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COUNTEREXAMPLES IN MEASURE AND INTEGRATION

Often it is more instructive to know 'what can go wrong' and to understand 'why a result fails' than to plod through yet another piece of theory. In this text, the authors gather more than 300 counterexamples – some of them both surprising and amusing – showing the limitations, hidden traps and pitfalls of measure and integration. Many examples are put into context, explaining the relevant parts of the theory, and pointing out further reading.

The text starts with a self-contained, non-technical overview on the fundamentals of measure and integration. A companion to the successful undergraduate textbook *Measures, Integrals and Martingales*, it is accessible to advanced undergraduate students, requiring only modest prerequisites. More specialized concepts are briefly summarized at the beginning of each chapter, allowing for self-study as well as supplementary reading for any course covering measures and integrals. For researchers, the text provides ample examples and warnings as to the limitations of general measure theory.

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Preface

A **counterexample** /ˈkaʊntərɪɡˌzɑːmpl/ is *an example that opposes or contradicts an idea or theory.*[1] It is fair to say that the word 'counterexample' is not too common in everyday language, but rather a concept from philosophy and, of course, mathematics. In mathematics, there are proofs and examples, and while an example, say, of some $x \in A$ satisfying $x \in B$ does not prove $A \subseteq B$, the *counter*example of some $x_0 \in B$ such that $x_0 \notin A$ disproves $A \subseteq B$; in other words, it proves that $A \subseteq B$ does not hold. This observation shows that there is no sharp distinction between example and counterexample, and we do not give a definition of what a counterexample should or could be (you may want to consult Lakatos [94] instead), but assume the more pragmatic point of view of a working mathematician. If we want to solve a problem, we look at the same time for a proof and for counterexamples which help us to capture and delineate the subject matter.

The same is also true for the student of mathematics, who will gain a better understanding of a theorem or theory if he knows its limitations – which may be expressed in the form of counterexamples. The present collection of (counter-)examples grew out of our own experience, in the classroom and on `stackexchange.com`, where we are often asked after the 'how' and 'why' of many a result. This explains the wide range of examples, from the fairly obvious to rather intricate constructions. The choice of the examples reflects, naturally, our own taste. We decided to include only those counterexamples which could be dealt with in a couple of pages (or less) and which are not too pathological – one can, indeed, destroy almost anything by the choice of the underlying topology. We intend the present volume as a companion to our textbook *Measures, Integrals and Martingales* [MIMS], which means that most examples are from elementary measure and integration, not touching on integration on

[1] Oxford dictionaries `https://en.oxforddictionaries.com/definition/counterexample`, accessed 11-May-2019.

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groups (Haar measure) or on really deep axiomatic issues (e.g. as in descriptive set theory, see Kechris [89], and the advanced constructive theory of functions, see Kharazishvili [91, 92]).

This book is intended as supplementary reading for a course in measure and integration theory, or for seminars and reading courses where students can explore certain aspects of the theory by themselves. Where appropriate, we have added comments putting the example into context and pointing the reader to further literature. We think that this book will also be useful for lecturers and tutors in teaching measure and integration, and for researchers who may discover new and sometimes unexpected phenomena. Readers are assumed to have basic knowledge of functional analysis, point-set topology and, of course, measure and integration theory. For novices, there is a panorama of measure and integration which gives a non-technical overview on the subject and can serve, to some extent, as a first introduction. The overall presentation is as self-contained as possible; in order to make the text easy to access, we use only a few standard references – Schilling [MIMS] and Bogachev [19] for measure and integration, Rudin [151] and Yosida [202] for functional analysis, and Willard [200] and Engelking [53] for topology.

Some of the counterexamples are famous, many are more or less well known, and a few are of our own making. When we could trace the origin of an example, we have given references and attached names, but most entries are 'standard' examples which seem to have been in the public domain for ages; having said this, we acknowledge a huge debt to many anonymous authors and we do apologize if we have failed to give proper credit. The three classic counterexample books by Gelbaum & Olmsted [65], Steen & Seebach [172], and Stoyanov [180] were both inspiration and encouragement. We hope that this book lives up to their high standards.

It is a pleasure to acknowledge the interest and skill of our editor, Roger Astley, in the preparation of this book and Cambridge University Press for the excellent book design. Many colleagues have contributed to this text with comments and suggestions, in particular M. Auer, R. Baumgarth, G. Berschneider, N.H. Bingham – for the famous full red-ink treatment, C.-S. Deng, D.E. Edmunds and C. Goldie – for most helpful discussions, Y. Ishikawa, N. Jacob – for access to his legendary library, Y. Mishura and N. Sandrić. We thank our colleagues and friends who suffered for quite a while from our destructive search for counterexamples (*Do you know an example of a measure which fails to … ?*), strange functions and many outer-worldly excursions – and our families who have us back in real life.

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User's Guide

This book is not intended for linear reading – although this might well be possible – but invites the reader to browse, to read selectively and to look things up. We have, therefore, organized the material in self-contained chapters which treat different aspects of measure and integration theory. We assume that the reader has a basic knowledge of abstract measure and integration; the outline given in the 'panorama' (Chapter 1) is intended to refresh the reader's memory, to fix notation and to give a first non-technical introduction to the subject. The cross-reference [☞ $n.m$] appearing in the margin points towards essential counterexamples to the (positive) result at hand. Some supplementary material which is not always part of the mathematical curriculum is collected in Chapter 2; look it up once you need it. ☞ $n.m$

Cross-referencing. Throughout the text, [☞ $n.m$] and Example $n.m$ refers to counterexample m in Chapter n. Theorem $n.m$, Definition $n.m$, etc. point to the respective theorem, definition, etc. in the 'panorama' (Chapter 1) or the 'refresher' (Chapter 2). Equation m in Chapter n is denoted by ($n.m$). At the beginning of each chapter, we recall more specialized results and definitions which are particular to that chapter; these are numbered locally as $5A$, $5B$, $5C$, ... (for Chapter 5, say) and they are mostly used within that chapter. Theorems, lemmas and corollaries may also appear in a counterexample; if needed, we use again local numbering 1, 2, 3, ...

Finding stuff. Following Gelbaum & Olmsted [65] we have organized the examples by theme and all counterexamples are listed in the list of contents by (hopefully) meaningful names. We begin with examples on Riemann integration (Chapter 3) and move on to various aspects of the (abstract) Lebesgue integral (Chapters 4–19). The chapters on Lebesgue integration follow 'The way

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of integration' (alluding to Fig. 1.3 in Chapter 1), i.e. beginning with measurable sets and σ-algebras to set functions, measurable functions, to integrals and theorems on integration. The subject index helps to find definitions, theorems and concepts, but it does not refer to specific counterexamples.

Notation. We tried to avoid specialized notation and we use commonly accepted standard notation, e.g. as in [MIMS]. The following list is intended to aid cross-referencing, so notation that is specific to a single section is generally not listed; numbers following entries are page numbers.

Unless otherwise stated, binary operations between functions such as $f \pm g$, $f \cdot g$, $f \wedge g$, $f \vee g$, comparisons $f \leqslant g$, $f < g$ or limiting relations $f_n \xrightarrow[n\to\infty]{} f$, $\lim_n f_n$, $\liminf_n f_n$, $\limsup f_n$, $\sup_i f_i$ or $\inf_i f_i$ are always understood pointwise.

General notation

positive	always in the sense $\geqslant 0$
negative	always in the sense $\leqslant 0$
increasing	$x \leqslant y \Rightarrow f(x) \leqslant f(y)$
decreasing	$x \leqslant y \Rightarrow f(x) \geqslant f(y)$
countable	finite or countably infinite
$\mathbb{N}$	natural numbers: $1, 2, 3, \dots$
$\mathbb{N}_0$	positive integers: $0, 1, 2, \dots$
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	integer, rational, real, complex numbers
$\overline{\mathbb{R}}$	$[-\infty, +\infty]$ (two-point compactification), 11, 38
$\inf \emptyset$, $\sup \emptyset$	$\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$
$a \vee b$, $a \wedge b$	$\max\{a, b\}$, $\min\{a, b\}$
$\gcd(\cdot, \cdot)$	greatest common divisor
$\aleph_0$	cardinality of $\mathbb{N}$, 44
$\mathfrak{c}$	cardinality of $\mathbb{R}$, 44
ω_0	first infinite ordinal, ordinal number of $\mathbb{N}$, 44
ω_1	first uncountable ordinal, 45, 46
$\Omega = [0, \omega_1]$	ordinal space, 45, 46
$\Omega_0 = [0, \omega_1)$	countable ordinals, 45, 46

Sets and set operations

$A \uplus B$	union of disjoint sets
$A \vartriangle B$	$(A \setminus B) \uplus (B \setminus A)$
A^c	complement of A
$\overline{A}$	closure of A, 37
A°	open interior of A, 37
$A_n \uparrow A$	$A_n \subseteq A_{n+1}$, $A = \bigcup_n A_n$
$A_n \downarrow A$	$A_n \supseteq A_{n+1}$, $A = \bigcap_n A_n$
$\#A$	cardinality of A
$B_r(x)$	open (metric) ball $\{y\,;\ d(x, y) < r\}$

Families of sets

$\mathscr{A}, \mathscr{B}, \mathscr{C}$	generic families of sets
$\mathscr{A}^*$	μ^* measurable sets, 28
	completion, 9
$\mathscr{A} \otimes \mathscr{B}$	product σ-algebra, 10, 21
$\mathscr{B}(X)$	Borel sets in X, 9
$\mathscr{L}(X)$	Lebesgue sets in X, 10
$\mathscr{O}(X)$	open sets in X, 36
$\mathscr{P}(X)$	all subsets of X
$\sigma(\mathscr{F})$	σ-algebra generated by $\mathscr{F}$, 9
$\sigma(\phi)$, $\sigma(\phi_i, i \in I)$	σ-algebra generated by the map(s) ϕ, resp. ϕ_i, 9

Measures and integrals

μ, ν	generic (positive) measures
μ_*, μ^*	inner and outer measure, 182, 100
δ_x	Dirac measure in x, 8
λ, λ^d	Lebesgue measure, 9
$\zeta, \zeta_X, \#(\cdot)$	counting measure on X, 8
$\mu \circ f^{-1}, f_*\mu$	image or push-forward measure, 8, 24

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$\mu \times \nu$	product of measures, 10, 21
$\mu * \nu$	convolution, 26
$\mu \ll \nu$	absolute continuity, 29
$\mu \perp \nu$	singular measures, 29
$\frac{d\nu}{d\mu}$	Radon–Nikodým derivative, 29
$\operatorname{supp} \mu$	support of a measure, 123
$\overline{\int}$, $\underline{\int}$	upper, lower R-integral, 2

Functions and spaces

$\mathbb{1}_A$	$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases}$
$\operatorname{sgn}(x)$	$\mathbb{1}_{(0,\infty)}(x) - \mathbb{1}_{(-\infty,0)}(x)$
$f(A)$	$\{f(x)\,;\ x \in A\}$
$f^{-1}(\mathscr{B})$	$\{f^{-1}(B)\,;\ B \in \mathscr{B}\}$
f^+	$\max\{f(x), 0\}$ positive part
f^-	$-\min\{f(x), 0\}$ negative part
$\{f \in B\}$	$\{x\,;\ f(x) \in B\}$
$\{f \geqslant \lambda\}$	$\{x\,;\ f(x) \geqslant \lambda\}$, etc.
$f * g$	convolution, 26
$\operatorname{supp} f$	support $\overline{\{f \neq 0\}}$
$C(X)$	continuous functions on X
$C_b(X)$	bounded — —
$C_c(X)$	— — with compact support
L^p, L^∞	Lebesgue spaces, 15
$\|f\|_p, \|f\|_{L^p}$	$(\int \|f\|^p \, d\mu)^{1/p}$, $p < \infty$
$\|f\|_\infty, \|f\|_{L^\infty}$	$\operatorname{esssup} f :=$ $\inf\{c\,;\ \mu\{\|f\| \geqslant c\} = 0\}$, 14

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List of Topics and Phenomena

Topic	Possible consequence	Example
μ is not finite	▶ μ not continuous from above	[☞ 5.10]
	▶ range of μ not closed	[☞ 6.17]
	▶ Jensen's inequality does not hold	
	▶ $p \leqslant q \not\Rightarrow L^q \subseteq L^p$	[☞ 18.1]
	▶ no series test for integrability	[☞ 10.5]
	▶ Egorov's theorem fails	[☞ 11.11]
	▶ convergence in probability $\not\Rightarrow$ convergence in measure	[☞ 11.5]
μ is not σ-finite	▶ no unique product measure	[☞ 16.1]
	▶ Fubini's and Tonelli's theorem fail	[☞ 16.8–16.18]
	▶ Radon–Nikodým's theorem fails	[☞ 17.1–17.3]
	▶ Lebesgue's decomposition theorem fails	[☞ 17.11]
	▶ there is no positive integrable function	[☞ 10.20]
	▶ limits in probability not unique	[☞ 11.6]
	▶ $f_n \to f$ in probability $\not\Rightarrow f_{n_k} \to f$ a.e.	[☞ 11.7]
	▶ L^p, $1 \leqslant p < \infty$, not separable	[☞ 18.14]
	▶ $(X, \mathscr{A}^*, \mu^*\|_{\mathscr{A}^*}) \neq$ completion of $(X, \mathscr{A}, \mu)$	[☞ 9.9]
	▶ trace of a regular measure not regular	[☞ 9.22]
μ does not have the finite subset property	▶ $\int_A f\,d\mu = \int_A g\,d\mu$ for all $A \not\Rightarrow f = g$ a.e.	[☞ 10.23]
	▶ $\exists f \in L^\infty$ s.t. $\Lambda_f(g) = \int fg\,d\mu$, $g \in L^1$, satisfies $\\|\Lambda_f\\| < \\|f\\|_{L^\infty}$	[☞ 18.21]
	▶ $\int \|fg\|\,d\mu \leqslant C\\|g\\|_{L^q} \not\Rightarrow f \in L^p$	[☞ 18.19]
μ is not locally finite	▶ $C_b(X) \cap L^p(\mu)$ not dense in $L^p(\mu)$, $1 \leqslant p < \infty$	[☞ 18.16]
μ is not regular	▶ Lusin's theorem fails	[☞ 13.16]
	▶ $C_b(X) \cap L^p(\mu)$ not dense in $L^p(\mu)$, $1 \leqslant p < \infty$	[☞ 18.16]
	▶ there exists $\nu \neq \mu$ s.t. $\int f\,d\mu = \int f\,d\nu$ for all $f \in C_c(X)$	[☞ 18.26, 18.27]

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Topic	Possible consequence	Example
X is not separable	▶ $\mathscr{B}(X)$ not generated by open balls ▶ $\mathscr{B}(X)$ not countably generated ▶ $\operatorname{supp}\mu \neq$ smallest closed set F such that $\mu(X \setminus F) = 0$ ▶ $\mu(X) \neq \mu(\operatorname{supp}\mu)$ ▶ finite measures not tight	[☞ 4.14] [☞ 4.10] [☞ 6.2] [☞ 6.3] [☞ 5.26]
X is not a metric space	▶ compact sets not Borel ▶ fewer Baire sets than Borel sets ▶ pointwise limits of measurable functions not measurable ▶ finite measures not outer regular ▶ inner compact regular $\not\Rightarrow$ inner regular	[☞ 4.15] [☞ 4.24] [☞ 8.16] [☞ 9.18] [☞ 9.18]
X is not σ-compact	▶ locally finite $\not\Rightarrow$ σ-finite ▶ inner regular $\not\Rightarrow$ inner compact regular	[☞ 5.16] [☞ 9.20]
X is not locally convex	▶ only trivial dual space ▶ no Bochner integral	[☞ 18.11] [☞ 18.34]
X has cardinality $> \mathfrak{c}$	▶ the diagonal is not in $\mathscr{P}(X) \otimes \mathscr{P}(X)$ ▶ $\mathscr{B}(X) \otimes \mathscr{B}(X) \neq \mathscr{B}(X \times X)$ ▶ metric not jointly measurable with respect to $\mathscr{B}(X) \otimes \mathscr{B}(X)$	[☞ 15.9] [☞ 15.6] [☞ 15.10]
$\mathscr{A}$ is too small, e.g. discrete	▶ 'few' measurable functions $f : X \to \mathbb{R}$ ▶ factorization lemma fails	[☞ 8.2] [☞ 8.20]
$\mathscr{A}$ is too big, e.g. discrete	▶ 'many' measurable functions $f : X \to \mathbb{R}$ ▶ 'few' non-atomic measures	[☞ 8.1] [☞ 6.15]
$\mathscr{A}$ not countably generated	▶ two-valued measures which are not a point mass	[☞ 6.10]
role of 'small' sets	▶ Lebesgue null sets may be uncountable/of second category ▶ $B + B = \mathbb{R}$ for a Lebesgue null set B ▶ $2^{\mathfrak{c}}$ many Lebesgue sets but 'only' $\mathfrak{c}$ many Borel sets ▶ $f' = 0$ a.e. $\not\Rightarrow$ f constant ▶ f a.e. continuous $\not\Rightarrow$ $f = g$ a.e. for g continuous ▶ $\mu_n \to \mu$ weakly $\not\Rightarrow$ $\mu_n(B) \to \mu(B)$ for all B ▶ support of a probability measure may have measure 0	[☞ 7.4, 7.9] [☞ 7.27] [☞ 4.20] [☞ 2.6, 14.5] [☞ 13.1] [☞ 19.5] [☞ 6.3]
lack of countability	▶ many theorems fail for nets, e.g. classical convergence theorems, Egorov's and Lévy's continuity theorem ▶ uncountable supremum of measurable functions are not measurable ▶ $\mathscr{B}(X)^{\otimes I}$ is 'small' for I uncountable	[☞ 12.9] [☞ 11.12, 19.12] [☞ 8.18] [☞ 15.8, 4.17]

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	▶ projective limit of consistent family may not exist	[☞ 16.21]
	▶ $t \longmapsto f(t,x)$ cts. $\forall x$ $\not\Rightarrow t \longmapsto \int f(t,x)\,\mu(dx)$ cts.	[☞ 14.12]
lack of uniform integrability	▶ $(f_n)_{n\in\mathbb{N}} \subseteq L^1$, $f_n \to 0$ a.e. $\not\Rightarrow \int f_n \to 0$	[☞ 12.1, 12.7]
	▶ $f_n \to f$, $f_n' \to g$ everywhere $\not\Rightarrow f' = g$ a.e.	[☞ 14.8]
	▶ $f_n \to f$ in probability $\not\Rightarrow f_{n_k} \to f$ in measure	[☞ 11.8]
	▶ sequential weak compactness fails	[☞ 18.29]
L^1, L^∞ are special	▶ L^∞ separable if, and only if, $\dim L^\infty < \infty$	[☞ 18.15]
	▶ $(L^1)^* \supsetneqq L^\infty$	[☞ 18.21, 18.22]
	▶ $(L^\infty)^* \supsetneqq L^1$	[☞ 18.24]
	▶ not uniformly convex	[☞ 18.32]
⚠ atom	▶ comparison: different definitions of atom	[☞ 6.11]
⚠ absolute continuity	▶ comparison: different definitions of absolute continuity	[☞ 17.4]
⚠ convergence in measure	▶ comparison: convergence in measure vs. in probability	[☞ p. 20 Fig. 1.4 pp. 221, 226–227]
⚠ weak convergence	▶ weak convergence of measures is not weak convergence in the sense of functional analysis	[☞ p. 371, 19.7]
⚠ Baire σ-algebra	▶ comparison of different definitions of Baire sets	[☞ 4.23]

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