Noncommutative Function-Theoretic Operator Theory and Applications

This concise monograph explores how core ideas in Hardy-space function theory and operator theory continue to be useful and informative in new settings, leading to new insights for noncommutative multivariable operator theory. Beginning with a review of the confluence of system-theory ideas and reproducing-kernel techniques, the book then covers representations of backward-shift-invariant subspaces in the Hardy space as ranges of observability operators, and representations for forward-shift-invariant subspaces via a Beurling–Lax representer equal to the transfer function of the linear system. This pair of backward-shift-invariant and forward-shift-invariant subspaces form a generalized orthogonal decomposition of the ambient Hardy space. All this leads to the de Branges–Rovnyak model theory and characteristic operator function for a Hilbert-space contraction operator. The chapters that follow generalize the system theory and reproducing-kernel techniques to enable an extension of the ideas above to weighted Bergman-space multivariable settings.

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Preface

This book aims to present Hardy-space function theory and related operator theory in various multivariable, free noncommutative, weighted Bergman-space settings with an emphasis on system theory and reproducing-kernel approaches. The authors have been working together in this area for the last couple of decades. The idea is to focus on the settings related to the unit-ball extensions of the classical unit-disk case that came up some 10 years ago in the course of examining possible system-theoretic interpretations of Bergman-inner functions as transfer functions of certain discrete-time linear systems and, on the other hand, as characteristic functions of hypercontractive Hilbert-space operators.

The early stages of the study started within the single-variable setting and were particularly inspired by the papers of Hedenmalm and Olofsson. From the very beginning, we were planning to present the results in the form suitable for subsequent multivariable (commutative and non-commutative) extensions. The book presents the multivariable noncommutative part of the theory; some single-variable results have already appeared in journal publications of the authors, while the noncommutative multivariable results have only been exposed at various conferences and meetings. After the first draft of the book was prepared and posted, it took almost two years to revise it, filling in the gaps and displaying additional connections between different parts of the theory. Portions of the commutative part of the theory are already appearing in the literature, in particular, in papers of Eschmeier, Sarkar, and others.

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