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Algebraic Number Theory for Beginners

This book introduces algebraic number theory through the problem of generalizing "unique prime factorization" from ordinary integers to more general domains. Solving polynomial equations in integers leads naturally to these domains, but unique prime factorization may be lost in the process. To restore it, we need Dedekind's concept of *ideals*. However, one still needs the supporting concepts of algebraic number field and algebraic integer, and the supporting theory of rings, vector spaces, and modules. It was left to Emmy Noether to encapsulate the properties of rings that make unique prime factorization possible, in what we now call *Dedekind rings*. The book develops the theory of these concepts, following their history, motivating each conceptual step by pointing to its origins, and focusing on the goal of unique prime factorization with a minimum of distraction or prerequisites. This makes for a self-contained, easy-to-read book, short enough for a one-semester course.

JOHN STILLWELL is the author of many books on mathematics; among the best known are *Mathematics and Its History, Naive Lie Theory*, and *Elements of Mathematics*. He is a member of the inaugural class of Fellows of the American Mathematical Society and winner of the Chauvenet Prize for mathematical exposition.

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Following a Path from Euclid to Noether

JOHN STILLWELL
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To my grandchildren, Ida and Isaac

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Preface

The history of mathematics, like the life of each individual mathematician, is a story that begins with concrete experience and (generally) ends at high levels of abstraction. A good example, which we follow in this book, is the story of arithmetic. It begins with *counting*, then *adding* and *multiplying*; then it symbolizes this experience in *equations*. Next, it investigates equations via the abstract structures of *groups*, *rings*, and *fields*, and so on, to higher and higher levels of abstraction. This is a typical story, but the story alone does not explain why abstraction is necessary – or why it ever happened at all.

The reason is that abstract structures distill the essence of many concrete structures, enabling us to see past a mass of distracting details. For example, it is an impossible task to list all the facts about addition and multiplication of numbers, and some specific questions about them were not answered for hundreds of years. Mathematicians have been able to answer some of the hard questions only by working with abstract concepts that encapsulate the nature of addition and multiplication.

The art of algebra is the art of abstraction: choosing concepts that distill the essence of questions that interest us. To some extent the proof that we have chosen the "right" concepts is in the pudding. The right concepts answer many questions and make the answers seem obvious. But a concept may be "right" in the sharper sense that we can prove it is a *necessary part of the answer*. That is, an answer or solution exists only in structures that exemplify the concept in question.

A famous example is the discovery by Galois of the group concept, which explains which polynomial equations have solutions by radicals. Galois associated a group – now called the *Galois group* – with each equation and showed that an equation is solvable by radicals if and only if its Galois group

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has a certain property, now called *solvability*. Thus the concept of solvable group is the "right" concept to explain solvability of equations.

In this book we study a second famous example: Dedekind's theory of rings and ideals, which explains the phenomenon of unique prime factorization in arithmetic and its generalizations. Again, there is an abstract algebraic concept – now called a *Dedekind domain* – that exactly captures the property of unique prime factorization. Dedekind domains are an equally good example of the power of abstraction, and in some ways easier than the group concept, since their algebra is *commutative*. They also have a natural motivation as an outgrowth of arithmetic – which is why our path starts with Euclid.

The material in the book may be found in comprehensive graduate algebra texts, such as Zariski and Samuel (1958), Jacobson (1985), and Rotman (2015), but it is hard work to extract it from them. I prefer not to be comprehensive, so as to tell the story with only the essential abstractions, and to make it sufficiently self-contained to be accessible to undergraduates. This means including enough number theory to motivate the problem of unique prime factorization, which we do in the first three chapters. These chapters introduce algebraic numbers to solve classical equations such as the Pell equation, and the concepts of ring and field that abstract the algebra *of* these numbers.

Accessibility to undergraduates, in my opinion, also means including the *linear algebra* needed to view number fields and number rings as vector spaces and modules. I realize that this opinion is somewhat controversial. Modern books on algebraic number theory commonly assume linear algebra is already known, and indeed, every undergraduate takes a course in linear algebra these days. But linear algebra is a multifaceted subject, and I doubt that many undergraduates know the subject from the viewpoint needed here, which varies the base field (or base *ring*) and relies on the trace, determinant, characteristic polynomial, and discriminant. Those who do may skip the parts where these topics are covered, but I believe they should at least be skimmed in order to see where linear algebra fits in the bigger algebraic picture.

In fact the book closest to this one could be the classic telling of the story by Dedekind in 1877, which may be seen in English translation in Dedekind (1996). Dedekind's account is at a lower level of generality than ours, being concerned only with the needs of number theory, but it follows a similar path. The advantage of raising the level of generality is that one sees how close Dedekind came to the ultimate setting for unique prime factorization. As Emmy Noether used to say: "Es steht alles schon bei Dedekind." ("Everything is already in Dedekind.")

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I should say, however, that I raise the level of generality only in easy stages, when it becomes necessary. As in the history of the subject, the general case appears only after the important special cases.

To make the book useful to undergraduates and instructors, I have included many exercises, distributed in small batches at the end of most sections. These range from routine exercises, which test and reinforce understanding of new concepts, to exercise "packages" leading to substantial theorems. These theorems are often concrete consequences of the abstract machinery developed in the main text. The aim of each "package" is to reach an interesting goal by a sequence of easy steps, so the exercises include commentary to explain what the goal is and (in some cases) where to look for help later in the book.

Although many important and useful results occur in exercises, it should be stressed that these results are *not assumed* in the main text. In a few cases they are later *used* in the main text, but only after the main text has proved them.

In fact, the technical prerequisites for this book are small, since the whole point is to grow a big abstract structure from ideas in arithmetic. High school algebra should suffice, if it includes the matrix concept, and otherwise undergraduate linear algebra as far as matrices. Apart from these technical skills, however, the reader will also need sufficient mathematical maturity to be comfortable with abstractions. In most cases this will mean a couple of years of undergraduate mathematics, even a first course in abstract algebra. This book carries commutative algebra far beyond the typical first course, but it certainly will not hurt to have a first impression of fields, rings, and ideals.

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