measure	380
19.15 A net of Dirac measures converging weakly to a non-Dirac measure	381
19.16 $f_n \mu \rightarrow f \mu$ weakly does not imply $f_n \rightarrow f$ in probability	381
19.17 $f_n \mu \rightarrow f \mu$ weakly does not imply $f_n \rightarrow f$ weakly in $L^1(\mu)$	382
19.18 $f_n \rightarrow f$ weakly in $L^p(\mu)$ for $p > 1$ does not imply $f_n \mu \rightarrow f \mu$ weakly	383
<i>References</i>	385
<i>Index</i>	394