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Introduction

1.1 Function-Theoretic Operator Theory on Vectorial Hardy Spaces, Reproducing Kernel Hilbert Spaces, and Discrete-Time Linear Systems: Background

Arguably the synthesis of Hardy space function theory with operator theory begins with the famous paper of Beurling [50] making the connection between invariant subspaces for the shift operator on the Hardy space H^2 and inner functions, including the canonical factorization of any H^2-function as the product of an outer function, a Blaschke product, and a singular inner function. Around the same time appeared work of Livšic [122], obtaining triangular models for operators close to self-adjoint (see Kriete [117] and Vinnikov [174] for updates) and finding a characteristic function as a unitary invariant for a class of operators close to being unitary [122]. Ensuing work of Sz.-Nagy–Foias–Bercovici-Kércy [171] and of de Branges–Rovnyak [58, 59] further developed a model theory based on a characteristic function for operators close to being unitary. The work of Sz.-Nagy–Foias made an explicit connection with dilation theory, while that of de Branges–Rovnyak went beyond Hardy spaces by involving more general reproducing kernel Hilbert spaces only contractively included in a larger ambient reproducing kernel Hilbert space. However, as emphasized by Helton [99, 100], Fuhrmann [87], and others, at least implicit in a lot of this work in function-theoretic operator theory were connections with systems theory. In particular, the explicit formula for the Sz.-Nagy–Foias characteristic function is recognizable as having the form of a transfer function for a conservative discrete-time linear system; the fact that a rational inner function has such a realization can be traced to the engineering circuit-theory literature from the 1950s (see [99]), and the de Branges–Rovnyak model theory can be developed from a system-theory perspective (see [34]). For a thorough overview of all these connections between Hardy-space function theory, operator theory, and systems theory and connections with still other applications in engineering and harmonic analysis as of 2002, we refer to the two-volume treatise of Nikolski [132, 133].

In the ensuing decades, there has been much work extending these approaches to the context of multivariable function theory synthesized with multivariable operator theory. In particular, there have been contributions to multivariable operator theory with a distinctive reproducing kernel flavor, in both concrete and abstract commutative settings (Arveson [19] and Bhattacharyya, Eschmeier, Sarkar, and collaborators [51, 53–55, 80, 91, 163, 164], freely noncommutative settings (Ball–Bolotnikov–Fang [30, 33]), and sometimes with interplay between the two settings (Davidson–Pitts [69], Ball–Bolotnikov [22], Jury–Martin [108], Salomon–Shalit–Shamovich [160], Hartz [94]). For the free noncommutative setting, there is now a notion of reproducing kernel and associated reproducing kernel Hilbert space on a noncommutative Reinhardt-domain setting (Ball-Vinnikov and collaborators [43, 45]) as well as on more general free noncommutative domains (Ball–Marx–Vinnikov [42]), which fits into the framework of a general noncommutative function theory [9, 109]. There has also been work using system-theory ideas to push multivariable operator theory in new directions (Ball, Bolotnikov, Vinnikov, and collaborators [22–24, 44, 46] and Olofsson [134, 136]). Of course there is some overlap between the systems-theory approach and the reproducing kernel approach, Let us mention one instance of such an overlap: What we have called *observability operator* here and elsewhere in our system-theory approach is essentially the same as what is known as *Gelu Poisson kernel* in the terminology of Gelu Popescu (see e.g. [147, 151–153]).

Apart from the connection with dilation theory, characteristic functions, and operator model theory which we develop here, the multi-shift setting for the study of operator tuples has been a core area of study in operator theory, beginning with the work of Shields [166] and culminating in the recent beefy papers of Chavan, Trivedi, and collaborators [62, 92]. Our goal here is to lay out systematically the free noncommutative function theory for a class of weighted Bergman spaces on a full free Fock space and the associated Sz.-Nagy–Foias style model theory for the class of operators which can be modeled as the compression to a joint $*$-invariant subspace of the shift operator tuples on such a space.

Our primary tool is the system-theory approach outlined above, but there will also be a nontrivial use of reproducing kernel techniques, specifically of the notion of formal noncommutative reproducing kernel Hilbert space developed in [43, 45]. In fact, we shall see that most of the basic results can be derived via either approach, but there is at least one instance (see

Theorem 9.2.20) where the systems-theory approach leads to some additional information not attainable via the purely reproducing kernel approach.

1.2 The Synthesis of the Systems-Theory and Reproducing Kernel Approaches

1.2.1 The Systems-Theory Approach

By way of motivation for the more general noncommutative, multivariable settings to come, we now illustrate in some detail the system-theory approach to function-theoretic operator theory for the classical setting.

For $\mathcal{X}$ and $\mathcal{Y}$, any pair of Hilbert spaces, we use the notation $\mathcal{L}(\mathcal{X},\mathcal{Y})$ to denote the space of bounded, linear operators from $\mathcal{X}$ to $\mathcal{Y}$, shortening the notation $\mathcal{L}(\mathcal{X},\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$. We start with the classical discrete-time linear system

$$\Sigma(\mathbf{U}): \quad \begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \quad\;\; y(k) = Cx(k) + Du(k) \end{cases} \tag{1.2.1}$$

with $x(k)$ taking values in the *state space* $\mathcal{X}$, $u(k)$ taking values in the *input space* $\mathcal{U}$, and $y(k)$ taking values in the *output space* $\mathcal{Y}$, where $\mathcal{U}$, $\mathcal{Y}$, and $\mathcal{X}$ are given Hilbert spaces and where the *system matrix* (sometimes also called *colligation matrix* or *connection matrix*) of the system

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

is a given bounded linear operator. If we let the system evolve on the nonnegative integers $n \in \mathbb{Z}_+$, then the whole trajectory $\{u(n), x(n), y(n)\}_{n\in\mathbb{Z}_+}$ is determined from the input signal $\{u(n)\}_{n\in\mathbb{Z}_+}$ and the initial state $x(0) = x$, according to the formulas

$$\begin{aligned} x(k) &= A^k x + \sum_{j=0}^{k-1} A^{k-1-j} Bu(j), \\ y(k) &= CA^k x + \sum_{j=0}^{k-1} CA^{k-1-j} Bu(k) + Du(k). \end{aligned} \tag{1.2.2}$$

Application of the Z-transform

$$\{f(k)\}_{k\in\mathbb{Z}_+} \mapsto \widehat{f}(\lambda) = \sum_{k=0}^{\infty} f(k)\lambda^k$$

to the system equations (1.2.1) converts the expressions (1.2.2) to the so-called frequency-domain formulas

$$\begin{aligned}\widehat{x}(\lambda) &= (I-\lambda A)^{-1}x + \lambda(I-\lambda A)^{-1}B\widehat{u}(\lambda),\\ \widehat{y}(\lambda) &= C(I-\lambda A)^{-1}x + \big[D+\lambda C(I-\lambda A)^{-1}B\big]\widehat{u}(\lambda)\\ &= \mathcal{O}_{C,A}x + \Theta_{\mathbf{U}}(\lambda)\widehat{u}(\lambda), \end{aligned} \tag{1.2.3}$$

where

$$\mathcal{O}_{C,A}\colon\ x \mapsto \sum_{k=0}^{\infty}(CA^k x)\,\lambda^k = C(I-\lambda A)^{-1}x \tag{1.2.4}$$

is the observability operator and where

$$\Theta_{\mathbf{U}}(\lambda) = D + \lambda C(I-\lambda A)^{-1}B$$

is the *transfer function* of the system Σ given by (1.2.1). In particular, if the input signal $\{u(n)\}_{n\in\mathbb{Z}_+}$ is taken to be zero, the resulting output $\{y(n)\}_{n\in\mathbb{Z}_+}$ is given by $y = \mathcal{O}_{C,A}x(0)$. If $\mathcal{O}_{C,A}$ is injective, i.e., if (C,A) satisfies the *observability condition*

$$\bigcap_{k=0}^{\infty}\operatorname{Ker} CA^k = \{0\}, \tag{1.2.5}$$

we say that the output pair (C,A) is *observable*. In case $\mathcal{O}_{C,A}$ is bounded as an operator from $\mathcal{X}$ into the standard vector-valued Hardy space of the unit disk

$$H^2_{\mathcal{Y}} = \Big\{ f(\lambda) = \sum_{k\ge 0} f_k\lambda^k\colon\ \sum_{k\ge 0}\|f_k\|^2_{\mathcal{Y}} < \infty \Big\},$$

we say that the pair (C,A) is *output stable*. Let us mention that it is possible to give a complete characterization as to when a given output pair (C,A) is output stable in terms of the existence of a positive-semidefinite solution of a linear-matrix-inequality (here actually a Stein inequality) determined uniquely by the pair (C,A) (see Theorem 4.0.1 for the precise statement).

The case where the operator system matrix $\mathbf{U}$ is *isometric*, or more generally just *contractive*, is of special interest. In system-theoretic terms, the isometric property of $\mathbf{U}$ has the interpretation that the system $\Sigma(\mathbf{U})$ is *conservative* in the sense that the energy stored by the state at time k ($\|x(k+1)\|^2 - \|x(k)\|^2$) is exactly compensated by the net energy put into the system from the outside environment ($\|u(k)\|^2 - \|y(k)\|^2$). In case $\mathbf{U}$ is contractive, the system $\Sigma(\mathbf{U})$ is said to be *dissipative* in the sense that the net energy ($\|x(k+1)\|^2 - \|x(k)\|^2$) stored by the state at time k is no more than the net energy put into the system from the outside environment ($\|u(k)\|^2 - \|y(k)\|^2$) at time k. In case the system is dissipative (i.e., $\|\mathbf{U}\| \leq 1$), the transfer function $\Theta_{\mathbf{U}}$ is in the *Schur class* $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ (the class of contractive $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued analytic functions Θ on the open unit disk $\mathbb{D}$), and moreover the observability operator $\mathcal{O}_{C,A} \colon \mathcal{X} \to H^2_{\mathcal{Y}}$ is contractive. Conversely, if Θ is in the Schur class, then Θ has a realization as $\Theta = \Theta_{\mathbf{U}}$ as in (1.2.1) with $\Sigma(\mathbf{U})$ dissipative (in fact, even conservative).

Given any holomorphic $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function Θ on the unit disk, we associate the multiplication operator $M_\Theta \colon f(z) \mapsto \Theta(z) f(z)$ (or $f \mapsto \Theta \cdot f$ for short). Then, the operator-theoretic significance of the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ is that the multiplication operator M_Θ is a contraction from $H^2_{\mathcal{U}}$ to $H^2_{\mathcal{Y}}$ exactly when Θ is in the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$.

If $\mathbf{U}$ is isometric and, in addition, the state-space operator A is *strongly stable* in the sense that $\|A^n x\| \to 0$ as $n \to \infty$ for each $x \in \mathcal{X}$, then the observability operator is a partial isometry (even an isometry in case (C, A) is observable) and the transfer function $\Theta_{\mathbf{U}}$ is *inner* (the boundary values $\Theta_{\mathbf{U}}(\zeta)$ existing as strong radial limits from inside $\mathbb{D}$ for almost every ζ on the unit circle $\mathbb{T}$ are isometric operators from $\mathcal{U}$ to $\mathcal{Y}$), or equivalently, the multiplication operator $M_\Theta \colon H^2_{\mathcal{U}} \to H^2_{\mathcal{Y}}$ is isometric. Conversely: *any inner function Θ arises in this way as $\Theta = \Theta_{\mathbf{U}}$ with $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ isometric with A strongly stable.*

We say that a subspace $\mathcal{M} \subset H^2_{\mathcal{Y}}$ is *shift-invariant* if $f \in \mathcal{M} \Rightarrow S_{\mathcal{Y}} f \in \mathcal{M}$, where $S_{\mathcal{Y}}$ is the shift operator given as the coordinate multiplication operator on $H^2_{\mathcal{Y}}$

$$S_{\mathcal{Y}} = M_\lambda \colon f(\lambda) \mapsto \lambda f(\lambda).$$

Note that if Θ is inner, then $\mathcal{M} := M_\Theta H^2_{\mathcal{U}} = \Theta \cdot H^2_{\mathcal{U}}$ is a shift-invariant subspace for $S_{\mathcal{Y}}$; the content of the Beurling–Lax theorem is that conversely, any such invariant subspace can be represented in this way. Similarly, we say that the subspace $\mathcal{N} \subset H^2_{\mathcal{Y}}$ is *backward-shift-invariant* if $f \in \mathcal{N} \Rightarrow S^*_{\mathcal{Y}} f \in \mathcal{N}$, where the backward-shift operator $S^*_{\mathcal{Y}}$, the Hilbert-space adjoint of the forward-shift operator $S_{\mathcal{Y}}$, works out to be

$$S^*_{\mathcal{Y}} \colon f(\lambda) \mapsto [f(\lambda) - f(0)]/\lambda.$$

The computation

$$S^*_{\mathcal{Y}} \colon C(I - \lambda A)^{-1} x \mapsto \lambda^{-1}[C(I - \lambda A)^{-1} - C]x = C(I - \lambda A)^{-1} A x$$

shows that, for any output-stable pair (C, A), the space $\operatorname{Ran} \mathcal{O}_{C,A}$ is $S^*_{\mathcal{Y}}$-invariant. Conversely, *if $\mathcal{M}^\perp \subset H^2_{\mathcal{Y}}$ is $S^*_{\mathcal{Y}}$-invariant, then there is an output pair* (C, A) (with $C^*C = I - A^*A$) *so that* $\mathcal{M}^\perp = \operatorname{Ran} \mathcal{O}_{C,A}$. Furthermore, in case $\mathbf{U}$ is unitary with A strongly stable, then the set of possible Z-transformed output signals $\widehat{y}(\lambda)$ appearing in the form in the transformed system equations as in (1.2.3) is all of $H^2_{\mathcal{Y}}$ and the additive decomposition of $\widehat{y}(\lambda)$ appearing in (1.2.3) is orthogonal:

$$H^2_{\mathcal{Y}} = \operatorname{Ran} \mathcal{O}_{C,A} \oplus \operatorname{Ran} M_{\Theta_{\mathbf{U}}}.$$

1.2.2 Realization Formulas for Reproducing Kernels

More generally, following the reproducing kernel approach of de Branges–Rovnyak [58, 59], as enhanced in the work of the authors and collaborators [30–34], it is of interest to consider also the case where the $S_{\mathcal{Y}}$-invariant subspace $\mathcal{M}$ carries its own norm distinct from the norm inherited from the ambient space $H^2_{\mathcal{Y}}$ but with the prescription that the inclusion map $\iota \colon \mathcal{M} \to H^2_{\mathcal{Y}}$ be contractive. Then a generalization of the Beurling–Lax theorem due to de Branges–Rovnyak says that one can always find a contractive multiplier (i.e., a Schur-class function, not necessarily inner) Θ so that $\mathcal{M} = \Theta \cdot H^2_{\mathcal{U}}$, with *lifted norm* given by

$$\|\Theta f\| = \inf\{\|g\| \colon g \in H^2_{\mathcal{U}} \ \text{such that} \ \Theta \cdot g = \Theta \cdot f\}.$$

In this context, there is a generalization of orthogonal complement denoted by $\mathcal{M}^{[\perp]}$, which we call the *Brangesian complement* (see Section 3.1.1 for details), which is also contractively included in $H^2_{\mathcal{Y}}$ and provides a linear decomposition

$$H^2_{\mathcal{Y}} = \mathcal{M} + \mathcal{M}^{[\perp]}$$

that is neither orthogonal nor even a direct-sum decomposition but does have a canonical minimality property, making the space $\mathcal{M}^{[\perp]}$ uniquely determined by $\mathcal{M}$ (see [28, 162] or Section 3.1.1). Here $\mathcal{M}^{[\perp]}$ also carries its own norm with the inclusion map into $H^2_{\mathcal{Y}}$ contractive. Furthermore, for the case where $\mathcal{M} = \Theta \cdot H^2_{\mathcal{U}}$ for a Schur-class function Θ, $\mathcal{M}^{[\perp]}$ is a reproducing kernel Hilbert space with reproducing kernel $K_{\mathcal{M}^{[\perp]}}$ given by

$$K_{\mathcal{M}^{[\perp]}}(\lambda, \mu) = \frac{I_{\mathcal{Y}} - \Theta(\lambda)\Theta(\mu)^*}{1 - \lambda\overline{\mu}}$$

and as a lifted-norm space is induced by the operator $(I - M_\Theta M_\Theta^*)^{\frac{1}{2}}$,

$$\mathcal{M}^{[\perp]} = \operatorname{Ran}(I - M_\Theta M_\Theta^*)^{\frac{1}{2}}$$

with

$$\|(I - M_\Theta M_\Theta^*)^{\frac{1}{2}} f\| = \min\left\{\|g\|_{H^2_{\mathcal{Y}}} \colon (I - M_\Theta M_\Theta^*)^{\frac{1}{2}} g = (I - M_\Theta M_\Theta^*)^{\frac{1}{2}} f\right\}.$$

Alternatively, one can start with an $S_{\mathcal{Y}}^*$-invariant subspace $\mathcal{N}$ contractively included in $H^2_{\mathcal{Y}}$ and find a contractive output pair (C, A) (so $A^*A + C^*C \preceq I_{\mathcal{X}}$ where $A \in \mathcal{L}(\mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$) so that $\mathcal{N}$ is the range of the observability operator $\mathcal{O}_{C,A}$. Then, $\mathcal{N}$ is itself a reproducing kernel Hilbert space with reproducing kernel

$$K_{\mathcal{N}}(\lambda, \mu) = C(I_{\mathcal{X}} - \lambda A)^{-1}(I_{\mathcal{X}} - \overline{\mu} A^*)^{-1} C^*.$$

If one then solves the factorization problem for injective $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$,

$$\begin{bmatrix} B \\ D \end{bmatrix} \begin{bmatrix} B^* & D^* \end{bmatrix} = \begin{bmatrix} I_{\mathcal{X}} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} A^* & C^* \end{bmatrix} \tag{1.2.6}$$

and then lets $\mathbf{U}$ be the system matrix $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, then $\mathbf{U}$ is unitary with associated transfer function $\Theta_{\mathbf{U}}(\lambda) = D + \lambda C(I - \lambda A)^{-1} B$ giving rise to the contractive multiplier $\Theta_{\mathbf{U}}$ generating the Brangesian complement of $\mathcal{N}$,

$$\mathcal{N}^{[\perp]} = \Theta_{\mathbf{U}} \cdot H^2_{\mathcal{U}},$$

or equivalently, solving the kernel factorization problem

$$\frac{I_{\mathcal{Y}}}{1 - \lambda\overline{\mu}} - C(I - \lambda A)^{-1}(I - \overline{\mu} A^*)^{-1} C^* = \frac{\Theta(\lambda)\Theta(\mu)^*}{1 - \lambda\overline{\mu}}. \tag{1.2.7}$$

Then, the space $H^2_{\mathcal{Y}}$ has an additive decomposition

$$H^2_{\mathcal{Y}} = \mathcal{N} + \mathcal{N}^{[\perp]} = \operatorname{Ran} \mathcal{O}_{C,A} + M_{\Theta_{\mathbf{U}}} H^2_{\mathcal{U}} \tag{1.2.8}$$

corresponding again to the additive decomposition of $\widehat{y} \in H^2_{\mathcal{Y}}$ in (1.2.3), but this time not orthogonal nor a direct sum but rather a Brangesian minimal decomposition. In case one of $\operatorname{Ran} \mathcal{O}_{C,A}$ or $\operatorname{Ran} M_{\Theta_{\mathbf{U}}}$ is contained in $H^2_{\mathcal{Y}}$ isometrically, then they both are isometrically included and the decomposition (1.2.8) is orthogonal, and we recover most of the results discussed above derived via the systems theory approach.

1.2.3 Connections with Operator Model Theory

If we start with a contraction operator T on a Hilbert space $\mathcal{X}$, we can always form the isometric output pair $(C, A) := (D_{T^*}, T^*)$, where $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$ is the *defect operator* of T^*, here viewed as an operator from $\mathcal{X}$ to the *defect space* $\mathcal{Y} := \mathcal{D}_{T^*} = \overline{\operatorname{Ran}}\,(I - TT^*)^{\frac{1}{2}}$. Then, we may form the observability operator $\mathcal{O}_{D_{T^*}, T^*} \colon X \to H^2_{\mathcal{D}_{T^*}}$. If we assume that T is *completely noncoisometric* (c.n.c. for short), then $\mathcal{O}_{D_{T^*}, T^*}$ is one-to-one. Since (D_T^*, T^*) is an isometric output pair, one can show that the observability operator is isometric exactly when T is *pure* (i.e., T^* is strongly stable). Then, the solution of the factorization problem (1.2.6) with $(C, A) = (D_{T^*}, T^*)$ leads to

$$\begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} D_T \\ -T \end{bmatrix} : \mathcal{D}_T \to \begin{bmatrix} \mathcal{X} \\ \mathcal{D}_{T^*} \end{bmatrix}$$

giving rise to the unitary system matrix

$$\mathbf{U}_T = \begin{bmatrix} T^* & D_T \\ D_{T^*} & -T \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{D}_T \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{X} \\ \mathcal{D}_{T^*} \end{bmatrix}$$

with associated transfer function

$$\Theta_T(\lambda) = [-T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T]|_{\mathcal{D}_T} : \ \mathcal{D}_T \to \mathcal{D}_{T^*} \qquad (1.2.9)$$

equal to the Sz.-Nagy–Foias as well as the de Branges–Rovnyak characteristic function for the c.n.c. contraction operator T. Furthermore, the observability operator $\mathcal{O}_{D_{T^*}, T^*}$ is isometric exactly when T^* is strongly stable, or equivalently, when Θ_T is inner. In this case, T^* is unitarily equivalent to the restriction of the backward shift $S^*_{\mathcal{D}_{T^*}}$ to its invariant subspace $\mathcal{N} = \operatorname{Ran} \mathcal{O}_{D_{T^*}, T^*} \subset H^2_{\mathcal{D}_{T^*}}$. In case T^* is not strongly stable, it is still the case that T^* is unitarily equivalent to $S_{\mathcal{D}_{T^*}}$ restricted to an invariant subspace $\mathcal{N} = \operatorname{Ran} \mathcal{O}_{D_{T^*}, T^*} \subset H^2_{\mathcal{D}_{T^*}}$, but in this case we have only a contractive containment of $\mathcal{N}$ in the ambient space $H^2_{\mathcal{D}_{T^*}}$. In this case, we can still see that T dilates to an isometry $S_{\mathcal{D}_{T^*}} \oplus V$ on a space $H^2_{\mathcal{D}_{T^*}} \oplus \mathcal{W}$, where V is a unitary operator on the Hilbert space $\mathcal{W}$, i.e., there is a subspace $\widetilde{\mathcal{N}}$ of $H^2_{\mathcal{D}_{T^*}} \oplus \mathcal{W}$ so that T^* is unitarily equivalent to $(S_{\mathcal{D}_{T^*}} \oplus V)^*|_{\mathcal{N}}$. The model theory of Sz.-Nagy–Foias–Bercovici–Kércy [171] gives a functional model for T that is embedded isometrically in a functional model for $(S_{\mathcal{D}_{T^*}} \oplus V)^*|_{\mathcal{N}}$ via a somewhat different approach, whereby one first proves the Sz.-Nagy dilation theorem and finds a model for T inside the geometry of a functional model for the unitary dilation of T. Section 1.5 of the paper of Douglas [73] obtains a model for the isometric (and then by further extension unitary) dilation $S_{\mathcal{D}_{T^*}} \oplus V$ of T by finding a complementary embedding operator Q so that the operator $\begin{bmatrix} \mathcal{O}_{D_{T^*}, T^*} \\ Q \end{bmatrix}$ defines an embedding of $\mathcal{X}$ into a direct-sum space $\begin{bmatrix} H^2_{\mathcal{D}_{T^*}} \\ \overline{\operatorname{Ran}}\, Q \end{bmatrix}$.

We should point out that the Sz.-Nagy–Foias model theory actually applies to the more general situation of a *completely non-unitary* (c.n.u. for short) contraction operator, but explaining this additional feature does not fit into

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our narrative here; for a sample of the difficulties in handling the c.n.u. class in more general multivariable settings, we refer to work of the authors and Vinnikov [22, 46].

1.2.4 Summary

In summary, we have the following themes connecting vectorial Hardy-space function theory, conservative/dissipative discrete-time linear systems, and model theory for Hilbert-space contraction operators:

1. **Backward-shift-invariant subspaces and ranges of observability operators:** A backward-shift-invariant subspace of $H^2_{\mathcal{Y}}$ arises as the range of some isometric observability operator. More generally, a contractively included backward-shift-invariant subspace of $H^2_{\mathcal{Y}}$ arises as the lifted-norm space associated with a contractive observability operator. Moreover, it is possible to characterize in terms of existence of a solution to a certain linear-matrix-inequality when a given output pair (C, A) gives rise to an observability operator $\mathcal{O}_{C,A}$ mapping $\mathcal{X}$ boundedly into $H^2_{\mathcal{Y}}$. The special case $C = I_{\mathcal{X}}$ corresponds to exponential stability for A.
2. **Forward-shift-invariant subspaces and contractive multipliers:** A forward-shift-invariant subspace of $H^2_{\mathcal{Y}}$ has Beurling–Lax representation $\mathcal{M} = M_\Theta \cdot H^2_{\mathcal{U}}$ for some inner multiplier Θ from $H^2_{\mathcal{U}}$ to $H^2_{\mathcal{Y}}$. More generally, a contractively included forward-shift-invariant subspaces of $H^2_{\mathcal{Y}}$ has a lifted-norm Beurling–Lax representation $\Theta \cdot H^2_{\mathcal{U}}$ for a contractive multiplier Θ from $H^2_{\mathcal{U}}$ to $H^2_{\mathcal{Y}}$.
3. **Hardy-space decompositions in backward- and forward-shift-invariant subspaces:** In the case of a conservative linear system with strongly stable state operator A (i.e., $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is isometric and also $\|A^n x\| \to 0$ as $n \to \infty$ for each $x \in \mathcal{X}$), the observability operator $\mathcal{O}_{C,A} \colon \mathcal{X} \to H^2_{\mathcal{Y}}$ and the transfer-function multiplier operator $M_{\Theta_{\mathbf{U}}}$ are isometric, and one has the orthogonal decomposition of the form
$$H^2_{\mathcal{Y}} = \operatorname{Ran} \mathcal{O}_{C,A} \oplus \operatorname{Ran} M_{\Theta_{\mathbf{U}}} \oplus \mathcal{W} \tag{1.2.10}$$
for a shift-invariant subspace $\mathcal{W}$; if it is the case that $\mathbf{U}$ is also coisometric, then $\mathcal{W} = \{0\}$. Conversely, if $\mathcal{M} \subset H^2_{\mathcal{Y}}$ is $S_{\mathcal{Y}}$-invariant (and hence $\mathcal{M}^\perp \subset H^2_{\mathcal{Y}}$ is $S^*_{\mathcal{Y}}$-invariant), then there is a unitary $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ with A strongly stable such that $\mathcal{M}^\perp = \operatorname{Ran} \mathcal{O}_{C,A}$ and $\mathcal{M} = \Theta_{\mathbf{U}} \cdot H^2_{\mathcal{U}}$.
   More generally, if $\mathbf{U}$ is merely contractive (rather than isometric or unitary) and/or A is not strongly stable, then a linear decomposition of the form (1.2.10) holds but as a Brangesian minimal decomposition rather

than as a Hilbert-space orthogonal decomposition. If **U** is coisometric, it is again the case that the space $\mathcal{W}$ is zero.

4. **Model theory for Hilbert-space contraction operators:** An inner (and more generally, contractive) multiplier $M_\Theta \colon H^2_{\mathcal{U}} \to H^2_{\mathcal{Y}}$ arises as the Sz.-Nagy–Foias/de Branges–Rovnyak characteristic function for some c.n.c. Hilbert-space contraction operator T that in turn induces a canonical functional model for the operator T which also exhibits a unitary dilation $S \oplus V$ for T.

Much work has been done to extend this set of ideas, particularly themes #2 and #4 (the operator-model theory aspects without the system-theoretic connections) to more general settings, e.g.,

(i) to Bergman spaces and hypercontraction operators; see Agler [4], Müller [130], Müller–Vasilescu [131], Hedenmalm–Korenblum–Zhu [97], Duren–Schuster [76]),
(ii) to the Drury–Arveson space and commutative row-contractive operator tuples; see Bhattacharyya–Eschmeier–Sarkar [53, 54], Bhattacharyya–Sarkar [55], and Ball–Bolotnikov [22],
(iii) to more general domains in $\mathbb{C}^d$ than the ball and associated more general commutative operator tuples; see Athavale [20], Curto–Vasilescu [67, 68], Timotin [173], Pott [155], Ambrozie–Engliš–Müller [16], Arazy–Engliš [18].
(iv) to the full Fock space and freely noncommutative row-contractive operator tuples, possibly also constrained to lie in a prescribed noncommutative operator variety; see Bunce [60], Frazho [85], Popescu [141–144, 147, 148],
(v) to a more general formalism of representations of certain operator algebras based on tensor-algebra constructions; see Muhly–Solel [126–129], and
(vi) to noncommutative hypercontractive operator tuples modeled on noncommutative varieties (see Popescu [151–153]) as well as a weighted version of the tensor-algebra context (see Muhly–Solel [129]).

Identification of a characteristic function defined by a formula of the Sz.-Nagy–Foias type (1.2.9) (the main thrust of theme #4 above) can be found (i) for the Bergman space setting first in the work of Olofsson [134–136] and then followed up by the authors [23, 24] and Eschmeier [79], (ii) for the Drury–Arveson space setting earlier in the work of Bhattacharyya et al. [53–55], (iii) for the full Fock space in the work of