

Cambridge University Press 978-1-316-51910-3 — Spaces of Measures and their Applications to Structured Population Models Christian Düll , Piotr Gwiazda , Anna Marciniak-Czochra , Jakub Skrzeczkowski

Frontmatter

**More Information** 

# CAMBRIDGE MONOGRAPHS ON APPLIED AND COMPUTATIONAL MATHEMATICS

#### Series Editors

M. ABLOWITZ, S. DAVIS, J. HINCH, A. ISERLES, J. OCKENDON, P. OLVER

**Spaces of Measures and their Applications to Structured Population Models** 



978-1-316-51910-3 — Spaces of Measures and their Applications to Structured Population Models Christian Düll , Piotr Gwiazda , Anna Marciniak-Czochra , Jakub Skrzeczkowski Frontmatter

More Information

The Cambridge Monographs on Applied and Computational Mathematics series reflects the crucial role of mathematical and computational techniques in contemporary science. The series publishes expositions on all aspects of applicable and numerical mathematics, with an emphasis on new developments in this fast-moving area of research.

State-of-the-art methods and algorithms as well as modern mathematical descriptions of physical and mechanical ideas are presented in a manner suited to graduate research students and professionals alike. Sound pedagogical presentation is a prerequisite. It is intended that books in the series will serve to inform a new generation of researchers.

A complete list of books in the series can be found at www.cambridge.org/mathematics.

Recent titles include the following:

- 20. Greedy approximation, Vladimir Temlyakov
- 21. Spectral methods for time-dependent problems, *Jan Hesthaven, Sigal Gottlieb & David Gottlieb*
- 22. The mathematical foundations of mixing, *Rob Sturman*, *Julio M. Ottino & Stephen Wiggins*
- 23. Curve and surface reconstruction, Tamal K. Dey
- 24. Learning theory, Felipe Cucker & Ding Xuan Zhou
- 25. Algebraic geometry and statistical learning theory, Sumio Watanabe
- 26. A practical guide to the invariant calculus, Elizabeth Louise Mansfield
- 27. Difference equations by differential equation methods, Peter E. Hydon
- 28. Multiscale methods for Fredholm integral equations, *Zhongying Chen, Charles A. Micchelli & Yuesheng Xu*
- 29. Partial differential equation methods for image inpainting, *Carola–Bibiane Schönlieb*
- 30. Volterra integral equations, Hermann Brunner
- 31. Symmetry, phase modulation and nonlinear waves, Thomas J. Bridges
- 32. Multivariate approximation, V. Temlyakov
- 33. Mathematical modelling of the human cardiovascular system, *Alfio Quarteroni*, *Luca Dede'*, *Andrea Manzoni & Christian Vergara*
- 34. Numerical bifurcation analysis of maps, Yuri A. Kuznetsov & Hil G.E. Meijer
- 35. Operator-adapted wavelets, fast solvers, and numerical homogenization, *Houman Owhadi & Clint Scovel*



Cambridge University Press 978-1-316-51910-3 — Spaces of Measures and their Applications to Structured Population Models Christian Düll , Piotr Gwiazda , Anna Marciniak-Czochra , Jakub Skrzeczkowski Frontmatter More Information

# Spaces of Measures and their Applications to Structured Population Models

CHRISTIAN DÜLL Universität Heidelberg

PIOTR GWIAZDA
Polska Akademia Nauk (PAN), Warsaw

ANNA MARCINIAK-CZOCHRA
Universität Heidelberg

JAKUB SKRZECZKOWSKI Uniwersytet Warszawski, Poland





Cambridge University Press 978-1-316-51910-3 — Spaces of Measures and their Applications to Structured Population Models Christian Düll , Piotr Gwiazda , Anna Marciniak-Czochra , Jakub Skrzeczkowski Frontmatter

**More Information** 

# CAMBRIDGE UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom
One Liberty Plaza, 20th Floor, New York, NY 10006, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781316519103
DOI: 10.1017/9781009004770

© Christian Düll, Piotr Gwiazda, Anna Marciniak-Czochra and Jakub Skrzeczkowski 2022

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2022

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data

Names: Düll, Christian, author. | Gwiazda, Piotr, author. |

Marciniak-Czochra, Anna, author. | Skrzeczkowski, Jakub, author.

Title: Spaces of measures and their applications to structured population
models / Christian Düll, Piotr Gwiazda, Anna Marciniak-Czochra, Jakub Skrzeczkowski.

Description: Cambridge; New York, NY: Cambridge University Press, 2022. |

Series: Cambridge monographs on applied and computational mathematics;

36 | Includes bibliographical references and index.

Identifiers: LCCN 2021029775 (print) | LCCN 2021029776 (ebook)

| ISBN 9781316519103 (hardback) | ISBN 9781009004770 (epub)

Subjects: LCSH: Population–Mathematical models. | Biology–Mathematical

models. | Radon measures. | Metric spaces. | Lipschitz spaces. |
Functions of bounded variation. | BISAC: MATHEMATICS / General
Classification: LCC HB849.51 .D85 2022 (print) | LCC HB849.51 (ebook) | DDC 304.601/51-dc23

LC record available at https://lccn.loc.gov/2021029775 LC ebook record available at https://lccn.loc.gov/2021029776

ISBN 978-1-316-51910-3 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.



Cambridge University Press 978-1-316-51910-3 — Spaces of Measures and their Applications to Structured Population Models Christian Düll , Piotr Gwiazda , Anna Marciniak-Czochra , Jakub Skrzeczkowski Frontmatter **More Information** 

## Contents

	Preface		<i>page</i> vii
	Notation		
	Introduction		
1	Analytical Setting		
	1.1	Preliminaries	10
	1.2	$\mathcal{M}(S)$ as a Normed Space	15
	1.3	The Positive Cone $\mathcal{M}^+(S)$	26
	1.4	Relation to the Wasserstein Distance	42
2	Structured Population Models on the State Space $\mathbb{R}^+$		
	2.1	Introduction of a Structured Population Model	58
	2.2	Analysis of the Linear Model	63
	2.3	The Nonlinear Problem	77
	2.4	Uniqueness of a Measure Solution	80
	2.5	Splitting Operator Approach	82
	2.6	Measure Solutions with Semigroup Splitting Technique	90
	2.7	Further Notes and References	99
3	Structured Population Models on Proper Spaces		
	3.1	Two-Parameter Families of Homeomorphisms	101
	3.2	Linear Structured Population Model on a Metric Space	108
	3.3	The Nonlinear Model	119
	3.4	Equivalence of the Generalised Model and the PDE Mode	el 128
	3.5	Applications	136
4	Numerical Methods for Structured Population Models		
	4.1	Finite Difference Schemes	148
	4.2	Algorithms to Compute Distances Between Measures	150
	43	The Escalator Boxcar Train Algorithm	155



Cambridge University Press 978-1-316-51910-3 — Spaces of Measures and their Applications to Structured Population Models Christian Düll , Piotr Gwiazda , Anna Marciniak-Czochra , Jakub Skrzeczkowski

More Information

Frontmatter

vi		Contents	
	4.4	Particle Method Combined with Splitting	
		Operator Technique	177
	4.5	Further Notes and References	183
5	Recent Developments and Future Perspectives		186
	5.1	Asymptotic Analysis	186
	5.2	Differentiability and Optimal Control Problems	195
	5.3	Irregular Transport	203
	5.4	Further Notes and References	207
Appendix A		Topology, Compactness and Proper Spaces	209
Appendix B		Functional Analysis	215
Appendix C		Bounded Lipschitz and Hölder Functions	220
Appendix D		Results on Approximation with Polynomials	229
Appendix E		Differential Geometry	233
Appendix F		Measure Theory	235
Appendix G		Weaker Topologies on Spaces of Measures	242
Appendix H		The Bochner Integral	269
Appendix I		Semigroups	272
App	endix J	Supplement to Chapter 2	278
App	endix K	Technical Proofs from Chapter 3	282
	Refere	ences	289
Index			307



Cambridge University Press 978-1-316-51910-3 — Spaces of Measures and their Applications to Structured Population Models Christian Düll , Piotr Gwiazda , Anna Marciniak-Czochra , Jakub Skrzeczkowski Frontmatter

More Information

### **Preface**

The idea of this book was developed over several years of joint research and numerous discussions with colleagues whose focus is in functional analysis, partial differential equations, transport processes, probability theory or mathematical modelling.

The development of the mathematical framework for the analysis of structured population models in measure spaces presented in this book was supported by many projects, starting with the Emmy Noether Project funded by the German Research Council (DFG) and by the European Research Council Starting Grant for Anna Marciniak-Czochra and the International PhD Programme "Mathematical Methods in Natural Sciences" (Foundation for Polish Science) coordinated by Piotr Gwiazda. The latter led to several PhD projects that accelerated the progress of development of analytic and numerical theory. We express thanks to Agnieszka Ulikowska and Jędrzej Jabłoński, for significant contributions.

Many people have contributed to the development of this comprehensive framework. The range of individuals involved is far too wide to be able to list them all. In particular, however, we would like to thank our colleagues and collaborators José A. Carrillo, Rinaldo Colombo, Karolina Kropielnicka, Sander C. Hille, Marie Doumic, Benoît Perthame, Agnieszka Świerczewska-Gwiazda, Emil Wiedemann, Tomasz Dębiec, Horst R. Thieme, Azmy S. Ackleh, Nicolas Saintier and Tomasz Szarek, who inspired us and helped in clarifying analytical concepts and anchoring them in relevant areas of analysis and probability theory.

Many aspects of the book have been influenced by insightful questions and suggestions from the students who worked through the book as a part of the Master's course "Spaces of Radon Measures: Functional Analysis and Applications to PDEs" given at Heidelberg University in the summer term



Cambridge University Press 978-1-316-51910-3 — Spaces of Measures and their Applications to Structured Population Models Christian Düll , Piotr Gwiazda , Anna Marciniak-Czochra , Jakub Skrzeczkowski Frontmatter

More Information

viii Preface

of 2020. The course was part of the research-oriented teaching programme of the Heidelberg STRUCTURES Excellence Cluster supported by Deutsche Forschungsgemeinschaft (DFG) under Germany's Excellence Strategy EXC-2181/1 - 390900948. It attracted mathematics and scientific computing Master's students interested in functional analysis and its applications to partial differential equations, especially in the context of the modelling of transport and growth processes. Teaching the subject helped significantly to improve the quality of the book. The students detected and helped us to remove many errors, asked interesting questions, made careful comments and frequently demanded clarifying examples. We want to express our gratitude for help to Joris Edelmann, Alina Feldmann, Lukas Huber, Sumet Khumphairan, Finn Münnich, Tuan Tung Nguyen, Achita Prasertwaree, Camillo Tissot, Kay Zaradny and Mikuláš Zindulka.

The final shape of the book has profited from fruitful discussions with Marek Kimmel and the constructive criticism and suggestions we received from anonymous reviewers and colleagues from the Heidelberg Group on Applied Analysis and Modeling in Biosciences, especially Maria V. Barbarossa and Johannes Kammerer. We also acknowledge the support of our friends Zuzanna Szymańska and Błażej Miasojedow from Warsaw, who detected glitches in our numerical pseudo-codes. During preparation of this book, Piotr Gwiazda and Jakub Skrzeczkowski were supported by the National Science Center, Poland, through projects nos. 2018/30/M/ST1/00423 and 2019/35/N/ST1/03459, respectively.

**Keywords:** structured population model, bounded Lipschitz distance, nonnegative Radon measures.

**AMS Subject Classification:** 26A16, 28A33, 35B30, 35F10, 35L50, 35R02, 65M12.



978-1-316-51910-3 — Spaces of Measures and their Applications to Structured Population Models Christian Düll , Piotr Gwiazda , Anna Marciniak-Czochra , Jakub Skrzeczkowski Frontmatter

More Information

### **Notation**

#### **General Notation**

We use the following notation for sets. Let (S, d) be a metric space. Then

 $\mathcal{B}(S)$ , the Borel  $\sigma$ -algebra of S, i.e. the  $\sigma$ -algebra generated by the open sets of S.

 $B_r(s) := \{x \in S \mid d(x, s) \le r\}$ , the open set with radius r and centre s.

 $\mathbb{N} = \{1, 2, 3, \ldots\}.$ 

 $\mathbb{N}_0 = \{0, 1, 2, \ldots\}.$ 

 $\mathbb{R}^+ = [0, \infty).$ 

 $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}.$ 

 $\mathbb{S}^d$ , the unit sphere  $\{x \in \mathbb{R}^{d+1} \mid ||x|| = 1\}$ .

In general, we will use a superscript + to indicate nonnegativity.

### **Special Functions and Spaces**

Here we present a selection of spaces and functions that appear several times throughout the book and whose definition the reader might want to look up once in a while. Note that special spaces that occur only locally in one section are not listed here.

**1**, constant function  $\mathbf{1}(x) = 1$  for all x.

 $\mathrm{BC}^{\alpha,1}$  ( $[0,T] \times \mathcal{M}^+(\mathbb{R}^+); X$ ), the space of X-valued, bounded functions defined on  $[0,T] \times \mathcal{M}^+(\mathbb{R}^+)$  that are  $\alpha$ -Hölder continuous in t and Lipschitz continuous in  $\mathcal{M}^+(\mathbb{R}^+)$ , see Section 3.5.

BL(S), the space of bounded Lipschitz functions  $S \to \mathbb{R}$ .

 $BL(S)_{+}^{*}$ , the space of positive linear functionals on BL(S).



978-1-316-51910-3 — Spaces of Measures and their Applications to Structured Population Models Christian Düll , Piotr Gwiazda , Anna Marciniak-Czochra , Jakub Skrzeczkowski Frontmatter

More Information

x Notation

 $C_h(S)$ , the space of bounded continuous functions  $S \to \mathbb{R}$ .

 $C_c(S)$ , the space of continuous functions  $S \to \mathbb{R}$  with compact support in S.

 $C^k(S)$ , the space of k-times continuously differentiable functions  $S \to \mathbb{R}$ .

 $C_{ub}(S)$ , the space of uniformly continuous and bounded functions  $S \to \mathbb{R}$ .

 $C_0(S) = \frac{1}{C_c^0(S)} \|\cdot\|_{\infty}$ , see Definition 1.3.

 $C^0(S)$ , the space of continuous functions  $S \to \mathbb{R}$ .

 $C^0([0,T];(\mathcal{M}^+(S),\rho_F))$ , the space of continuous maps from the interval [0,T] to  $(\mathcal{M}^+(S),\rho_F)$ . A natural metric on such a space is defined in (3.23).

 $\chi_T$ , characteristic function of a set T,  $\chi_T(x) = \begin{cases} 1, & x \in T \\ 0, & x \notin T \end{cases}$ .

 $E = \overline{\mathcal{M}(S)}^{\|\cdot\|_{BL^*}}.$ 

Id, the identity map  $x \mapsto x$ .

 $L^1([0,T], BL(\mathbb{R}^d))$ , the Bochner space of functions with  $L^1$  regularity in time and BL regularity in space, cf. (3.1).

 $L^p$ , the usual space of functions whose pth power is integrable.

 $\mathbf{Lip}_1(S)$ , the space of 1-Lipschitz functions  $S \to \mathbb{R}$ .

 $\mathcal{L}(X,Y)$ , the space of bounded linear operators from X to Y.

 $\mathcal{M}(S)$ , the space of finite signed Radon measures.

 $\mathcal{M}^+(S)$ , the space of nonnegative finite signed Radon measures.

 $\mathcal{P}(S)$ , the set of probability measures on S.

 $S_{t,s}$ , a semigroup, see Definition I.4.

 $W^{k,p}$ , the usual Sobolev space of k-times weakly differentiable functions with all derivatives in  $L^p$ .

#### Norms and Seminorms

To keep track of the various norms and seminorms that are introduced in the book, we provide a list for the convenience of the reader.

 $|\cdot|_{Lip}$ , Lipschitz constant, see Definition 1.1.

 $\|\cdot\|_{\infty}$ , supremum norm, see Definition 1.1.

 $\|\cdot\|_{BL}$ , bounded Lipschitz norm, see Definition 1.1.

 $\|\cdot\|_{TV}$ , total variation norm, see Definition 1.19.

 $\|\cdot\|_{BL^*}$ , flat (or dual bounded Lipschitz) norm, see Definition 1.22.

 $\rho_F(\cdot,\cdot)$ , the flat metric induced by the flat norm.

 $W_p(\cdot, \cdot)$ , pth Wasserstein distance, see Definition 1.77.

 $[\cdot]_{\alpha}$ , Hölder seminorm, see Definition C.17.

 $\|\cdot\|_{\lambda}$ , Bielecki norm, see equation (2.15).



978-1-316-51910-3 — Spaces of Measures and their Applications to Structured Population Models Christian Düll , Piotr Gwiazda , Anna Marciniak-Czochra , Jakub Skrzeczkowski Frontmatter

**More Information** 

Notation xi

- $\|\cdot\|_{\mathrm{BC}^{\alpha,1}}$ , see Section 3.5.
- $\|\cdot\|_{BTM}$ , see Section 3.5.
- $\|\cdot\|_{C^k}$ , the usual norm in the space  $C^k$ , see Appendix D.
- $\langle \cdot, \cdot \rangle_{X^*, X}$ , the usual dual pairing between  $X^*$  and X. We will normally omit the spaces in the index if it is clear from the context which underlying spaces are meant.



Cambridge University Press 978-1-316-51910-3 — Spaces of Measures and their Applications to Structured Population Models Christian Düll , Piotr Gwiazda , Anna Marciniak-Czochra , Jakub Skrzeczkowski Frontmatter More Information