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Spaces of Measures and their Applications to Structured Population Models

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Preface

The idea of this book was developed over several years of joint research and numerous discussions with colleagues whose focus is in functional analysis, partial differential equations, transport processes, probability theory or mathematical modelling.

The development of the mathematical framework for the analysis of structured population models in measure spaces presented in this book was supported by many projects, starting with the Emmy Noether Project funded by the German Research Council (DFG) and by the European Research Council Starting Grant for Anna Marciniak-Czochra and the International PhD Programme “Mathematical Methods in Natural Sciences” (Foundation for Polish Science) coordinated by Piotr Gwiazda. The latter led to several PhD projects that accelerated the progress of development of analytic and numerical theory. We express thanks to Agnieszka Ulikowska and Jędrzej Jabłoński, for significant contributions.

Many people have contributed to the development of this comprehensive framework. The range of individuals involved is far too wide to be able to list them all. In particular, however, we would like to thank our colleagues and collaborators José A. Carrillo, Rinaldo Colombo, Karolina Kropielnicka, Sander C. Hille, Marie Doumic, Benoît Perthame, Agnieszka Świerczewska-Gwiazda, Emil Wiedemann, Tomasz Dębiec, Horst R. Thieme, Azmy S. Ackleh, Nicolas Saintier and Tomasz Szarek, who inspired us and helped in clarifying analytical concepts and anchoring them in relevant areas of analysis and probability theory.

Many aspects of the book have been influenced by insightful questions and suggestions from the students who worked through the book as a part of the Master’s course “Spaces of Radon Measures: Functional Analysis and Applications to PDEs” given at Heidelberg University in the summer term of 2020. The course was part of the research-oriented teaching programme of the Heidelberg STRUCTURES Excellence Cluster supported by Deutsche Forschungsgemeinschaft (DFG) under Germany's Excellence Strategy EXC-2181/1 - 390900948. It attracted mathematics and scientific computing Master's students interested in functional analysis and its applications to partial differential equations, especially in the context of the modelling of transport and growth processes. Teaching the subject helped significantly to improve the quality of the book. The students detected and helped us to remove many errors, asked interesting questions, made careful comments and frequently demanded clarifying examples. We want to express our gratitude for help to Joris Edelmann, Alina Feldmann, Lukas Huber, Sumet Khumphairan, Finn Münnich, Tuan Tung Nguyen, Achita Prasertwaree, Camillo Tissot, Kay Zaradny and Mikuláš Zindulka.

The final shape of the book has profited from fruitful discussions with Marek Kimmel and the constructive criticism and suggestions we received from anonymous reviewers and colleagues from the Heidelberg Group on Applied Analysis and Modeling in Biosciences, especially Maria V. Barbarossa and Johannes Kammerer. We also acknowledge the support of our friends Zuzanna Szymańska and Błażej Miasojedow from Warsaw, who detected glitches in our numerical pseudo-codes. During preparation of this book, Piotr Gwiazda and Jakub Skrzeczkowski were supported by the National Science Center, Poland, through projects nos. 2018/30/M/ST1/00423 and 2019/35/N/ST1/03459, respectively.

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AMS Subject Classification: 26A16, 28A33, 35B30, 35F10, 35L50, 35R02, 65M12.

Notation

General Notation

We use the following notation for sets. Let (S, d) be a metric space. Then

$\mathcal{B}(S)$, the Borel σ-algebra of S, i.e. the σ-algebra generated by the open sets of S.
$B_r(s) := \{x \in S \mid d(x, s) \leq r\}$, the open set with radius r and centre s.
$\mathbb{N} = \{1, 2, 3, \ldots\}$.
$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.
$\mathbb{R}^+ = [0, \infty)$.
$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.
$\mathbb{S}^d$, the unit sphere $\{x \in \mathbb{R}^{d+1} \mid \|x\| = 1\}$.

In general, we will use a superscript + to indicate nonnegativity.

Special Functions and Spaces

Here we present a selection of spaces and functions that appear several times throughout the book and whose definition the reader might want to look up once in a while. Note that special spaces that occur only locally in one section are not listed here.

$\mathbf{1}$, constant function $\mathbf{1}(x) = 1$ for all x.
$\mathrm{BC}^{\alpha,1}([0,T] \times \mathcal{M}^+(\mathbb{R}^+); X)$, the space of X-valued, bounded functions defined on $[0,T] \times \mathcal{M}^+(\mathbb{R}^+)$ that are α-Hölder continuous in t and Lipschitz continuous in $\mathcal{M}^+(\mathbb{R}^+)$, see Section 3.5.
$\mathrm{BL}(S)$, the space of bounded Lipschitz functions $S \to \mathbb{R}$.
$\mathrm{BL}(S)^*_+$, the space of positive linear functionals on $\mathrm{BL}(S)$.

$C_b(S)$, the space of bounded continuous functions $S \to \mathbb{R}$.
$C_c(S)$, the space of continuous functions $S \to \mathbb{R}$ with compact support in S.
$C^k(S)$, the space of k-times continuously differentiable functions $S \to \mathbb{R}$.
$C_{ub}(S)$, the space of uniformly continuous and bounded functions $S \to \mathbb{R}$.
$C_0(S) = \overline{C_c^0(S)}^{\|\cdot\|_\infty}$, see Definition 1.3.
$C^0(S)$, the space of continuous functions $S \to \mathbb{R}$.
$C^0([0,T]; (\mathcal{M}^+(S), \rho_F))$, the space of continuous maps from the interval $[0,T]$ to $(\mathcal{M}^+(S), \rho_F)$. A natural metric on such a space is defined in (3.23).
χ_T, characteristic function of a set T, $\chi_T(x) = \begin{cases} 1, & x \in T \\ 0, & x \notin T \end{cases}$.
$E = \overline{\mathcal{M}(S)}^{\|\cdot\|_{BL^*}}$.
Id, the identity map $x \mapsto x$.
$L^1([0,T], \mathrm{BL}(\mathbb{R}^d))$, the Bochner space of functions with L^1 regularity in time and BL regularity in space, cf. (3.1).
L^p, the usual space of functions whose pth power is integrable.
$\mathbf{Lip}_1(S)$, the space of 1-Lipschitz functions $S \to \mathbb{R}$.
$\mathcal{L}(X,Y)$, the space of bounded linear operators from X to Y.
$\mathcal{M}(S)$, the space of finite signed Radon measures.
$\mathcal{M}^+(S)$, the space of nonnegative finite signed Radon measures.
$\mathcal{P}(S)$, the set of probability measures on S.
$\mathcal{S}_{t,s}$, a semigroup, see Definition I.4.
$W^{k,p}$, the usual Sobolev space of k-times weakly differentiable functions with all derivatives in L^p.

Norms and Seminorms

To keep track of the various norms and seminorms that are introduced in the book, we provide a list for the convenience of the reader.

$|\cdot|_{\mathbf{Lip}}$, Lipschitz constant, see Definition 1.1.
$\|\cdot\|_\infty$, supremum norm, see Definition 1.1.
$\|\cdot\|_{\mathrm{BL}}$, bounded Lipschitz norm, see Definition 1.1.
$\|\cdot\|_{\mathrm{TV}}$, total variation norm, see Definition 1.19.
$\|\cdot\|_{\mathrm{BL}^*}$, flat (or dual bounded Lipschitz) norm, see Definition 1.22.
$\rho_F(\cdot,\cdot)$, the flat metric induced by the flat norm.
$W_p(\cdot,\cdot)$, pth Wasserstein distance, see Definition 1.77.
$[\cdot]_\alpha$, Hölder seminorm, see Definition C.17.
$\|\cdot\|_\lambda$, Bielecki norm, see equation (2.15).

$\|\cdot\|_{BC^{\alpha,1}}$, see Section 3.5.
$\|\cdot\|_{BTM}$, see Section 3.5.
$\|\cdot\|_{C^k}$, the usual norm in the space C^k, see Appendix D.
$\langle\cdot,\cdot\rangle_{X^*,X}$, the usual dual pairing between X^* and X. We will normally omit the spaces in the index if it is clear from the context which underlying spaces are meant.