A COURSE OF MODERN ANALYSIS

This classic work has been a unique resource for thousands of mathematicians, scientists and engineers since its first appearance in 1902. Never out of print, its continuing value lies in its thorough and exhaustive treatment of special functions of mathematical physics and the analysis of differential equations from which they emerge. The book also is of historical value as it was the first book in English to introduce the then modern methods of complex analysis.

This fifth edition preserves the style and content of the original, but it has been supplemented with more recent results and references where appropriate. All the formulas have been checked and many corrections made. A complete bibliographical search has been conducted to present the references in modern form for ease of use. A new foreword by Professor S. J. Patterson sketches the circumstances of the book's genesis and explains the reasons for its longevity. A welcome addition to any mathematician's bookshelf, this will allow a whole new generation to experience the beauty contained in this text.

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(Left): Edmund Taylor Whittaker (1873–1956); (Right): George Neville Watson (1886–1965): Universal History Archive/Contributor/Getty Images.

A COURSE OF MODERN ANALYSIS

Fifth Edition

An introduction to the general theory of infinite processes and of analytic functions with an account of the principal transcendental functions

E.T. WHITTAKER AND G.N. WATSON

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Foreword

S.J. Patterson

There are few books which remain in print and in constant use for over a century; "Whittaker and Watson" belongs to this select group. In fact there were two books with the title "A Course in Modern Analysis", the first in 1902 by Edmund Whittaker alone, a textbook with a very specific agenda, and then the joint work, first published in 1915 as a second edition. It is an extension of the first edition but in such a fashion that it becomes a handbook for those working in analysis. As late as 1966 J.T. Whittaker, the son of E.T. Whittaker, wrote in his Biographical Memoir of Fellows of the Royal Society (i.e. obituary) of G.N. Watson that there were still those who preferred the first edition but added that for most readers the later edition was to be preferred. Indeed the joint work is superior in many different ways.

The first edition was written at a time when there was a movement for reform in mathematics at Cambridge. Edmund Whittaker's mentor Andrew Forsyth was one of the driving forces in this movement and had himself written a *Theory of Functions* (1893) which was, in its time, very influential but is now scarcely remembered. In the course of the nineteenth century the mathematics education had become centered around the Mathematical Tripos, an intensely competitive examination. Competitions and sports were salient features of Victorian Britain, a move away from the older system of patronage and towards a meritocracy. The reader familiar with Gilbert and Sullivan operettas will think of the Modern Major-General in *The Pirates of Penzance*. The Tripos had become not only a sport but a spectator sport, followed extensively in middle-class England[1]. The result of this system was that the colleges were in competition with one another and employed coaches to prepare the talented students for the Tripos. They developed the skills needed to answer difficult questions quickly and accurately – many Tripos questions can be found in the exercises in *Whittaker and Watson*. The Tripos system did not encourage the students to become mathematicians and separated them from the professors who were generally very well informed about the developments on the Continent. It was a very inward-looking, self-reproducing system. The system on the Continent, especially in the German universities, was quite different. The professors there sought contact with the students, either as note-takers for lectures or in seminar talks, and actively supported those by whom they were most impressed. The students vied with one another for the attention of the professor, a different and more fruitful form of competition. This

[1] Some idea of this may be gleaned from G.B. Shaw's play *Mrs Warren's Profession*, written in 1893 but held back by censorship until 1902. In this play Mrs Warren's daughter Vivie has distinguished herself in Cambridge – she tied with the third Wrangler, described as a "magnificent achievement" by a character who has no mathematical background. She herself could not be ranked as a Wrangler as she was female. She would have been a contemporary of Grace Chisholm, later Grace Chisholm Young, whose family background was by no means as colourful as that of the fictional Vivie Warren.

system allowed the likes of Weierstrass and Klein to build up groups of talented and highly motivated students. It had become evident to Andrew Forsyth and others that Cambridge was missing out on the developments abroad because of the concentration on the Tripos system[2].

It is interesting to read what Whittaker himself wrote about the situation at the end of the nineteenth century in Cambridge and so of the conditions under which *Whittaker and Watson* was written. We quote from his Royal Society Obituary Notice (1942) of Andrew Russell Forsyth:

> He had for some time past realized, as no one else did, the most serious deficiency of the Cambridge school, namely its ignorance of what had been and was being done on the continent of Europe. The college lecturers could not read German, and did not read French.
>
> ⋮
>
> The schools of Göttingen and Berlin to a great extent ignored each other (Berlin said that Göttingen proved nothing, and Göttingen retorted that Berlin had no ideas) and both of them ignored French work.
>
> But Cambridge had hitherto ignored them all: and the time was ripe for Forsyth's book. The younger men, even undergraduates, had heard in his lectures of the extraordinary riches and beauty of the domain beyond Tripos mathematics, and were eager to enter into it. From the day of its publication in 1893, the face of Cambridge was changed: the majority of the pure mathematicians who took their degrees in the next twenty years became function-theorists.

and further

> As head of the Cambridge school of mathematics he was conspicuously successful. British mathematicians were already indebted to him for the first introduction of the symbolic invariant-theory, the Weierstrassian elliptic functions, the Cauchy–Hermite applications of contour-integration, the Riemannian treatment of algebraic functions, the theory of entire functions, and the theory of automorphic functions: and the importation of novelties continued to occupy his attention. A great traveller and a good linguist, he loved to meet eminent foreigners and invite them to enjoy Trinity hospitality: and in this way his post-graduate students had opportunities of becoming known personally to such men as Felix Klein (who came frequently), Mittag-Leffler, Darboux and Poincaré. To the students themselves, he was devoted: young men fresh from the narrow examination routine of the Tripos were invited to his rooms and told of the latest research papers: and under his fostering care, many of the wranglers of the period 1894–1910 became original workers of distinction.

The two authors were very different people. Edmund Whittaker (1874–1956) went on from Cambridge in 1906 to become the Royal Astronomer in Ireland (then still a part of the

[2] For his arguments see A. Forsyth: Old Tripos Days at Cambridge, *Math. Gazette* **19** 162–179 (1935). For a dissenting opinion see K. Pearson: Old Tripos Days at Cambridge, as seen from another viewpoint, *Math. Gazette* **20** 27–36 (1936).

United Kingdom) and Director of Dunsink Observatory, thereby following in the footsteps of William Rowan Hamilton. In 1985, on the occasion of the bicentenary of Dunsink, the then Director, Patrick A. Wayman, singled out Whittaker as the greatest director aside from Hamilton and one who, despite his relatively short tenure of office, 1906–1912, had achieved most for the Observatory[3]. This appointment brought out his skills as an administrator. Following this he moved to Edinburgh where he exerted his influence to guide mathematics there into the new century. Some indication of the success is given by the fact that it was W.V.D. Hodge, a student of his, who, at the International Congress of Mathematicians in 1954, invited the International Mathematical Union to hold the next Congress in Edinburgh. Whittaker himself did not live to experience the event which reflected the status in which Edinburgh was held at the end of his life.

George Neville Watson (1886–1965) on the other hand was a retiring scholar who, after leaving Cambridge, at least in the flesh, spent four years (1914–1918) in London, and then became professor in Birmingham where he remained for the rest of his life[4], living a relatively withdrawn life devoted to his mathematical work and with stamp-collecting and the study of the history of railways as hobbies. His early work was very much in the direction of E.W. Barnes and A.G. Greenhill. After Ramanujan's death he took over from Hardy the analysis of many of Ramanujan's unpublished papers, especially those connected with the theory of modular forms and functions, and of complex multiplication. It is worth remarking that Greenhill, a student and ardent admirer of James Clerk Maxwell and primarily an applied mathematician, concerned himself with the computation of singular moduli, and it was probably he who aroused Ramanujan's interest in this topic. Watson's work in this area is, besides his books, that for which he is best remembered today.

Both authors wrote other books that are still used today. In Whittaker's case these are his *A Treatise on the Analytical Dynamics of Particles & Rigid Bodies*, reprinted in 1999, with a foreword by Sir William McCrea in the CUP series "Cambridge Mathematical Library", a source of much mathematics which is difficult to find elsewhere, and his *History of Theories of the Aether and Electricity* which, despite some unconventional views, is an invaluable source on the history of these parts of physics and the associated mathematics.

Watson, on the other hand, wrote his *A Treatise on the Theory of Bessel Functions*, published in 1922, which like *Whittaker and Watson* has not been out of print since its appearance. On coming across it for the first time as a student I was taken aback by such a thick book being devoted to what seemed to be a very circumscribed subject. One of the Fellows of my college, a physicist, replying to a fellow student who had made a similar observation, declared that it was a work of genius and he would have been proud to have written something like it. In the course of the years I have had recourse to it over and over again and would now concur with this opinion.

Watson's *Bessel Functions*, like *Whittaker and Watson*, despite being somewhat old-fashioned, has retained a freshness and relevance that has made both of them classics. Unlike many other books of this period the terminology, although not the style, is that of today. It is less a *Cours d'Analyse* and more of a *Handbuch der Funktionentheorie*. Perhaps my own experiences can illuminate this. My copy was given to me in 1967 by my mathematics teacher,

[3] *Irish Astronomical Journal* **17** 177–178 (1986).
[4] It is worth noting that from 1924 on E.W. Barnes was a disputative Bishop of Birmingham.

Mr Cecil Hawe, after I had been awarded a place to study mathematics in Cambridge. He had bought it 20 years earlier as a student. During my student years *the* textbook on second year analysis was J. Dieudonné's *Foundations of Modern Analysis*. People then were prone to be a bit supercilious at least about the "modern" in the title of *Whittaker and Watson*.[5] At that time it lay on my bookshelf unused. Five years later I was coming to terms with the theory of non-analytic automorphic forms, especially with Selberg's theory of Eisenstein series. At this point I discovered how useful a book it was, both for the treatment of Bessel functions and for that of the hypergeometric function. It also has a very useful chapter on Fredholm's theory of integral equations which Selberg had used. In the years since then several other chapters have proved useful, and ones I thought I knew became useful in novel ways. It became a constant companion. This was mainly in connection with doing mathematics but it also proved its worth in teaching – for example the chapter on Fourier series gives very useful results which can be obtained by relatively elementary methods and are suitable for undergraduate lectures. Dieudonné's book is tremendous for the university teacher; it gives the fundamentals of analysis in a concentrated form, something very useful when one has an overloaded syllabus and a limited number of hours to teach it in. On the other hand it is much less useful as a "*Handbuch*" for the working analyst, at least in my experience. Nor was it written for this purpose. *Whittaker and Watson* started, in the first edition, as such a book for teaching but in the second and later editions became that book which has remained on the bookshelves of generations of working mathematicians, be they formally mathematicians, natural scientists or engineers.

One aspect that probably contributed to the long popularity of *Whittaker and Watson* is the fact that it is not overloaded with many of the topics that are within range of the text. Thus, for example, the authors do not go into the arithmetic theory of the Riemann zeta-function beyond the Euler product over primes. Whereas they discuss the 24 solutions to the hypergeometric equation in terms of the hypergeometric series from Riemann's point of view they do not go into H.A. Schwarz' beautiful solution of Gauss' problem as to which of these functions is algebraic. Schwarz' theory is covered in Forsyth's *Function Theory*. The decision to leave this out must have been difficult for Whittaker for it is a topic close to his early research. Finally they touch on the theory of Hilbert spaces only very lightly, just enough for their purposes. On the other hand Fredholm's theory, well treated here, has often been pushed aside by the theory of Hilbert spaces in other texts and it is a topic about which an analyst should be aware.

So, gentle reader, you have in your hands a book which has been useful and instructive to those working in mathematics for well over a hundred years. The language is perhaps a little quaint but it is a pleasure to peruse. May you too profit from this new edition.

[5] B.L. v.d. Waerden's *Moderne Algebra* became simply *Algebra* from the 1955 edition on; with either name it remains a great text on algebra.

Preface to the Fifth Edition

In 1896 Edmund Whittaker was elected to a Fellowship at Trinity College, Cambridge. Amongst other duties, he was employed to teach students, many of whom would later become distinguished figures in science and mathematics. These included G.H. Hardy, Arthur Eddington, James Jeans, J.E. Littlewood and a certain G. Neville Watson. His course on mathematical analysis changed the way the subject was taught, and he decided to write a book. So was born *A Course of Modern Analysis*, which was first published in 1902. It introduced students to functions of a complex variable, to the 'methods and processes of higher mathematical analysis', much of which was then fairly modern, and above all to special functions associated with equations that were used to describe physical phenomena. It was one of the first books in English to describe material developed *on the continent*, mostly in France and Germany. Its breadth and depth of coverage were unparalleled at the time and it became an instant classic. A second edition was called for, but in 1906 Whittaker had left Cambridge, moving first to Dublin, and then in 1912 to Edinburgh. His various duties, and no doubt, the moves themselves, impeded work on the new edition, and Whittaker gratefully accepted the offer from Watson to help him. A greatly expanded second edition duly appeared in 1915. The third edition, published five years later, was also enlarged by the addition of chapters, but the fourth edition was not much more than a corrected reprint with added references. I do not know if a fifth edition was ever planned. Both authors remained active for many years (Watson wrote, amongst other publications, the definitive *Treatise on Bessel Functions*), but perhaps they had nothing more to say to warrant a new edition. Nevertheless, the book remained a classic, being continually in print and reissued in paperback, first in 1963, and again, in 1996, as a volume of the *Cambridge Mathematical Library*. It never lost its appeal and occupied a unique place in the heart and work of many mathematicians (in particular, me) as an indispensable reference.

The original editions were typeset using 'hot metal', and over the years successive reprintings led to the degrading of the original plates. Photographic printing methods slowed this decline, but David Tranah at Cambridge University Press had the idea to halt, indeed reverse, the degradation, by rekeying the book and at the same time updating it with new references and commentary. He spoke to me about this, and we agreed that if he arranged for the rekeying into LaTeX, I would do the updating. I did not need much persuading: it has been a labor of love. So much so that I have preserved the archaic spelling of the original, along with the Peano decimal system of numbering paragraphs, as described by Watson in the Preface to the fourth edition! This will make it straightforward for users of this fifth edition to refer to the previous one. I have however decided to create a complete reference list and to refer readers to that rather than to items in footnotes, items that were often hard to identify. Many of these items are now available in digital libraries and so for many people will be easier to access than they were in the authors' time.

I have made no substantial changes to the text: in particular, the original idea of adding commentaries on the text was abandoned. I have checked and rechecked the mathematics, and I have added some additional references. I have also written an introduction that describes what's in the book and how it may be used in contemporary teaching of analysis. I have also provided summaries of each chapter, and, within them, make mention of more recent work where appropriate.

As I said, preparing this edition has been a labor of love. I have also learned a lot of mathematics, evidence of the enduring quality and value of the original work. It has been a rewarding experience to edit *A Course of Modern Analysis*: I hope that it will be equally rewarding for readers.

Victor H. Moll
2020, New Orleans

Preface to the Fourth Edition

Advantage has been taken of the preparation of the fourth edition of this work to add a few additional references and to make a number of corrections of minor errors.

Our thanks are due to a number of our readers for pointing out errors and misprints, and in particular we are grateful to Mr E. T. Copson, Lecturer in Mathematics in the University of Edinburgh, for the trouble which he has taken in supplying us with a somewhat lengthy list.

E. T. W.
G. N. W.
June 18, 1927

The decimal system of paragraphing, introduced by Peano, is adopted in this work. The integral part of the decimal represents the number of the chapter and the fractional parts are arranged in each chapter in order of magnitude. Thus, e.g., on pp. 187, 188[6], §9.632 precedes §9.7 [because 9.632 < 9.7.]

G.N.W.
July 1920

[6] in the fourth edition

Preface to the Third Edition

Advantage has been taken of the preparation of the third edition of this work to add a chapter on Ellipsoidal Harmonics and Lamé's Equation and to rearrange the chapter on Trigonometric Series so that the parts which are used in Applied Mathematics come at the beginning of the chapter. A number of minor errors have been corrected and we have endeavoured to make the references more complete.

Our thanks are due to Miss Wrinch for reading the greater part of the proofs and to the staff of the University Press for much courtesy and consideration during the progress of the printing.

E. T. W.
G. N. W.
July, 1920

Preface to the Second Edition

When the first edition of my *Course of Modern Analysis* became exhausted, and the Syndics of the Press invited me to prepare a second edition, I determined to introduce many new features into the work. The pressure of other duties prevented me for some time from carrying out this plan, and it seemed as if the appearance of the new edition might be indefinitely postponed. At this juncture, my friend and former pupil, Mr G. N. Watson, offered to share the work of preparation; and, with his cooperation, it has now been completed.

The appearance of several treatises on the Theory of Convergence, such as Mr Hardy's *Course of Pure Mathematics* and, more particularly, Dr Bromwich's *Theory of Infinite Series*, led us to consider the desirability of omitting the first four chapters of this work; but we finally decided to retain all that was necessary for subsequent developments in order to make the book complete in itself. The concise account which will be found in these chapters is by no means exhaustive, although we believe it to be fairly complete. For the discussion of Infinite Series on their own merits, we may refer to the work of Dr Bromwich.

The new chapters of Riemann Integration, on Integral Equations, and on the Riemann Zeta-Function, are entirely due to Mr Watson: he has revised and improved the new chapters which I had myself drafted and he has enlarged or partly rewritten much of the matter which appeared in the original work. It is therefore fitting that our names should stand together on the title-page.

Grateful acknowledgement must be made to Mr W. H. A. Lawrence, B.A., and Mr C. E. Winn, B.A., Scholars of Trinity College, who with great kindness and care have read the proof-sheets, to Miss Wrinch, Scholar of Girton College, who assisted in preparing the index, and to Mr Littlewood, who read the early chapters in manuscript and made helpful criticisms. Thanks are due also to many readers of the first edition who supplied corrections to it; and to the staff of the University Press for much courtesy and consideration during the progress of the printing.

E.T. Whittaker
July 1915

Preface to the First Edition

The first half of this book contains an account of those methods and processes of higher mathematical analysis, which seem to be of greatest importance at the present time; as will be seen by a glance at the table of contents, it is chiefly concerned with the properties of infinite series and complex integrals and their applications to the analytical expression of functions. A discussion of infinite determinants and of asymptotic expansions has been included, as it seemed to be called for by the value of these theories in connexion with linear differential equations and astronomy.

In the second half of the book, the methods of the earlier part are applied in order to furnish the theory of the principal functions of analysis – the Gamma, Legendre, Bessel, Hypergeometric, and Elliptic Functions. An account has also been given of those solutions of the partial differential equations of mathematical physics which can be constructed by the help of these functions.

My grateful thanks are due to two members of Trinity College, Rev. E. M. Radford, M.A. (now of St John's School, Leatherhead), and Mr J. E. Wright, B.A., who with great kindness and care have read the proof-sheets; and to Professor Forsyth, for many helpful consultations during the progress of the work. My great indebtedness to Dr Hobson's memoirs on Legendre functions must be specially mentioned here; and I must thank the staff of the University Press for their excellent cooperation in the production of the volume.

E. T. WHITTAKER
Cambridge
1902 August 5

Introduction

The book is divided into two distinct parts. **Part I. The Processes of Analysis** discusses topics that have become standard in beginning courses. Of course the emphasis is in concrete examples and regrettably, this is different nowadays. Moreover the quality and level of the problems presented in this part is higher than what appears in more modern texts. During the second part of the last century, the tendency in introductory Analysis texts was to emphasize the topological aspects of the material. For obvious reasons, this is absent in the present text. There are 11 chapters in Part I.

For a student in an American university, the material presented here is roughly distributed along the following lines:

- Chapter 1 (Complex Numbers)
- Chapter 2 (The Theory of Convergence)
- Chapter 3 (Continuous Functions and Uniform Convergence)
- Chapter 4 (The Theory of Riemann Integration)

are covered in *Real Analysis* courses.

- Chapter 5 (The Fundamental Properties of Analytic Functions; Taylor's, Laurent's and Liouville's Theorems)
- Chapter 6 (The Theory of Residues, Applications to the Evaluations of Definite Integrals)
- Chapter 7 (The Expansion of Functions in Infinite Series)

are covered in *Complex Analysis*. These courses usually cover the more elementary aspects of

- Chapter 12 (The Gamma-Function)

appearing in Part II.

Most undergraduate programs also include basic parts of

- Chapter 9 (Fourier Series and Trigonometric Series)
- Chapter 10 (Linear Differential Equations)

and some of them will expose the student to the elementary parts of

- Chapter 8 (Asymptotic Expansions and Summable Series)
- Chapter 11 (Integral Equations)

The material covered in Part II is mostly absent from a generic graduate program. Students interested in Number Theory will be exposed to some parts of the contents in

- Chapter 12 (The Gamma-Function)
- Chapter 13 (The Zeta-Function of Riemann)
- Chapter 14 (The Hypergeometric Function)

and a glimpse of

- Chapter 17 (Bessel Functions)
- Chapter 20 (Elliptic Functions. General Theorems and the Weierstrassian Functions)
- Chapter 21 (The Theta-Functions)
- Chapter 22 (The Jacobian Elliptic Functions).

Students interested in Applied Mathematics will be exposed to

- Chapter 15 (Legendre Functions)
- Chapter 16 (The Confluent Hypergeometric Function)
- Chapter 18 (The Equations of Mathematical Physics)

and some parts of

- Chapter 19 (Mathieu Functions)
- Chapter 23 (Ellipsoidal Harmonics and Lamé's Equation)

It is perfectly possible to complete a graduate education without touching upon the topics in Part II. For instance, in the most commonly used textbooks for Analysis, such as Royden [565] and Wheeden and Zygmund [666] there is no mention of special functions. On the complex variables side, in Ahlfors [13] and Greene–Krantz [260] one finds some discussion on the Gamma function, but not much more.

This is not a new phenomenon. Fleix Klein [377] in 1928 (quoted in [91, p. 209]) writes '*When I was a student, Abelian functions were, as an effect of the Jacobian tradition, considered the uncontested summit of mathematics, and each of us was ambitious to make progress in this field. And now? The younger generation hardly knows Abelian functions.*

During the last two decades, the trend towards the abstraction is being complemented by a group of researchers who emphasize concrete examples as developed by Whittaker and Watson. Among the factors influencing this return to the classics one should include[7] the appearance of symbolic languages and algorithms producing automatic proofs of identities. The work initiated by Wilf and Zeilberger, described in [518], shows that many identities have *automatic proofs*. A second influential factor is the monumental work by B. Berndt, G. Andrews and collaborators to provide context and proofs of all results appearing in S. Ramanujan's work. This has produced a collection of books, starting with [60] and currently at [25]. The third example in this list is the work developed by J. M. Borwein and his collaborators in the propagation of *Experimental Mathematics*. In the volumes [88, 89] the authors present their ideas on how to transform mathematics into a subject, similar in flavor to other experimental sciences. The point of view expressed in the three examples mentioned above has attracted a new generation of researchers to get involved in this point of view type of mathematics. This is just one direction in which Whittaker and Watson has been a profound influence in modern authors.

[7] This list is clearly a subjective one.

The remainder of this chapter outlines the content of the book and a comparison with modern practices.

The first part is named **The Processes of Analysis**. It consists of 11 chapters. A brief description of each chapter is provided next.

Chapter 1: Complex Numbers. The authors begin with an informal description of positive integers and move on to rational numbers. Stating that *from the logical standpoint it is improper to introduce geometrical intuition to supply deficiencies in arithmetical arguments*, they adopt Dedekind's point of view on the construction of real numbers as classes of rational numbers, later called *Dedekind's cuts*. An example is given to show that there is no rational number whose square is 2. The arithmetic of real numbers is defined in terms of these cuts. Complex numbers are then introduced with a short description of *Argand diagrams*. The current treatment offers two alternatives: some authors present the real number from a collection of axioms (as an ordered infinite field) and other approach them from Cauchy's theory of sequences: *a real number is an equivalence class of Cauchy sequences of rational numbers*. The reader will find the first point of view in [304] and the second one is presented in [599].

Chapter 2. The Theory of Convergence. This chapter introduces the notion of convergence of sequences of real or complex numbers starting with the definition of $\lim_{n\to\infty} x_n = L$ currently given in introductory texts. The authors then consider monotone sequences of real numbers and show that, for bounded sequences, there is a natural Dedekind cut (that is, a real number) associated to them. A presentation of Bolzano's theorem *a bounded sequence of real numbers contains a limit point* and Cauchy's formulation of the completeness of real numbers; that is, the existence of the limit of a sequence in terms of elements being arbitrarily close, is discussed. These ideas are then illustrated in the analysis of convergence of series. The discussion begins with *Dirichlet's test for convergence*: *Assume a_n is a sequence of complex numbers and f_n is a sequence of positive real numbers. If the partial sums $\sum_{n=1}^{p} a_n$ are uniformly bounded and f_n is decreasing and converges to* 0*, then $\sum_{n=1}^{\infty} a_n f_n$ converges*. This is used to give examples of convergence of Fourier series (discussed in detail in Chapter 9). The convergence of the geometric series $\sum_{n=1}^{\infty} x^n$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$, for real s, are presented in detail. This last series defines the *Riemann zeta function* $\zeta(s)$, discussed in Chapter 13. The elementary ratio test states that $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} |a_{n+1}/a_n| < 1$ and diverges if the limit is strictly above 1. A discussion of the case when the limit is 1 is presented and illustrated with the convergence analysis of the *hypergeometric series* (presented in detail in Chapter 14). The chapter contains some standard material on the convergence of power series as well as some topics not usually found in modern textbooks: discussion on double series, convergence of infinite products and infinite determinants. The final exercise[8] in this chapter presents the evaluation of an infinite determinant considered by Hill in his analysis of the Schrödinger

[8] In this book, Examples are often what are normally known as Exercises and are numbered by section, i.e., 'Example a.b.c'. At the end of most chapters are Miscellaneous Examples, all of which are Exercises, and which are numbered by chapter: thus 'Example a.b'. This is how to distinguish them.

equation with periodic potential (this is now called the *Hill equation*). The reader will find in [451] and [536] information about this equation.

Chapter 3. Continuous Functions and Uniform Convergence. This chapter also discusses functions $f(x, y)$ of two real variables as well as functions of one complex variable $g(z)$. The notion of *uniform convergence* of a series is discussed in the context of the limiting function of a series of functions. This is normally covered in every introductory course in Analysis. The classical M-test of Weierstrass is presented. The reader will also find a test for uniform convergence, due to Hardy, and its application to the convergence of Fourier series. The chapter also contains a discussion of the series

$$g(z) = \sum_{m,n} \frac{1}{(z + 2m\omega_1 + 2n\omega_2)^\alpha}$$

which will be used to analyze the Weierstrass $\wp$-function: one of the fundamental *elliptic functions* (discussed in Chapter 20). The chapter contains a discussion on the fact that a continuous function defined of a compact set (in the modern terminology) attains its maximum/minimum value. This is nowadays a standard result in elementary analysis courses.

Chapter 4. The Theory of Riemann Integration. The authors present the notion of the Riemann integral on a finite interval $[a, b]$, as it is currently done: as limiting values of upper and lower sums. The fact that a continuous function is integrable is presented. The case with finite number of discontinuities is given as an exercise. Basic results, such as integration by parts, differentiation with respect to the limits of integration, differentiation with respect to a parameter, the mean value theorem for integrals and the representation of a double integral as iterated integral are presented. This material has become standard. The chapter also contains a discussion on integrals defined on an infinite interval. There is a variety of tests to determine convergence and criteria that can be used to evaluate the integrals. Two examples of integral representations of the *beta integral* (discussed in Chapter 12) are presented. A basic introduction to *complex integration* is given at the end of the chapter; the reader is referred to Watson [650] for more details. This material is included in basic textbooks in Complex Analysis (for instance, see [13, 26, 155, 260, 455, 552]).

Chapter 5. The Fundamental Properties of Analytic Functions; Taylor's, Laurent's and Liouville's Theorems. This chapter presents the basic properties of analytic functions that have become standard in elementary books in complex analysis. These include the Cauchy–Riemann equations and Cauchy's theorem on the vanishing of the integral of an analytic function taken over a closed contour. This is used to provide an integral representation as

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{z - \xi}\, d\xi$$

where Γ is a closed contour containing ξ in its interior. This is then used to establish classical results on analytic functions such as bounds on the derivatives and Taylor theorem. There is also a small discussion on the process of analytic continuation and many-valued functions. This chapter contains also basic properties on functions having poles as isolated singularities: Laurent's theorem on expansions and Liouville's theorem on the fact that every entire function that is bounded must be constant (a result that plays an important role in the presentation of elliptic functions in Chapter 20). The Bessel function J_n, defined by its integral representation

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - x \sin\theta)\, d\theta$$

makes its appearance in an exercise. This function is discussed in detail in Chapter 17. The chapter also contains a proof of the following fact: *any function that is analytic, including at ∞, except for a number of non-essential singularities, must be a rational function*. This has become a standard result. It represents the most elementary example of characterizing *functions of rational character on a Riemann surface*. This is the case of $\mathbb{P}^1$, the Riemann sphere. The next example corresponds to the torus $\mathbb{C}/\mathbb{L}$, where $\mathbb{L}$ is a lattice. This is the class of elliptic functions described in Chapters 20, 21 and 22. The reader is referred to [461, 553, 600, 665] for more details.

Chapter 6. The Theory of Residues: Application to the Evaluation of Definite Integrals. This chapter presents application of Cauchy's integral representation of functions analytic except for a certain number of poles. Most of the material discussed here has become standard. One of the central concepts is that of the *residue* of a function at a pole $z = z_k$, defined as the coefficient of $(z - z_k)^{-1}$ in the expansion of f near $z = z_k$. As a first sign of the importance of these residues is the statement that the integral of $f(z)$ over the boundary of a domain Ω is given by the sum of the residues of f inside Ω, the so-called *argument principle* which gives the difference between zeros and poles of a function as a contour integral. This chapter also presents methods based on residues to evaluate a variety of definite integrals including rational functions of $\cos\theta$, $\sin\theta$ over $[0, 2\pi]$, integrals over the whole real line via deformation of a semicircle, integrals involving some of the kernels such as $1/(e^{2\pi z} - 1)$ (coming from the Fermi–Dirac distribution in Statistical Mechanics) and $1/(1-2a\cos x + a^2)$ related to Legendre polynomials (discussed in Chapter 15). An important function makes its appearance as Exercise 17:

$$\psi(t) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi t},$$

introduced by Poisson in 1823. The exercise outlines a proof of the transformation rule

$$\psi(t) = t^{-1/2}\psi(1/t)$$

known as *Poisson summation formula*. It plays a fundamental role in many problems in Number Theory, including the proof of the *prime number theorem*. This states that, for $x > 0$, the number of primes up to x, denoted by $\pi(x)$, has the asymptotic behavior $\pi(x) \sim x/\log x$ as $x \to \infty$. The reader will find in [492] how to use contour integration and the function $\psi(t)$ to provide a proof of the asymptotic behavior of $\psi(t)$. This function reappears in Chapter 21 in the study of *theta functions*.

Chapter 7. The Expansion of Functions in Infinite Series. This chapter begins with a result of Darboux on the expansion of an analytic function defined on a region Ω. For points a, x, with the segment from a to x contained in Ω, one has the expansion

$$\phi^{(n)}(0)[f(z)-f(0)] = \sum_{k=1}^{n}(-1)^{k-1}(z-a)^k\left[\phi^{(n-k)}(1)f^{(m)}(z)-\phi^{n-k}(0)f^{(k)}(a)\right] + (-1)^n(z-a)^{n+1}\int_0^1 \phi(t)f^{(n+1)}(a+t(z-a))\,dt,$$

for any polynomial ϕ. The formula is then applied to the *Bernoulli polynomials* currently defined by the generating function

$$\frac{te^{zt}}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n(z)}{n!}t^n.$$

(The text employs the notation $\phi_n(t)$ without giving the value for $n=0$.) Darboux's theorem then becomes the classical *Euler–MacLaurin summation formula*

$$\sum_{j=0}^{n} f(j) = \int_0^n f(x)\,dx + \frac{f(n)+f(0)}{2} + \sum_{k=1}^{\lfloor p/2\rfloor}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n)-f^{(2k-1)}(0)\right] + (-1)^{p-1}\int_0^n f^{(p)}(x)\frac{B_p(x-\lfloor x\rfloor)}{p!}\,dx.$$

The quantity $x-\lfloor x\rfloor$ is the *fractional part* of x, denoted by $\{x\}$. This formula is used to estimate partial sums of series of values of an analytic function in terms of the corresponding integrals. The important example of the *Riemann zeta function* $\zeta(s)$ is presented in Chapter 13.

The chapter contains a couple of examples of expansions of one function in terms of another one. The first one, due to Bürmann, starts with an analytic function $\phi(z)$ defined on a region and $\phi(a)=b$ with $\phi'(a)\neq 0$. Define $\psi(z)=(z-a)/(\phi(z)-a)$, then one obtains the expansion

$$f(z) = f(a) + \sum_{k=1}^{n-1}\frac{[\phi(z)-b]^k}{k!}\left(\frac{d}{da}\right)^{k-1}\left[f'(a)\psi^k(a)\right] + R_n$$

where the error term has the integral representation

$$R_n = \frac{1}{2\pi i}\int_a^z\int_\gamma\left[\frac{\phi(z)-b}{\phi(t)-b}\right]^{n-1}\frac{f'(t)\phi'(z)}{\phi(t)-\phi(z)}\,dt\,dz,$$

where γ is a contour in the t-plane, enclosing a and t and such that, for any μ interior to γ, the equation $\phi(t)=\phi(\mu)$ has a unique solution $t=\mu$. The discussion also contains results of Teixeira on conditions for the convergence of the series for $f(z)$ obtained by letting $n\to\infty$. This type of result also contains an expansion of Lagrange for solutions of the equation $\mu = a + t\phi(\mu)$, for analytic function ϕ satisfying $|t\phi(z)| < |z-a|$. The theorem states that any analytic function f of the solution μ can be expanded as

$$f(\mu) = f(a) + \sum_{n=1}^{\infty}\frac{t^n}{n!}\left(\frac{d}{da}\right)^{n-1}\left[f'(a)\phi^n(a)\right].$$

This expansion has interesting applications in Combinatorics; see [681] for details. The last type of series expansion described here corresponds to the classical partial fraction expansions of a rational function and its extensions to trigonometric functions.

The results of this chapter are then used to prove representations of an entire function f in the form

$$f(z) = f(0)e^{G(z)} \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{a_n}\right) e^{g_n(z)} \right\}^{m_n}$$

where a_n is a zero of f of multiplicity m_n and $G(z)$ is an entire function. The function $g_n(z)$ is a polynomial, introduced by Weierstrass, which makes the product converge. An application to $1/\Gamma(z)$ is discussed in Chapter 12.

Chapter 8. Asymptotic Expansions and Summable Series. This chapter presents an introduction to the basic concepts behind asymptotic expansion. The initial example considers $f(x) = \int_x^\infty t^{-1} e^{x-t}\, dt$. A direct integration by parts shows that the sum $S_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, k!}{x^{k+1}}$ satisfies, for fixed x, the inequality $|f(x) - S_n(x)| \leq n!/x^{n+1}$. Therefore, for $x \geq 2n$, one obtains $|f(x) - S_n(x)| < 1/n^2 2^{n+1}$. It follows that the integral f can be evaluated with great accuracy for large values of x by computing the partial sum of the divergent series $S_n(x)$. This type of behavior is written as $f(x) \sim \sum_{n=0}^{\infty} A_n x^{-n}$ and the series is called *the asymptotic expansion* of f.

The chapter covers the basic properties of asymptotic series: such expansions can be multiplied and integrated *but not* differentiated. Examples of asymptotic expansions of special functions appear in later chapters: for the Gamma function in Chapter 12 and for the Bessel function in Chapter 17.

The final part of the chapter deals with *summation methods*, concentrating on methods assigning a value to a function given by a power series outside its circle of convergence D. The first example, due to Borel, starts with the identity

$$\sum_{n=0}^{\infty} a_n z^n = \int_0^\infty e^{-t}\phi(tz)\, dt \text{ where } \phi(u) = \sum_{n=0}^{\infty} \frac{a_n}{n!} u^n \text{ valid for } z \in D.$$

The series $\sum_{n=0}^{\infty} a_n z^n$ is said to be *Borel summable* if the integral on the right converges for z outside D. For such z, the Borel sum of the series is assigned to be the value of the integral. The discussion continues with *Cesàro summability*, a notion to be discussed in the context of Fourier series in Chapter 9. Extensions by Riesz and Hardy are mentioned. More details on asymptotic expansions can be found in [468, 508].

Chapter 9. Fourier Series and Trigonometric Series. The authors discuss *trigonometrical series* defined as series of the form

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

for two sequences of real numbers $\{a_n\}$ and $\{b_n\}$. Such series are named *Fourier series* if there is a function f, with finite integral over $(-\pi, \pi)$, such that the coefficients are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt.$$

The chapter contains a variety of results dealing with conditions under which the Fourier series associated to a function f converges to f. These include Dirichlet's theorem stating that, under some technical conditions, the Fourier series converges to $\frac{1}{2}\left[f(x+0)+f(x-0)\right]$. This is followed by Fejer's theorem that the Fourier series is Césaro summable at all points where the limits $f(x \pm 0)$ exist. The proofs are based on the analysis of the so-called Dirichlet–Féjer kernel. Examples are provided where there is not a single analytic expression for the Fourier series. The notion of orthogonality of the sequence of trigonometric functions makes an implicit appearance in all the proofs. The so-called Riemann–Lebesgue theorem, on the behavior of Fourier coefficients, is established. This result states that if $\psi(\theta)$ is integrable on the interval (a, b), then $\lim_{n\to\infty} \int_a^b \psi(\theta)\sin(\lambda\theta)\,d\theta = 0$. The chapter contains results on the function f which imply pointwise convergence of the Fourier series. The results of Dini and Jordan, with conditions on the expressions $f(x \pm 2\theta) - f(x \pm 0)$ near $\theta = 0$, are presented. The reader will find more information about convergence of Fourier series in [368] and in the treatise [690]. The results of Kolmogorov [381, 382] on an integrable function with a Fourier series diverging everywhere, as well as the theorem of Carleson [118] on the almost-everywhere convergence of the Fourier series of a continuous function, are some of the high points of this difficult subject.

The chapter also includes a discussion on the uniqueness of the representation of a Fourier series for a function f and also of the *Gibbs phenomenon* on the behavior of a Fourier series in a neighborhood of a point of discontinuity of f.

Chapter 10. Linear Differential Equations. This chapter discusses properties of solutions of second order linear differential equations

$$\frac{d^2u}{dz^2} + p(z)\frac{du}{dz} + q(z)u = 0,$$

where p, q are analytic functions of z except for a finite number of points. The discussion is local; that is, in a neighborhood of a point $c \in \mathbb{C}$. The points c are classified as *ordinary*, where the functions p, q are assumed to be analytic at c and otherwise *singular*.

The question of existence and uniqueness of solutions of the equation is discussed. The equation is transformed first into the form $\frac{d^2v}{dz^2} + J(z)v = 0$, by an elementary change of variables. Existence of solutions is obtained from an integral equation equivalent to the original problem. An iteration process is used to produce a sequence of analytic functions $\{v_n\}$. Then it is shown that, in a neighborhood of an ordinary point, this sequence converges uniformly to a solution of the equation. Uniqueness of the solution comes also from this process.

The solutions near an ordinary point are presented in the case of an *ordinary singular point*. These are points $c \in \mathbb{C}$ where p or q have a pole, but $(z-c)p(z)$ and $(z-c)^2q(z)$ are analytic functions in a deleted neighborhood of $z = c$. The so-called *method of Frobenius* is then used to seek formal series solutions in the form

$$u(z) = (z-c)^{\alpha}\left[1 + \sum_{n=1}^{\infty} a_n(z-c)^n\right].$$

The so-called *indicial equation* $\alpha^2 + (p_0 - 1)\alpha + q_0 = 0$ and its roots α_1, α_2, control the properties of these formal power series. The numbers p_0, q_0 are the leading terms of $(z-c)p(z)$ and $(z-c)^2q(z)$, respectively. It is shown that if α_1, α_2 do not differ by an integer, there are two formal solutions and these series actually converge and thus represent actual solutions. Otherwise one of the formal series is an actual solution and there is a procedure to obtain a second solution containing a logarithmic term. The reader will find in [151] all the details.

It is a remarkable fact that the behavior of the singularities determines the equation itself. For example, the most general differential equation of second order which has every point except a_1, a_2, a_3, a_4 and ∞ as ordinary points and these five points as regular points, must be of the form

$$\frac{d^2u}{dz^2} + \left\{ \sum_{r=1}^{4} \frac{1-\alpha_r-\beta_r}{z-a_r} \right\} \frac{du}{dz} + \left\{ \sum_{r=1}^{4} \frac{\alpha_r\beta_r}{(z-a_r)^2} + \frac{Az^2+Bz+C}{(z-a_1)(z-a_2)(z-a_3)(z-a_4)} \right\} u = 0,$$

for some constants α_r, β_r, A, B, C. F. Klein [376] describes how all the classical equations of Mathematical Physics appear in this class. Six classes, carrying the names of their discoverers (Lamé, Mathieu, Legendre, Bessel, Weber–Hermite and Stokes) are discussed in later chapters.

The chapter finally discusses the so-called Riemann P-function. This is a mechanism used to write a solution of an equation with three singular points and the corresponding roots of the indicial equation. Some examples of formal rules on P, which allow to transform a solution with expansion at one singularity to another are presented. The chapter concludes showing that a second order equation with three regular singular points may be converted to the *hypergeometric equation*. This is the subject of Chapter 14.

The modern theory of this program, to classify differential equations by their singularities, is its extension to nonlinear equations. A singularity of an ordinary differential equation is called *movable* if its location depends on the initial condition. An equation is called a *Painlevé equation* if its only movable singularities are poles. Poincaré and Fuchs proved that any first-order equation with this property may be transformed into the Ricatti equation or it may be solved in terms of the Weierstrass elliptic function. Painlevé considered the case of second order, transformed them into the form $u'' = R(u, u', z)$, where R is a rational function. Then he put them into 50 *canonical forms* and showed that all but six may be solved in terms of previously known functions. The six remaining cases gave rise to the *six Painlevé functions* $P_{\mathrm{I}}, \ldots, P_{\mathrm{VI}}$. See [261, 310, 336] for details. It is a remarkable fact that these functions, created for an analytic study, have recently appeared in a large variety of problems. See [37] and [562] for their appearance in combinatorial questions, [76, 636] for their relations to classical functions, [640] for connections to orthogonal polynomials, [632] for their appearance in Statistical Physics. The reader will find in [212] detailed information about their asymptotic behavior.

Chapter 11. Integral Equations. Given a function f, continuous on an interval $[a, b]$ and a *kernel* $K(x, y)$, say continuous on both variables or in the region $a \le y \le x \le b$ and vanishing for $y > x$, the equation

$$\phi(x) = f(x) + \lambda \int_a^b K(x,y)\phi(y)\,dy$$

for the unknown function ϕ, is called the *Fredholm integral equation of the second kind*. The solution presented in this chapter is based on the construction of functions $D(x, y, \lambda)$ and $D(\lambda)$, both entire in λ, as a series in which the nth-term consists of determinants of order $n \times n$ based on the function $K(x, y)$. The solution is then expressed as

$$\phi(x) = f(x) + \frac{1}{D(\lambda)} \int_a^b D(x,\xi,\lambda) f(\xi)\,d\xi.$$

In particular, in the homogeneous case $f \equiv 0$, there is a unique solution $\phi \equiv 0$ for those values of λ with $D(\lambda) \neq 0$. A process to obtain a solution for those values of λ with $D(\lambda) = 0$ is also described.

Volterra introduced the concept of *reciprocal functions* for a pair of functions $K(x, y)$ and $k(x, y; \lambda)$ satisfying the relation

$$K(x,y) + k(x,y;\lambda) = \lambda \int_a^b k(x,\xi;\lambda)K(\xi,y)d\xi.$$

Then the solution to the Fredholm equation is given by

$$f(x) = \phi(x) + \lambda \int_a^b k(x,\xi;\lambda)\phi(\xi)d\xi.$$

The last part of the chapter discusses the equation

$$\Phi(x) = f(x) + \lambda \int_a^b K(x,\xi)\Phi(\xi)\,d\xi$$

and the solution is expressed as a series in terms of a sequence of orthonormal functions and the sequence $\{\lambda_n\}$ of eigenvalues of the kernel $K(x, y)$. In detail, if $f(x) = \sum b_n\phi_n(x)$, then the solution Φ is given by $\Phi(x) = \sum \frac{b_n\lambda_n}{\lambda-\lambda_n}\phi_n(x)$.

The Fredholm equation is written formally as $\Phi = f+K\Phi$ and this gives $\Phi = f+Kf+K^2\Phi$. Iteration of this process gives the so-called *Neumann series* $\Phi = \sum_{n=0}^{\infty} K^n f$, expressing the unknown Φ in terms of iterations of the functional defined by the kernel K.

The study of Fredholm integral equations is one of the beginnings of modern Functional Analysis. The reader will find more details in P. Lax [415]. The ideas of Fredholm have many applications: the reader will find in H. P. McKean [460] a down-to-earth explanation of Fredholm's work and applications to *integrable systems* (such as the Korteweg–de Vries equation $u_t = u_{xxx} + 6uu_x$ and some special solutions called *solitons*), to the calculations of some integrals involving Brownian paths (such as P. Lévy's formula for the area generated by a two-dimensional Brownian path) and finally to explain the appearance of the so-called *sine kernel* in the limiting distribution of eigenvalues of random unitary matrices. This subject has some mysterious connections to the *Riemann hypothesis* as described by B. Conrey [154].

The second part of the book is called **The Transcendental Functions** and it consists of 12 chapters. A brief description of them is provided next.

Chapter 12. The Gamma-Function. This function, introduced by Euler, represents an extension of factorials $n!$ from positive integers to complex values of n. The presentation begins with the infinite product

$$P(z) = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

where $\gamma = \lim\limits_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right)$ is nowadays called the *Euler–Mascheroni constant*. The product is an entire function of $z \in \mathbb{C}$ and the Gamma function is defined by $\Gamma(z) = 1/P(z)$. Therefore $\Gamma(z)$ is an analytic function except for simple poles at $z = 0, -1, 2, \ldots$. The constant γ is identified as $-\Gamma'(1)$. The fact that Γ is a transcendental function is reflected by the fact, mentioned in this chapter, that Γ does not satisfy a differential equation with coefficients being rational functions of z. The chapter contains proofs of a couple of representations by Euler

$$\begin{aligned}\Gamma(z) &= \frac{1}{z} \prod_{n=1}^{\infty} \left(1 + \frac{1}{z}\right)^{z} \left(1 + \frac{z}{n}\right)^{-1} \\ &= \lim_{n\to\infty} \frac{(n-1)!}{z(z+1)\cdots(z+n-1)} n^{z}.\end{aligned}$$

The functional equation $\Gamma(z+1) = z\Gamma(z)$ follows directly from here. Using the value $\Gamma(1) = 1$, this leads to $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$, showing that Γ interpolates factorials.

The chapter also presents proofs of the *reflection formula*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

leading to the special value $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. There is also a discussion of the *multiplication formula* due to Gauss

$$\Gamma(nz) = (2\pi)^{-(n-1)/2} n^{-1/2+nz} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right)$$

and the special *duplication formula* of Legendre

$$\Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1} \Gamma(z)\Gamma(z + \tfrac{1}{2}).$$

This may be used to derive the relation $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{2\pi}{\sqrt{3}}$. Arithmetical properties of these values are difficult to establish. The reader is referred to [92] and [166] for an elementary presentation of the Gamma function, and to [106] for an introduction to issues of transcendence.

There are several integral representations of the Gamma function established in this chapter. Most of them appear in the collection of integrals by Gradshetyn and Ryzhik [258]. The first one, due to Euler, is

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt,$$

valid for $\operatorname{Re} z > 0$. This may be transformed to the logarithmic scale

$$\Gamma(z) = \int_0^1 \left(\log \frac{1}{x}\right)^{z-1} dx.$$

There is also a presentation of *Hankel's contour integral*

$$\Gamma(z) = -\frac{1}{2i\sin\pi z}\int_C (-t)^{z-1}e^{-t}\,dt, \quad z \notin \mathbb{Z}$$

where C is a thin contour enclosing the positive real axis.

The chapter also contains a discussion of two functions related to Γ: its logarithm $\log\Gamma(z)$ and the *digamma function*, $\psi(z) = \Gamma'(z)/\Gamma(z) = (\log\Gamma(z))'$. Integral representations for $\psi(z)$ include

$$\begin{aligned}\psi(z) &= \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}}\right)dt \\ &= \int_0^\infty \left(e^{-x} - \frac{1}{(1+x)^z}\right)\frac{dx}{x};\end{aligned}$$

the first one is due to Gauss and the second one to Dirichlet. The chapter also contains a multi-dimensional integral due to Dirichlet that can be reduced to a single variable problem:

$$\begin{aligned}&\int_{\mathbb{R}_+^n} f(t_1 + \cdots + t_n)t_1^{a_1-1}\cdots t_n^{a_n-1}dt_1\cdots dt_n \\ &\quad = \frac{\Gamma(a_1)\cdots\Gamma(a_n)}{\Gamma(a_1+\cdots+a_n)}\int_0^1 f(\tau)\tau^{a_1+\cdots+a_n-1}\,d\tau.\end{aligned}$$

Other multi-dimensional integrals appear in the modern literature. For a description of a remarkable example due to Selberg, the reader is referred to [214].

The properties of $\log\Gamma(z)$ presented in this chapter include a proof of the identity

$$\frac{d^2}{dz^2}\log\Gamma(z+1) = \sum_{k=1}^\infty \frac{1}{(z+k)^2},$$

showing that $\Gamma(z+1)$ is log-convex. This property, the functional equation and the value $\Gamma(1) = 1$ characterize the Gamma function. The reader will also find two integral representations due to Binet

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)\frac{e^{-tz}}{t}\,dt$$

and

$$\log\Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log 2\pi + 2\int_0^\infty \frac{\tan^{-1}(\frac{t}{z})}{e^{2\pi t}-1}\,dt.$$

Integrals involving $\log\Gamma(z)$ present interesting challenges. The value

$$\int_0^1 \log\Gamma(z)\,dz = \log\sqrt{2\pi}$$

due to Euler, may be obtained from the reflection formula for $\Gamma(z)$. The generalization of the previous evaluation to

$$L_n = \int_0^1 (\log\Gamma(z))^n\,dz$$

is discussed next. The value of L_2 is presented in [196] as an expression involving the

Riemann zeta function and its derivatives. The values of L_3 and L_4 were obtained in [38] and they involve more advanced objects: *multiple zeta values*. At the present time an evaluation of L_n, for $n \geq 5$, is an open question.

The chapter also contains a discussion of the asymptotic behavior of $\log \Gamma(z)$, as a generalization of Stirling's formula for factorials and also a proof of the expression for the Fourier series of $\log \Gamma(z)$ due to Kummer. The *Barnes G-function*, an important generalization of $\Gamma(z)$, appears in the exercises at the end of this chapter. A detailed presentation of these and other topics may be found in [20].

Chapter 13. The Zeta-Function of Riemann. For $s = \sigma + it \in \mathbb{C}$, the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is the *Riemann zeta function*. This had been considered by Euler for $s \in \mathbb{R}$. For $\delta > 0$, the series defines an analytic function of s on the half-plane $\sigma = \operatorname{Re} s \geq 1 + \delta$. The function admits the integral representation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1} e^{-x}}{1 - e^{-x}}\, dx.$$

Euler produced the infinite product

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

where the product extends over all prime numbers. This formula shows that $\zeta(s)$ has no zeros in the open half-plane $\operatorname{Re} s > 1$. The auxiliary function

$$\xi(s) = \frac{1}{2} \pi^{-s/2} s(s-1) \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

is analytic and satisfies the identity $\xi(s) = \xi(1-s)$. This function now shows that $\zeta(s)$ has no zeros for $\operatorname{Re} s < 0$, aside for the so-called *trivial zeros* at $s = -2, -4, -6, \ldots$ coming from the poles of $\Gamma(s/2)$. Thus all the non-trivial zeros lie on the strip $0 \leq \operatorname{Re} s \leq 1$. The *Riemann hypothesis* states that all the roots of $\zeta(s) = 0$ are on the *critical line* $\operatorname{Re} s = \frac{1}{2}$. At the end of §13.3 the authors state that:

It was conjectured by Riemann, but it has not yet been proved, that all the zeros of $\zeta(s)$ in this strip lie on the line $\sigma = \frac{1}{2}$; while it has quite recently been proved by Hardy [279] that an infinity of zeros of $\zeta(s)$ actually lie on $\sigma = \frac{1}{2}$. It is highly probable that Riemann's conjecture is correct, and the proof of it would have far-reaching consequences in the theory of Prime Numbers.

The reader will find in [93, 153, 458] more information about the Riemann hypothesis. In a remarkable new connection, it seems that the distribution of the zeros of $\zeta(s)$ is related to the eigenvalues of random matrices [367, 369].

This chapter establishes the identity

$$\zeta(2n) = \frac{(-1)^{n-1} B_{2n} (2\pi)^{2n}}{2\,(2n)!}$$

where $n \in \mathbb{N}$ and B_{2n} is the Bernoulli number. This is a generalization of the so-called *Basel* *problem* $\zeta(2) = \pi^2/6$. The solution of this problem won the young Euler instant fame. It follows that $\zeta(2n)$ is a rational multiple of π^{2n}, therefore this is a transcendental number. The arithmetic properties of the odd zeta values are more difficult to obtain. Apéry proved in 1979 that $\zeta(3)$ is not a rational number; see [27, 72, 689]. It is still unknown whether $\zeta(5)$ is irrational, but Zudilin [688] proved that one of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational. It is conjectured that *all odd zeta values* are irrational.

The literature contains a large variety of extensions of the Riemann zeta function. The chapter contains information about some of them: the *Hurwitz zeta function*

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \text{with } 0 < a \le 1$$

with integral representation

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}\, e^{-ax}}{1-e^{-x}}\, dx.$$

The chapter establishes the values of $\zeta(-m,a)$ in terms of derivatives of the Bernoulli polynomials and presents a proof of Lerch's theorem

$$\frac{d}{ds}\zeta(s,a)\bigg|_{s=0} = \log\left(\frac{\Gamma(a)}{\sqrt{2\pi}}\right).$$

The chapter mentions two further generalizations: one introduced by Lerch (see [414] for details)

$$\phi(x,a;s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i n x}}{(n+a)^s},$$

and another one by Barnes [43, 44, 45, 46]

$$\zeta_N(s,w \mid a_1,\dots,a_N) = \sum_{n_1,\dots,n_N} \frac{1}{(w+n_1a_1+\cdots+n_Na_N)^s}.$$

The reader will find in [566] more recent information on this function.

Chapter 14. The Hypergeometric Function. This function is defined by the series

$$F(a,b;c,z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n,$$

provided c is not a negative integer. Here $(u)_n = \Gamma(u+n)/\Gamma(u)$ is the Pochhammer symbol. The series converges for $|z| < 1$ and on the unit circle $|z| = 1$ if $\operatorname{Re}(c-a-b) > 0$. Many elementary functions can be expressed in hypergeometric form, for instance

$$F(1,1;1;z) = \frac{1}{1-z} \quad \text{and} \quad e^z = \lim_{b\to\infty} F\left(1,b;1;\frac{z}{b}\right).$$

The chapter begins with Gauss' evaluation $F(a,b;c;1)$ in the form

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

The function F satisfies the differential equation

$$z(1-z)\frac{d^2u}{dz^2} + [c-(a+b+1)z]\frac{du}{dz} - abu = 0.$$

This equation has 0, 1, ∞ as regular singular points and every other point is ordinary. The generalization to singular points at a, b, c with exponents given by $\{\alpha, \alpha'\}$, $\{\beta, \beta'\}$, $\{\gamma, \gamma'\}$, respectively, is the *Riemann differential equation*

$$\frac{d^2w}{dz^2} + \left[\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right]\frac{dw}{dz}$$
$$+ \left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right]\frac{w}{(z-a)(z-b)(z-c)} = 0.$$

It is shown that

$$\left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} F(\alpha+\beta+\gamma, \alpha+\beta'+\gamma; 1+\alpha-\alpha'); \frac{(z-a)(c-b)}{(z-b)(c-a)}$$

solves the Riemann differential equation. Using the invariance of this equation with respect to some permutations of the parameters (for example, the exchange of α and α') produces from $F(a, b; c; z)$ Kummer's 24 new solutions of Riemann's equation, for example

$$(1-z)^{-a}F\left(a, c-b; c; \frac{z}{z-1}\right) \quad \text{and} \quad (1-z)^{-b}F\left(c-a, b; c; \frac{z}{z-1}\right).$$

Since the solutions of a second-order differential equation form a two dimensional vector space, this type of transformation can be used to generate identities among hypergeometric series. The reader will find in [20, 339, 534, 641] more details on these ideas. The corresponding equation with *four regular singular points* at 0, 1, ∞, a is called the *Heun equation*

$$\frac{d^2u}{dx^2} + \left[\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-a}\right]\frac{du}{dx} + \left[\frac{\alpha\beta x - q}{x(x-1)(x-a)}\right]u = 0.$$

The corresponding process on the symmetries of the equation now gives 192 solutions. These are described in [452]. The reader will find in [190] an example of the appearance of Heun's equation in integrable systems.

The chapter also contains a presentation of *Barnes' integral representation*

$$F(a, b; c; z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s\, ds$$

and its use in producing an analytic continuation of the hypergeometric series. Finally, the identities of Clausen

$$\left[F(a, b; a+b+\tfrac{1}{2}; x)\right]^2 = {}_3F_2(2a, a+b, 2b; a+b+\tfrac{1}{2}, 2a+2b; x)$$

where ${}_3F_2$ is the analog of the hypergeometric series, now with three Pochhammer symbols on top and two in the bottom of the summand and Kummer's *quadratic transformation*

$$F(2a, 2b; a+b+\tfrac{1}{2}; x) = F(a, b; a+b+\tfrac{1}{2}; 4x(1-x))$$

appear as exercises in this chapter. The reader will find in [20] a detailed analysis of these topics.

Chapter 15. Legendre Functions. This chapter discusses *Legendre polynomials* $P_n(z)$ and some of their extensions. These days, the usual starting point for these functions is defining them as orthogonal polynomials on the interval $(-1, 1)$; that is,

$$\int_{-1}^{1} P_n(z)P_m(z)\,dz = 0 \quad \text{if } n \neq m,$$

plus some normalization in the case $n = m$. The starting point in this chapter is the generating function

$$(1 - 2zh + h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(z)h^n.$$

It is established from here that

$$P_n(z) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} z^{n-2r}$$

showing that $P_n(z)$ is a polynomial of degree n with leading coefficient $2^{-n}\binom{2n}{n}$.

The properties of these polynomials established in this chapter include

Rodriguez formula

$$P_n(z) = \frac{1}{2^n\, n!}\left(\frac{d}{dz}\right)^n (z^2 - 1)^n.$$

Legendre's differential equation The polynomials $P_n(z)$ are solutions of the differential equation

$$(1 - z^2)\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + n(n+1)u = 0.$$

In the new scale $x = z^2$, this equation takes its hypergeometric form

$$x(1-x)\frac{d^2y}{dx^2} + \frac{1}{2}(1-3x)\frac{du}{dx} + \frac{1}{4}n(n+1)u = 0.$$

The (more convenient) hypergeometric form $P_n(z) = {}_2F_1(n+1, -n; 1; \frac{1}{2}(1-z)$ is also established.

Recurrences The chapter presents proofs of the recurrences

$$(n+1)P_{n+1}(z) - (2n+1)zP_n(z)zP_n(z) - nzP_n(z) = 0$$

and

$$P'_{n+1}(z) - zP'_n(z) - (n+1)P_n(z) = 0.$$

Integral representations A variety of integral representations for the Legendre polynomials are presented:

- *Schläfli*:

$$P_n(z) = \frac{1}{2\pi i} \oint_C \frac{(t^2-1)^n}{2^n (t-z)^{n+1}}\, dt$$

where C is a contour enclosing z. This is then used to prove the *orthogonality relation*

$$\int_{-1}^{1} P_n(z) P_m(z)\, dz = \begin{cases} 0 & \text{if } n \neq m \\ 2/(2n+1) & \text{if } n = m. \end{cases}$$

- *Laplace*:

$$P_n(z) = \frac{1}{\pi} \int_0^{\pi} \left[z + (z^2-1)^{1/2} \cos\theta \right]^n d\theta$$

- *Mehler–Dirichlet*:

$$P_n(\cos\theta) = \frac{1}{\pi} \int_{-\theta}^{\theta} \frac{e^{(n+\frac{1}{2})i\varphi}}{(2\cos\varphi - 2\cos\theta)^{1/2}}\, d\varphi.$$

The formula of Schläfli given above is then used to extend the definition of $P_n(z)$ for $n \notin \mathbb{N}$. In order to obtain a single-valued function, the authors introduce a cut from -1 to $-\infty$ in the domain of integration.

Since the differential equation for the Legendre polynomials is of second order, it has a second solution independent of $P_n(z)$. This is called the *Legendre function of degree n of the second type*. It is denoted by $Q_n(z)$. The chapter discusses integral representations and other properties similar to those described for $P_n(z)$. For example, one has the hypergeometric expression

$$Q_n(z) = \frac{\sqrt{\pi}\,\Gamma(n+1)}{2^{n+1}\Gamma\left(n+\frac{3}{2}\right)} \frac{1}{z^{n+1}} F\left(\frac{n+1}{2}, \frac{n}{2}+1; n+\frac{3}{2}; z^{-2} \right).$$

One obtains from here

$$Q_0(z) = \frac{1}{2} \log \frac{z+1}{z-1}$$
$$Q_1(z) = \frac{1}{2} z \log \frac{z+1}{z-1} - 1.$$

In general $Q_n(z) = A_n(z) + B_n(z) \log \frac{z+1}{z-1}$ for polynomials A_n, B_n.

The chapter also includes further generalizations of the Legendre functions introduced by Ferrer and Hobson. These are called *associated Legendre functions*. Some of their properties are presented. There is also a discussion of the *addition theorem* for Legendre polynomial, as well as a short section on the Gegenbauer function. The reader will find in the *Digital Library of Mathematical Functions* developed at NIST [443] more information about these functions.

Chapter 16. The Confluent Hypergeometric Function. This chapter discusses the second-order differential equation with singularities at $\{0, \infty, c\}$ and corresponding exponents $\left\{\{\frac{1}{2}+m, \frac{1}{2}-m\}, \{-c, 0\}, \{c-k, k\}\right\}$ in the limiting situation $c \to \infty$. This is the case of *confluent singularities* (the limiting equation now has only two singularities: 0 and ∞, with 0 remaining regular and ∞ becomes an irregular singularity). After a change of variables to eliminate the first derivative term, the limiting equation becomes

$$\frac{d^2W}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right) W = 0.$$

This is called *Whittaker equation*.

The authors introduce the functions

$$M_{k,m}(z) = z^{1/2+m} e^{-z/2} \left(1 + \frac{\frac{1}{2} + m - k}{1!\,(2m+1)} z + \frac{(\frac{1}{2} + m - k)(\frac{3}{2} + m - k)}{2!\,(2m+1)(2m+2)} z^2 + \cdots\right)$$

and show that, when $2m \notin \mathbb{N}$, the functions $M_{k,m}(z)$ and $M_{k,-m}(z)$ form a fundamental set of solutions.

It turns out that it is more convenient to work with the functions $W_{k,m}(z)$ defined by the integral representation

$$W_{k,m}(z) = \frac{z^k e^{-k/2}}{\Gamma(\frac{1}{2} - k + m)} \int_0^\infty t^{-k-1/2+m} \left(1 + \frac{t}{z}\right)^{k-1/2+m} e^{-t}\, dt.$$

The reader is referred to [59, Chapter 6] for a readable description of the basic properties of these functions, called *Whittaker functions* in the literature.

The chapter also presents a selection of special functions that can be expressed in terms of $W_{k,m}(z)$. This includes the *incomplete gamma function*

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t}\, dt$$

that can be expressed as

$$\gamma(a, x) = \Gamma(a) - x^{(a-1)/2} e^{-x/2} W_{\frac{1}{2}(a-1), \frac{1}{2}a}(x),$$

as well as the *logarithmic integral function*, defined by

$$\mathrm{li}(z) = \int_0^x \frac{dt}{\log t} = -(-\log z)^{-1/2} z^{1/2} W_{-\frac{1}{2},0}(-\log z).$$

This function appears in the description of the asymptotic behavior of the function

$$\pi(x) = \text{number of primes } p \le x.$$

The *prime number theorem* may be written as $\pi(x) \sim \mathrm{li}(x)$ as $x \longrightarrow \infty$. See [191] for details.

The final example is the function

$$D_n(z) = 2^{n/2+1/4} z^{-1/2} W_{\frac{n}{2}+\frac{1}{4},-\frac{1}{4}}\left(\frac{z^2}{2}\right),$$

related in a simple manner to the *Hermite polynomials*, defined by

$$H_n(z) = (-1)^n e^{z^2/2} \left(\frac{d}{dz}\right)^n e^{-z^2/2}.$$

See [20] for information on this class of orthogonal polynomials.

Chapter 17. Bessel Functions. This chapter discusses the *Bessel functions* defined, for $n \in \mathbb{Z}$, by the expansion

$$\exp\left(\frac{z}{2}\left(t - \frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(z)t^n.$$

Some elementary properties of $J_n(z)$ are derived directly from this definition, such as $J_{-n}(z) = (-1)^n J_n(z)$, the series

$$J_n(z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!} \left(\frac{z}{2}\right)^{n+2r},$$

and the *addition theorem*

$$J_n(y+z) = \sum_{m=-\infty}^{\infty} J_m(y)J_{n-m}(z).$$

The Cauchy integral formula is then used to produce the representation

$$J_n(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^n \oint_C t^{-n-1} e^{t-z^2/4t}\,dt,$$

where C is a closed contour enclosing the origin. From here it is possible to extend the definition of $J_n(z)$ to values $n \notin \mathbb{Z}$ and produce the series representation

$$J_n(z) = \sum_{r=0}^{\infty} \frac{(-1)^r z^{n+2r}}{2^{n+2r} r!\Gamma(n+r+1)}.$$

This function is called the *Bessel function of the first kind of order n*. The integral representation of $J_n(z)$ is then used to show that $y(z) = J_n(z)$ is a solution of the differential equation

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(1 - \frac{n^2}{z^2}\right)y = 0,$$

called the *Bessel equation*. In the case $n \notin \mathbb{Z}$, the functions $J_n(z)$ and $J_{-n}(z)$ form a basis for the space of solutions. In the case $n \in \mathbb{Z}$ a second solution, independent of $J_n(z)$, is given by

$$Y_n(z) = \lim_{\varepsilon \to 0} 2\pi e^{\pi i(n+\varepsilon)} \left(\frac{J_{n+\varepsilon}(z)\cos(\pi(n+\varepsilon)) - J_{-(n+\varepsilon)}(z)}{\sin(2\pi(n+\varepsilon))}\right).$$

The functions $Y_n(z)$ are called the *Bessel functions of the second kind*.

This chapter also contains some information on some variations of the Bessel function such as

$$I_n(z) = i^{-n}J_n(iz) \quad \text{and} \quad K_n(z) = \frac{\pi}{2}\left[I_{-n}(z) - I_n(z)\right]\cot(\pi n).$$

Among the results presented here one finds

Recurrences such as

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z}J_n(z),$$
$$J_n'(z) = \frac{n}{z}J_n(z) - J_{n+1}(z)$$

and

$$z^{-n-1}J_{n+1}(z) = -\frac{1}{z}\frac{d}{dz}\left[z^{-n}J_n(z)\right]$$

which produces relations of Bessel functions of consecutive indices.

Zeros of Bessel functions it is shown that between any two non-zero consecutive roots of $J_n(z) = 0$ there is a unique root of $J_{n+1}(z) = 0$.

Integral representations such as

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - z\sin\theta)\,d\theta - \frac{\sin\pi n}{\pi}\int_0^{\infty} e^{-n\theta - z\sinh\theta}\,d\theta,$$

where, for $n \in \mathbb{Z}$, the second term vanishes.

Hankel representation in the form

$$J_n(z) = \frac{\Gamma(\frac{1}{2} - n)}{2\pi i\sqrt{\pi}}\left(\frac{z}{2}\right)^n \int_C (t^2-1)^{n-1/2}\cos(zt)\,dt$$

where C is a semi-infinite contour on the real line.

Evaluation of definite integrals such as one due to Mahler

$$K_0(x) = \int_0^{\infty}\frac{t}{1+t^2}J_0(tx)\,dt$$

and an example due to Sonine giving an expression for

$$\int_0^{\infty} x^{1-m}J_m(ax)J_m(bx)J_m(cx)\,dx.$$

A large selection of integrals involving Bessel functions may be found in [105], [258] and [544].

Series expansion The chapter also contains information about expansions of a function $f(z)$ in a series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n J_n(z) \quad \text{or} \quad f(z) = \sum_{n=0}^{\infty} a_n J_0(nz).$$

The reader will find in [20] and [59] more information on these functions at the level discussed in this chapter. Much more appears in the volume [653].

There are many problems whose solutions involve the Bessel functions. As a current problem of interest, consider the symmetric group $\mathfrak{S}_N$ of permutations π of N symbols. An increasing sequence of length k is a collection of indices $1 \le i_1 < \cdots < i_k \le N$ such that $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$. Define on $\mathfrak{S}_N$ a uniform probability distribution; that is, $\mathbb{P}(\pi) = 1/N!$ for each permutation π. Then the maximal length of an increasing subsequence of π is a random variable, denoted by $\ell_N(\pi)$, and its distribution is of interest. This is the *Ulam problem*. Introduce the centered and scaled function

$$\chi_N(\pi) = \frac{\ell_N(\pi) - 2\sqrt{N}}{N^{1/6}}.$$

Baik–Deift–Johannson [36] proved that $\lim_{N\to\infty} \mathbb{P}(\chi_N(\pi) \le x) = F(x)$, where $F(x)$, the so-called *Tracy–Widom distribution*, is given by

$$F(x) = \exp\left(-\int_x^\infty (y-x)u^2(y)\,dy\right).$$

Here $u(x)$ is the solution of the Painlevé P_{II} equation $u''(x) = 2u^3(x) + xu(x)$, with asymptotic behavior $u(x) \sim \text{Ai}(x)$ as $x \to \infty$. The *Airy function* $\text{Ai}(x)$ is defined by $\text{Ai}(x) = \sqrt{x}K_{1/3}\left(\frac{2}{3}x^{3/2}\right)/\pi\sqrt{3}$. The reader will find in [37] an introduction to this fascinating problem.

Chapter 18. The Equations of Mathematical Physics. This chapter contains a brief description of methods of solutions for the basic equations encountered in Mathematical Physics. The results are given for *Laplace's equation*

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

on a domain $\Omega \subset \mathbb{R}^3$. The chapter has a presentation on the physical problems modeled by this equation.

The results include the integral representation of the solution

$$V(x,y,z) = \int_{-\pi}^{\pi} f(z + ix\cos u + iy\sin u, u)\,du$$

as the 3-dimensional analog of the form $V(x,y) = f(x+iy) + g(x-iy)$ valid in the 2-dimensional case as well as an expression for $V(x,y,z)$ as a series with terms of the form

$$\int_{-\pi}^{\pi} (z + ix\cos u + iy\sin u)^n \begin{pmatrix}\cos mu\\ \sin mu\end{pmatrix} du.$$

This series is then converted into one of the form

$$V = \sum_{n=0}^{\infty} r^n \left\{A_n P_n(\cos\theta) + \sum_{m=1}^{\infty}\left(A_n^{(m)}\cos m\phi + B_n^{(m)}\sin m\phi\right)P_n^m(\cos\theta)\right\}$$

where P_n^m is Ferrer's version of the associated Legendre function.

The chapter also contains similar results for Laplace's equation on a sphere. For this type of domain, the authors obtain the formula

$$V(r,\theta,\phi) = \frac{a(a^2-r^2)}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi}\frac{f(\theta',\phi')\,\sin\theta'\,d\theta'\,d\phi'}{[r^2 - 2ar\{\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\} + a^2]^{3/2}},$$

and refer to Thompson and Tait [628] for further discussions on the *theory of Green's functions*. A similar analysis for an equation on a cylinder also appears in this chapter. In that case the Legendre functions are replaced by Bessel functions. Some of the material discussed in this chapter has become standard in basic textbooks in Mathematical Physics; see for instance [476].

Chapter 19. Mathieu Functions. This chapter discusses the *wave equation* $V_{tt} = c^2\Delta V$ and assuming a special form $V(x,y,t) = u(x,y)\cos(pt+\varepsilon)$ of the unknown V in a special system of coordinates (ξ, η) (introduced by Lamé) yields an equation for $u(x, y)$. Using the classical method of separation of variables $(u * x, y) = F(\xi)G(\eta)$ produces the equation

$$\frac{d^2 y}{dz^2} + (a + 16q\cos(2z))\, y = 0.$$

This is called *Mathieu's equation*. The value of a is determined by the periodicity condition $G(\eta + 2\pi) = G(\eta)$ and q is determined by a vanishing condition at the boundary. This type of equation is now called *Hill's equation*, considered by Hill [306] in a study on lunar motion. Details about this equation appear in [451] and connections to integrable systems appear in [459, 462, 463].

The authors show that $G(\eta)$ satisfies the integral equation

$$G(\eta) = \lambda \int_{-\pi}^{\pi} e^{k\cos\eta\cos\theta} G(\theta)\, d\theta$$

and this λ must be a characteristic value as described in Chapter 11.

A sequence of functions, named *Mathieu functions*, are introduced from the study of Mathieu's equation. In the case $q = 0$, the solutions are $\{1, \cos nz, \sin nz\}_{n\in\mathbb{N}}$, and via Fourier series the authors introduce functions $\{\mathrm{ce}_0(z, q), \mathrm{ce}_n(z, q), \mathrm{se}_n(z, q)\}_{n\in\mathbb{N}}$, reducing to the previous set as $q \to 0$. Some expressions for the first coefficients in the Fourier series of these functions are produced (it looks complicated to obtain exact expressions for them).

The authors present basic aspects of *Floquet theory* (more details appear in [451]). One looks for solution of Mathieu's equation in the form $y(z) = e^{\mu z}\phi(z)$, with ϕ periodic. The values of μ producing such solutions are obtained in terms of a determinant (called the *Hill determinant*). The modern theory yields these values in terms of a discriminant attached to the equation. The chapter also discusses results of Lindemann, transforming Mathieu's equation into the form

$$4\xi(1 - \xi)u'' + 2(1 - 2\xi)u' + (a - 16q + 32q\xi)u = 0.$$

This equation is *not* of hypergeometric type: the points 0, 1 are regular, but ∞ is an irregular singular point. Finally, the chapter includes some description of the asymptotic behavior of Mathieu functions. More details appear in [34] and [509].

Chapter 20. Elliptic Functions. General Theorems and the Weierstrassian Functions. Consider two complex numbers ω_1, ω_2 with non-real ratio. An *elliptic function* is a doubly-periodic functions: $f(z+2\omega_1) = f(z+2\omega_2) = f(z)$ where its singularities are at worst poles. The chapter discusses basic properties of the class $\mathcal{E}$ of elliptic functions. It is simple to verify that $\mathcal{E}$ is closed under differentiation and that the values of $f \in \mathcal{E}$ are determined by its values on the parallelogram with vertices 0, $2\omega_1$, $2\omega_1 + 2\omega_2$, $2\omega_2$. (Observe the factor of 2 in the periods.) This is called a *fundamental cell* and is denoted by $\mathbb{L}$. One may always assume that there are no poles of the function on the boundary of the cell. The first type of results deal with basic properties of an elliptic function:

(1) the number of poles is always finite; the same is true for the number of solutions of $f(z) = c$. This is independent of $c \in \mathbb{C}$ and is called the *degree* of the function f.
(2) any elliptic function without poles must be constant.

This result is used throughout the chapter to establish a large number of identities. The fundamental example

$$\wp(z) = \frac{1}{z^2} + \sum \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

where the sum runs over all non-zero $\omega = 2n\omega_1 + 2m\omega_2$, was introduced by Weierstrass. It is an elliptic function of order 2. It has a double pole at $z = 0$. It is an even function, so its zeros in the fundamental cell are of the form $\pm z_0$ mod $\mathbb{L}$. A remarkably recent formula for z_0 is given by Eichler and Zagier [192]. The $\wp$ function satisfies a differential equation

$$\left(\frac{d\wp(z)}{dz} \right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

where g_2, g_3 are the so-called *invariants* of the lattice $\mathbb{L}$. This function is then used to parametrize the algebraic curve $y^2 = 4x^3 + ax + b$, for $a, b \in \mathbb{C}$ with $a^3 + 27b^2 \neq 0$. The subject is also connected to differential equation by showing that if $y = \wp(z)$, then the inverse $z = \wp^{-1}(y)$ (given by an elliptic integral) can be written as the quotient of two solutions of

$$\frac{d^2v}{dy^2} + \left(\frac{3}{16} \sum_{r=1}^{3} (y - e_r)^{-2} - \frac{3}{8} y \prod_{r=1}^{3} (y - e_r)^{-1} \right) v = 0.$$

Here e_r are the roots of the cubic polynomial appearing in the differential equation for $\wp(z)$.

The *addition theorem*

$$\wp(z+y) = \frac{1}{4} \left[\frac{\wp'(z) - \wp'(y)}{\wp(z) - \wp(y)} \right]^2 - \wp(z) - \wp(y)$$

is established by a variety of methods. One presented by Abel deals with the intersection of the cubic curve $y^2 = 4x^3 + ax + b$ and a line and it is the basis for an *addition on the elliptic curve*, as the modern language states. Take two points $\mathfrak{a}$, $\mathfrak{b}$ on the curve and compute the line joining them. This line intersects the cubic at three points: the third is declared $-\mathfrak{a} \oplus \mathfrak{b}$. The points on the curve now form an abelian group: this is expected since the cubic may be identified with a torus $\mathbb{C}/\mathbb{L}$. The remarkable fact is that the addition of points preserves points with rational coordinates, so this set is also an abelian group. A theorem of Mordell and Weil states that this group is finitely generated. More information about the arithmetic of elliptic curves may be found in [331, 461, 592, 593]. The chapter also contains some information about two additional functions: the Weierstrass ζ-function, defined by $\zeta'(z) = -\wp(z)$ with $\lim_{z\to 0} \zeta(z) - 1/z = 0$ and the Weierstrass σ-function, defined by $(\log \sigma(z))' = \zeta(z)$ with $\lim_{z\to 0} \sigma(z)/z = 1$. These are the elliptic analogs of the cotangent and sine functions. The chapter contains some identities for them, for instance one due to Stickelberger: if $x + y + z = 0$, then

$$[\zeta(x) + \zeta(y) + \zeta(z)]^2 + \zeta'(x) + \zeta'(y) + \zeta'(z) = 0,$$

as well as the identity

$$\wp(z) - \wp(y) = -\frac{\sigma(z+y)\,\sigma(z-y)}{\sigma^2(z)\sigma^2(y)},$$

just to cite two of many. Among the many important results established in this chapter, we select three:

(1) any elliptic function f can be written in the form $R_1(\wp)+R_2(\wp)\wp'(z)$, with R_1, R_2 rational functions;
(2) every elliptic function f satisfies an *algebraic differential equation*;
(3) any curve of genus 1 can be parametrized by elliptic functions.

The chapter contains a brief discussion on *the uniformization of curves of higher genus*. This problem is discussed in detail in [7, 12, 477].

Chapter 21. The Theta-Functions. The study of the function

$$\vartheta(z,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niz}$$

with $q = \exp(\pi i\tau)$ and $\operatorname{Im}\tau > 0$ was initiated by Euler and perfected by Jacobi in [349]. This is an example of a *theta function*. It is a non-constant analytic function of $z \in \mathbb{C}$, so it cannot be elliptic, but it has a simple transformation rule under $z \mapsto z+\tau$. This chapter considers ϑ, relabelled as ϑ_1 as well as three other companion functions ϑ_2, ϑ_3 and ϑ_4. These functions have a single zero in the fundamental cell $\mathbb{L}$ and since they transform in a predictable manner under the elements of $\mathbb{L}$, it is easy to produce elliptic functions from them. This leads to a remarkable series of identities such as

$$\vartheta_3(z,q) = \vartheta_3(2z,q^4) + \vartheta_2(2z,q^4)$$

and

$$\vartheta_2^4(0,q) + \vartheta_4^4(0,q) = \vartheta_3^4(0,q)$$

that represents a parametrization of the Fermat projective curve $x^4 + y^4 = z^4$. The chapter also discusses the *addition theorem*

$$\vartheta_3(z+y)\vartheta_3(z-y)\vartheta_3^2(0) = \vartheta_3^2(y)\vartheta_3^2(z) + \vartheta_1^2(y)\vartheta_1^2(z)$$

(where the second variable q has been omitted) as well as an identity of Jacobi

$$\vartheta_1'(0) = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0).$$

This corresponds to the *triple product identity*, written as

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}p^2)(1+q^{2m-1}p^{-2}) = \sum_{n=-\infty}^{\infty} q^{n^2}p^{2n},$$

using the representation of theta functions as infinite products. The literature contains a variety of proofs of this fundamental identity; see Andrews [19] for a relatively simple one, Lewis [433] and Wright [683] for enumerative proofs and [311] for more general information on the so-called *q-series*. The chapter also shows that a quotient of theta functions ξ satisfies the differential equation

$$\left(\frac{d\xi}{d\tau}\right)^2 = \left(\vartheta_2^2(0) - \xi^2\vartheta_3^2(0)\right)\left(\vartheta_3^2(0) - \xi^2\vartheta_2^2(0)\right).$$

This is Jacobi's version of the differential equation satisfied by the Weierstrass $\wp$-function. The properties of solutions of this equation form the subject of the next chapter.

The reader will find in Baker [39, 40] a large amount of information on these functions from the point of view of the 19th century, Mumford [478, 479, 480] for a more modern point of view and [208, 209] for their connections to Riemann surfaces. Theta functions appeared scattered in the magnificent collection by Berndt [60, 61, 62, 63, 64] and Andrews–Berndt [21, 22, 23, 24, 25] on the formulas stated by Ramanujan.

Chapter 22. The Jacobian Elliptic Functions. Each elliptic function f has a *degree* attached to it. This is defined as the number of solutions to $f(z) = c$ in a fundamental cell. Constants have degree 0 and there are no functions of degree 1. A function of degree 2 either has a double pole (say at the origin) or two simple poles. The first case corresponds to the Weierstrass $\wp$ function described in Chapter 20. The second case is discussed in this chapter. The starting point is to show that any such function $y = y(u)$ may be written as a quotient of theta functions. From here the authors show that y must satisfy the equation

$$\left(\frac{dy}{du}\right)^2 = \left(1 - y^2\right)\left(1 - k^2 y^2\right)$$

where $k \in \mathbb{C}$ is the *modulus*. An expression for k as a ratio of null-values of theta functions is provided. Then $y = y(u)$ is seen to come from the inversion of the relation

$$u = \int_0^y \left(1 - t^2\right)^{-1/2}\left(1 - k^2 t^2\right)^{-1/2} dt$$

and, following Jacobi, the function y is called the *sinus amplitudinus* and is denoted by $y = \operatorname{sn}(u, k)$. This function becomes the trigonometrical $y = \sin u$ when $k \to 0$. Two companion functions $\operatorname{cn}(u, k)$ and $\operatorname{dn}(u, k)$ are also introduced. These functions satisfy a system of nonlinear differential equations

$$\dot{X} = YZ, \quad \dot{Y} = -ZX, \quad \dot{Z} = -k^2 XY,$$

and they are shown to parametrize the curve $\xi^2 = (1 - \eta^2)(1 - k^2\eta^2)$.

The chapter also contains an addition theorem for these functions, such as

$$\operatorname{sn}(u + v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \; dnu}{1 - k^2 \operatorname{sn}^2 u \; sn^2 v}$$

and other similar expressions.

The complete elliptic integral of the first kind $K(k)$ (and the complementary one $K'(k)$) appears here from $\operatorname{sn}(K(k), k) = 1$. The authors establish an expression for $K(k)$ in terms of theta values, prove Legendre's identity

$$\frac{d}{dk}\left(k(k')^2 \frac{dK}{dk}\right) = kK,$$

and present a discussion of the periods of the (Jacobian elliptic) functions sn, cn, dn in terms of elliptic integrals. The reader will find details of these properties in [90, 461]. Other results appearing here include product representations of Jacobi functions, the *Landen transformation* and several definite integrals involving these functions. There is also a discussion on the so-called *singular values*: these are special values of the modulus k such that the ratio

$K'(k)/K(k)$ has the form $(a + b\sqrt{n})/(c + d\sqrt{n})$ with a, b, c, d and $n \in \mathbb{Z}$. These values of k satisfy polynomial equations with integer coefficients. The authors state that the study of these equation *lies beyond the scope of this book*. The reader will find information about these equations in [90].

Chapter 23. Ellipsoidal Harmonics and Lamé's Equation. This chapter presents the basic theory of *ellipsoidal harmonics*. It begins with the expression

$$\Theta_p = \frac{x^2}{a^2 + \theta_p} + \frac{y^2}{b^2 + \theta_p} + \frac{z^2}{c^2 + \theta_p} - 1$$

where $a > b > c$ are the semi-axis of the ellipsoid $\Theta_p = 0$. A function of the form $\Pi_m(\Theta) = \Theta_1 \cdots \Theta_m$ is called an *ellipsoidal harmonic of the first species*. The chapter describes harmonic functions (that is, one satisfying $\Delta u = 0$) of this form. It turns out that every such function (with n even) has the form

$$\prod_{p=1}^{n/2} \left(\frac{x^2}{a^2 + \theta_p} + \frac{y^2}{b^2 + \theta_p} + \frac{z^2}{c^2 + \theta_p} - 1 \right)$$

where $\theta_1, \ldots, \theta_{n/2}$ are zeros of a polynomial $\Lambda(\theta)$ of degree $n/2$. This polynomial solves the *Lamé equation*

$$4\sqrt{(a^2 + \theta)(b^2 + \theta)(c^2 + \theta)} \frac{d}{d\theta} \left[\sqrt{(a^2 + \theta)(b^2 + \theta)(c^2 + \theta)} \frac{d\Lambda}{d\theta} \right] = [n(n+1)\theta + C]\, \Lambda(\theta).$$

The value C is constant and it is shown that there are $\frac{1}{2}n + 1$ possible choices. There are three other types of ellipsoidal harmonics with a similar theory behind them.

The chapter contains many versions of Lamé's equation: the *algebraic form*

$$\frac{d^2\Lambda}{d\lambda^2} + \frac{1}{2} \left(\frac{1}{a^2 + \lambda} + \frac{1}{b^2 + \lambda} + \frac{1}{c^2 + \lambda} \right) \frac{d\Lambda}{d\lambda} = \frac{[n(n+1)\lambda + C]\, \Lambda}{4(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}$$

as well as the *Weierstrass elliptic form*

$$\frac{d^2\Lambda}{du^2} = [n(n+1)\wp(u) + B]\, \Lambda$$

and finally the *Jacobi elliptic form*

$$\frac{d^2\Lambda}{d\alpha^2} = \left[n(n+1)k^2 \mathrm{sn}^2\alpha + A \right] \Lambda.$$

These equations are used to introduce *Lamé functions*. These are used to show that there are $2n + 1$ ellipsoidal harmonics that form a fundamental system of the harmonic functions of degree n.

The chapter contains a brief comment on work by Heun [300, 301] mentioning the study of an equation with *four* singular points. The reader will find in Ronveaux [563] more information about this equation.